Scientific Computing using Matlab

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Lecture 59
Method of Undetermined Coefficients

Hello viewers, welcome back to the course on Scientific Computing using Matlab. So, in the last lecture we have started with the Method of Undetermined Coefficients, so we will continue with that one.

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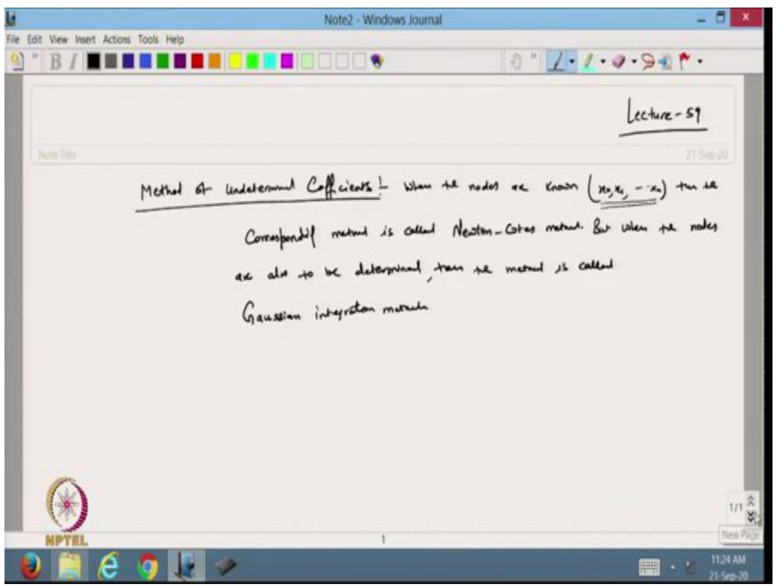

So, today we are going to discuss the method of Undetermined Coefficients. So, in this case we have discussed in the last lecture also, so when nodal values, or nodes I can say that, when the nodes are known, that nodes are this one x0, x1,...,xn, the value of this one. So, when the nodes are known then the corresponding method is called Newton-Codes method. But when the nodes are also to be determined, then the method is called Gaussian integration methods.

So, in this case this value, when the values of the nodes are known to us, then we will call it in the Newton-cotes methods otherwise it is called the Gaussian integration methods. So, let us do one example, how we can deal with the method of undetermined coefficients. So, let us take example 1. Determine the coefficient a, b, c and the error for, so this is the formula given to us,

$$\int_{-h}^{h} f(x)dx = af(-h) + bf(0) + cf(h)$$

. So, in this case if you see we have some domain, so this domain is my 0, this is my -h and this is h. So, I want to do the integration, some function is defined to mean this case. So, in this case I want to do the integration, so what I do is, it is given to me that this is the coefficient a, b, c, that is not known to me here.

And this is multiplying the value of the function f at -h, value the function at 0, and the value of the function at h, so in this case we want to find the value of a, b, c. Now, I do not know what is the value of function here, but I need it, so here we need to find 3 coefficients that are a, b, and c. So, this is the method of undetermined coefficients we do not know what is the value of a, b, c and we want to find out. So, this, I take it as equation number 1.

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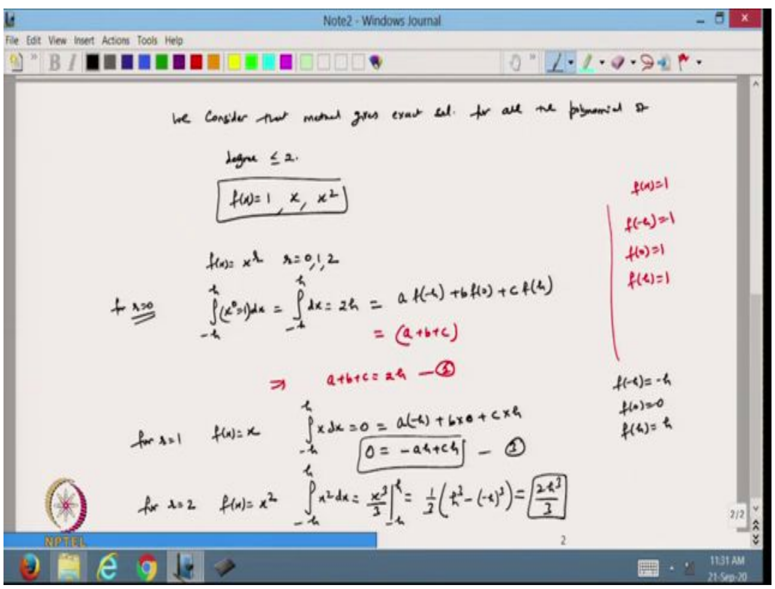

