Scientific Computing Using Matlab
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Lecture 58
Simpson's 3/8 Rule for Numerical Integration

Hello viewers. Welcome back to the course on Scientific Computing using Matlab. So, in the previous lecture we have discussed the Simpson one third rule and then we also discuss how the Simpson one third rule can be used as a composite formula.

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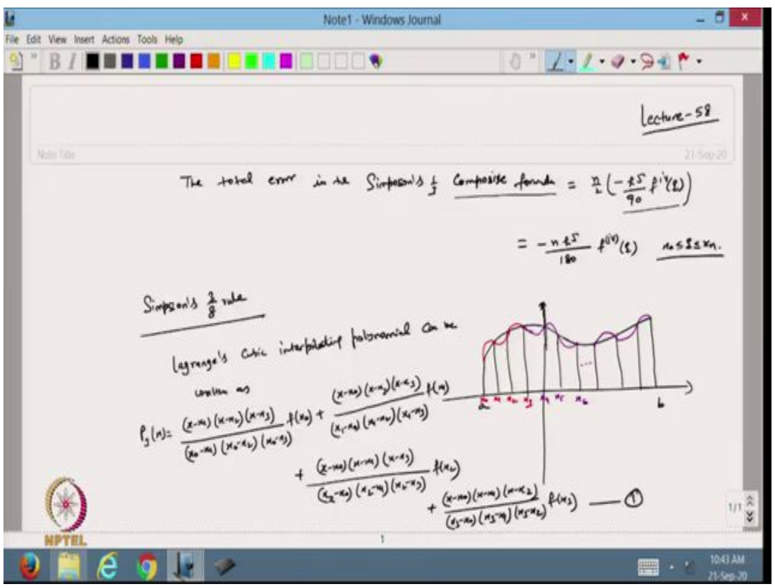

Now in the previous lecture the error, the total error in the Simpson one third for the composite formula, so that will be, because in this, in the previous case, in the Simpson one third rule, we apply the composite for n/2 times, so that is why instead of multiplying by n, we have to multiply by n/2 to the error, this one. So, this is the error for the Simpson one third rule and then I multiply it by n/2 because we are repeating this process n by 2 times.

So, from here I can write that this becomes $-\frac{nh^5}{180} f^{(iv)}(\xi), \quad x_0 \le \xi \le x_n$. So that will be our error corresponding to the composite formula. Now, after discussing the Simpson one third rule we will go for the another method that is called Simpson 3 by 8 rule. So, in this case the only change we are going to make is that as we have discussed in the previous lectures that this is my function from a to b.

And now in this case I divide the domain into n subintervals. Now instead of taking the 3 points now I choose the 4 points. So, this is the point suppose I choose and I want to interpolate the function from this point to this point with the cubic polynomial. So, the cubic polynomial passing through this point, suppose this is like this one. So, this is my cubic polynomial, I am able to find. So this is my x 0, this is x 1, x 2 and x 3.

The next will be again x 3, x 4, x 5, x 6 and from here suppose my cubic is now passing through this and it becomes, this is the function, this is my y axis and suppose it is, this is my cubic so I start with here going from here and then it is suppose this one, this value. So, this is my cubic, in the same way in the past I can have this type of cubic and this is so on. So, now in this case I am approximating the function from x naught to x3 by cubic. So, I know that the Lagrange cubic interpolating polynomial can be written as, so I represent it by p3(x).

So, this will be

$$p_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3))}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3))}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3))}{(x_1-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_1))}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

. So, this is my polynomial I can call it 1, so this is a cubic interpolating polynomial.

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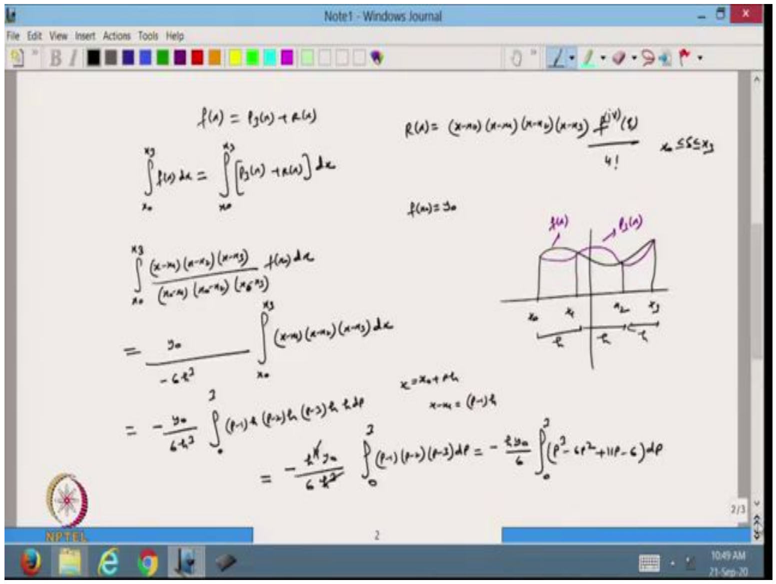