Now, I need to find 3 coefficients, so whenever I need to find 3 coefficients we consider that the method gives an exact solution for all the functions, all the polynomials I should write, of degree less than equal to 2. So, in this case I will consider that my f(x) is constant so that I could call it 1, then it is x, then it is x square. So, in this case I have the polynomial degree that is 0, 1 and 2.

Let us start doing that how we can find out. So for now in this we are considering, now we consider the function $f(x) = x^r, \quad r = 0, 1, 2$. So, that we have already discussed. Now, from minus h to h I take r is equal to, so 4 is equal to 0. The first one I take. So, this will be

$$\int_{-h}^{h} (x^0 = 1)dx = \int_{-h}^{h} dx = 2h = af(-h) + bf(0) + cf(h)$$

. Now, what is the meaning of f(-h)? So, let us discuss this one, what is the f(-h)? But I know that the value the function f(x) is just 1 for any value of x so it means that this is equal to 1 also, f(0)=1, f(h)=1.

So, from here I can write that a+b+c=2h. So, I call it equation number 2. Then for r=1, my function f(x)=x. Now from the left hand side -h to h x dx, so it is an odd function, and I am multiplying from -h to h so I know that this value would be 0. So, it is just this value.

And on the right hand side I will get a, what will be the value of f(-h). It means I am putting x=-h, so the value of the function will be -h, f(0)=0, f(h)=h. So, from here I can write that this is equal to a(-h)+b*0+c*h=-ah+ch=0.

So, this value is and that is equal to 0, so this is equal to 0. So, from here I can write that this is my next equation. Now for r=2, f(x) becomes x square, then if I multiply by putting the value of x square dx, then in that case I will get, $\int_{-h}^{h} x^2 dx = \frac{x^3}{3}\Big|_{-h}^{h} = \frac{2h^3}{3}$. So, this is the value I am getting after solving the integration.

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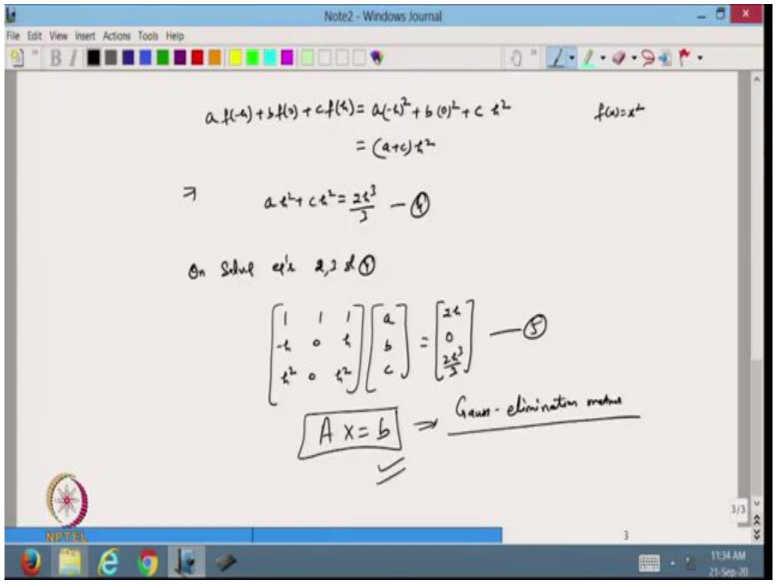

Now, on the right hand side I will get, now af(-h)+bf(0)+cf(h), so this one I need to find out. Now, from here $a(-h)^2 + b(0)^2 + ch^2 = (a+c)h^2$. Now from here I can write that this will be equal to $ah^2 + ch^2 = \frac{2h^3}{3}$.

So, that is the value, and now this is my equation number 4. Now suppose I have 3 variables this a, b, c and the three equations that we have now, so from solving equations 2, 3 and 4. So this one we can solve by the methods we have applied for solving the system of equations. So, this is basically a linear system equation. So, from here I can get, this is the system I will get, a, b, c. So, if I get the a, b, c then this value will be 1, 1, 1 here, because the first equation was a+b+c=2h.

Then the next one was -h 0 and that was h, and that was equal to 0 and the last one is I call it h square 0 and this is also h square because this is $ah^2 + ch^2 = \frac{2h^3}{3}$. So, this is equation number 5. So, this is the system of equations like Ax=b. And we know that we can solve it using maybe Gauss-elimination method and then we can find out the solution.

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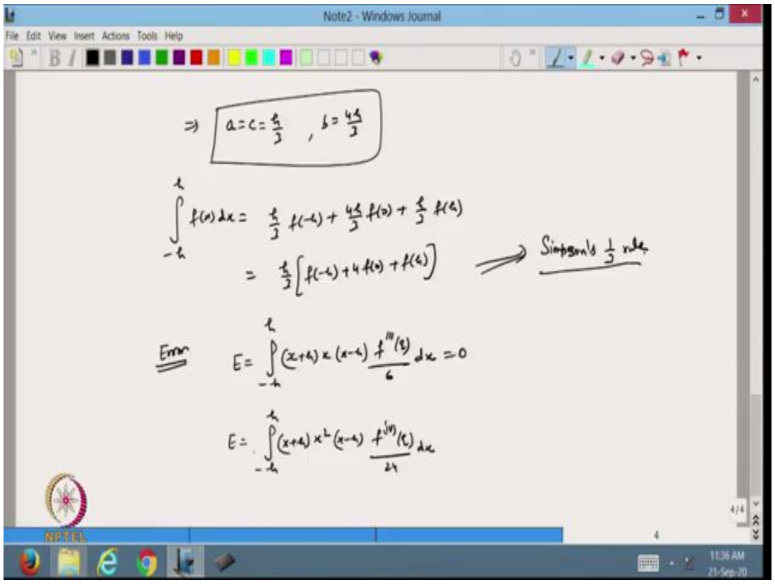

So, if we solve this one using the Gauss-elimination of any method I have discussed then from here I will get the value of a=c= h/3 and b= 4h/3 , so that is the value we got. So, if we calculate this value then, after calculating this value my integral will be equal to,

$$\int_{-h}^{h} f(x)dx = \frac{h}{3}f(-h) + \frac{4h}{3}f(0) + \frac{h}{3}f(h)$$
$$= \frac{h}{3}[f(-h) + 4f(0) + f(h)]$$

. And if you see from here then this is coming the same as Simpson one third rule. So, this is coming the same as a Simpson one third rule. Because in that case also we have the value of the function 3 points and we are able to find. So, this is similar as a Simpson one third rule. Now the error we want to find.

Now the error will be from minus h to h, so this is, because it is coming similar to Simpson one third rule so I know that the error will be $E = \int_{-h}^{h} (x+h)x(x-h)\frac{f'''(\xi)}{6}dx$. Now, if I remember from the Simpson one third rule then I know that this value is going to be 0 because we know that the Simpson one third rule is also giving exact roots, results for the cubic polynomial.

So, this will be equal to 0. So in that case we have to take the error from

$$E = \int_{-h}^{h} (x+h)x^2(x-h)\frac{f^{(iv)}(\xi)}{24}dx$$. So, this one we can calculate and now how we can calculate this one, there is a first thing is that we can do the integration and we are able to find the result.

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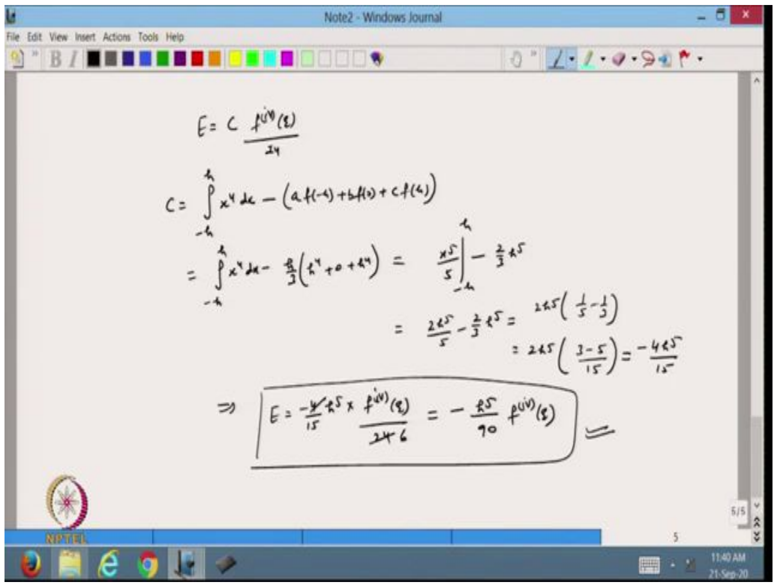

The other one we can do is that we can find that the error will be of course, it will be

$$E = \frac{Cf^{(iv)}(\xi)}{24}$$, so this C I want to find. So, this C we can find because I know that this is giving me the exact result for up to cubic, so if I do the, take the function x raised to power 4 dx, so it is not a cubic, it is a fourth derivative, fourth degree polynomial, so I know that then I can find the error. So, this will be $$C = \int_{-h}^{h} x^4 dx - (af(-h) + f(0) + cf(h))$$, the formula.

And I know the value of a, b, c so from here I can write

$$C = \int_{-h}^{h} x^4 dx - \frac{h}{3}(h^4 + 0 + h^4) = \frac{x^5}{5}\Big|_{-h}^{h} - \frac{2}{3}h^5$$

$$= \frac{2h^5}{5} - \frac{2}{3}h^5 = 2h^5(\frac{1}{3} - \frac{1}{5}) = \frac{-4h^5}{15}$$

So, this I am getting. So, from here my error will be, so C is this one, so it is

$E = \frac{-4h^5}{15}\frac{f^{(iv)}(\xi)}{24} = -\frac{h^5}{90}f^{(iv)}(\xi)$. So that is the error and it is the same as the Simpson one third rule. So in this case we have, alternately we found the error by taking the difference and then we are able to find the error that is the same as the Simpson one third rule. So this is the example we have solved.

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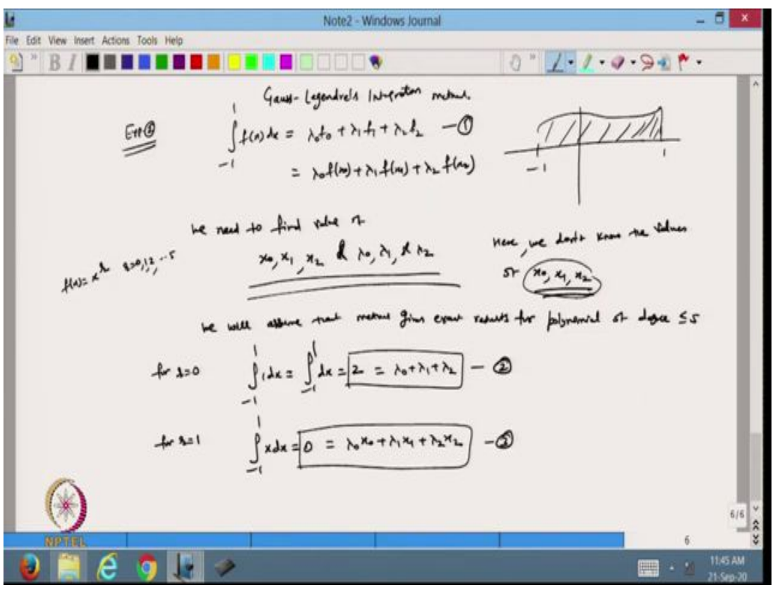

Similarly we can define another example, so let us take another example, example number 2. I have the integration $\int_{-1}^{1} f(x)dx = \lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2$, so this is given to me. Now, from here this is my -1 and this is 1. And my function is defined like this one and I want to find this integration, so this value.

Now, in this case I do not know that I should choose nodal values somewhere or I should choose any other nodal value, because the nodal value is also no-defined. So, in this case this is basically equal to $\lambda_0 f(x_0) + \lambda_1 f(x_1) + \lambda_2 f(x_2)$. So, here we do not know the values of x0, x1 and x2.

So, in this case we do not know the value of the nodal points, so once we are able to find the nodal values then this method will give you the same result as a Newton-cotes method but we do not know the value of this x0, x1, x2. So, this is a basically, if you see this is a Gauss, example of a Gauss-Legendre's integration method. So, in this case I have 6 values to find out, so we need to find values of x0, x1, x2 and the coefficient λ_0, λ_1, and λ_2.

So, 6 coefficients we need to find. So, we will assume that the method gives exact results for polynomials of degree less than equal to 5, so that we will consider. So in this case let us take that for, so my r will be in this case, it will be $f(x) = x^r, \quad r = 0, 1, 2, \ldots, 5$ So, for r=0, so if I take r=0 then do the constant, so from here I can write that this will become -1 to 1, f(x) will be 1.

So, this will be just a function $\int_{-1}^{1} dx = 2 = \lambda_0 + \lambda_1 + \lambda_2$. So that is the first equation I will get. So, this is the equation I am going to get. Then for r=1, I will get $\int_{-1}^{1} xdx$ and this is an odd function so its value comes 0. So, in this case, so this will be

$\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0$ so that is my another equation, third equation.

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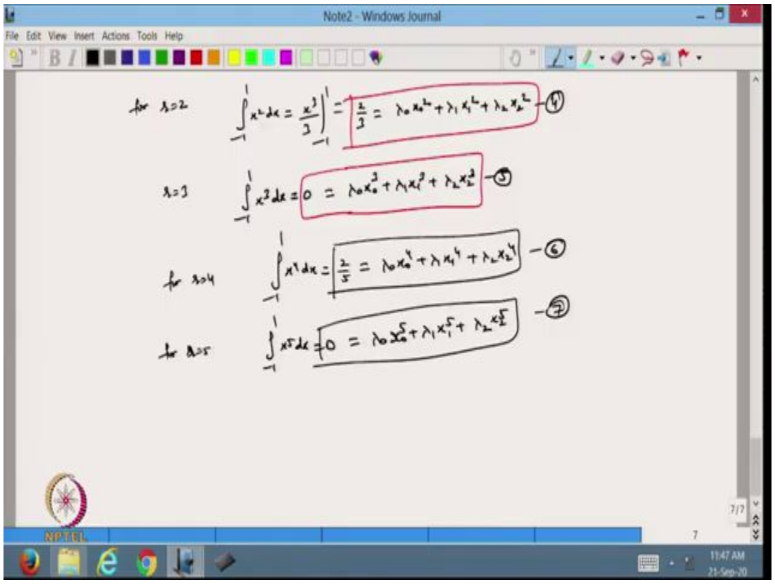

Then for r=2, so it is $\int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3} = \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2$, so this is the other equation we are going to get. So, this is my equation number 4. Similarly we can define r=3, this will be $\int_{-1}^{1} x^3 dx = 0 = \lambda_0 x_0^3 + \lambda_1 x_1^3 + \lambda_2 x_2^3$. So this is equation number 5, this is 4, this is 5. Then for r=4, this will be

$\int_{-1}^{1} x^4 dx = \frac{2}{5} = \lambda_0 x_0^4 + \lambda_1 x_1^4 + \lambda_2 x_2^4$, this is equation number 6.

And the last one for r=5, this is $\int_{-1}^{1} x^5 dx = 0 = \lambda_0 x_0^5 + \lambda_1 x_1^5 + \lambda_2 x_2^5$. So that is the last equation, so this is the seventh one. So we have 6 equations and 6 unknowns, so the unknowns I can take as $\lambda_0, \lambda_1, \lambda_2$, x0, x1, x2.

So, this is my vector with the 6 values, so we have 6 equations and 6 unknowns which can be solved using Gauss-elimination or other iterative methods because Gauss-elimination is a direct method, it is not a iterative method, using Gauss-elimination or iterative methods, not others, or iterative methods. So, this one we can do with the, the way we solve the system of question and suppose so we solve.

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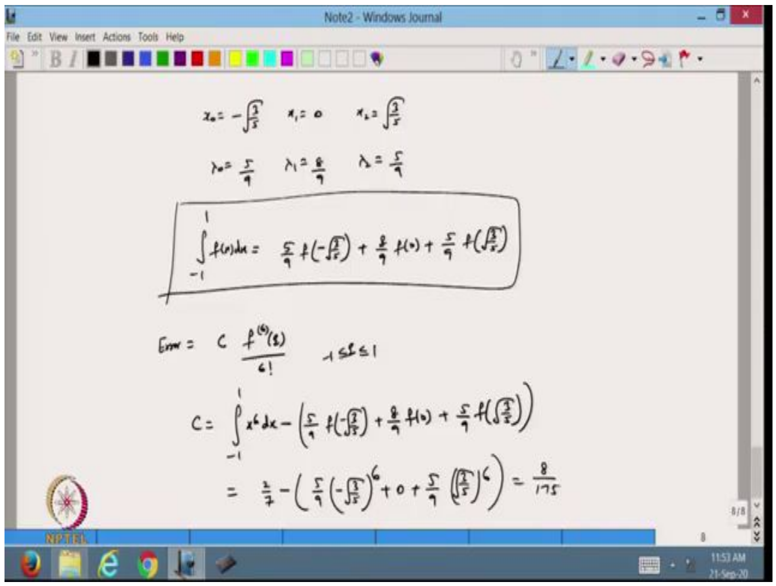

If we solve this one then the answer we are getting, that we are getting $x_0 = -\sqrt{\frac{3}{5}}$, $x_1 = 0$ and $x_2 = \sqrt{\frac{3}{5}}$. The value of $\lambda_0 = \frac{5}{9}$, $\lambda_1 = \frac{8}{9}$ and $\lambda_2 = \frac{5}{9}$. So from here we are able to find the value of all those 6 coefficients or the unknowns, so from here I can write

$$\int_{-1}^{1} f(x)dx = \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right),$$ this one.

And this under root 3 by 5 is lying between minus and m1. So that is the Gauss-Legendre integration formula for this function. So, using this one we are able to find that what is the nodal values because nodal values are not fixed in this case and then the corresponding coefficient we are able to find, so that is the way we can find out that how the nodal value can also be chosen with, so this is the way we can define undetermined coefficients.

So, from here if I want to find the error then the error associated this one, because from here I am able to find that this method is giving exact result up to the polynomial of degree 5 so the error term in this case will be, $Error = C\dfrac{f^{(6)}(\xi)}{6!}, \quad -1 \le \xi \le 1$ that I know. So this C I want to find, so C will be what, so it is

$$C = \int_{-1}^{1} x^6 dx - \left(\frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)\right)$$
$$= \frac{2}{7} - \left(\frac{5}{9}\left(-\sqrt{\frac{3}{5}}\right)^6 + 0 + \frac{5}{9}\left(\sqrt{\frac{3}{5}}\right)^6\right) = \frac{8}{175}$$

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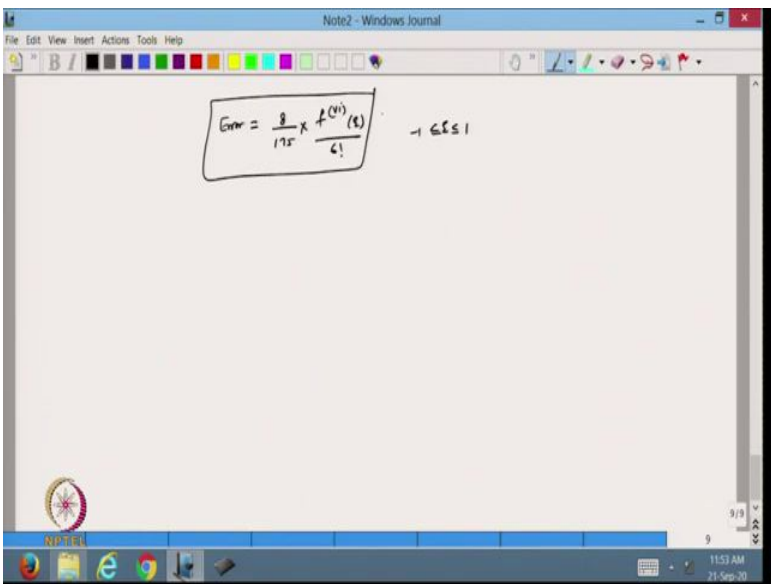

So from here I can say that the error corresponding to this will be

$$Error = \frac{8}{175} \times \frac{f^{(vi)}(\xi)}{6!}, \quad -1 \le \xi \le 1$$

. So this is giving me the exact result up to the fifth degree polynomial and that is the error introduced. So, this is the way we can find the errors and the method of undetermined coefficients. So, let me stop here today.

So, today we have discussed how we can apply the method of undetermined coefficients to find out the nodal value as well as the coefficients of that one and we also found that how the errors can be found out with the alternative methods. So that brings the end of this numerical integration. In the next lecture we will go for the Matlab codes, how the Matlab codes can be made for the methods we have studied in the numerical integration. So thanks for watching, thanks very much.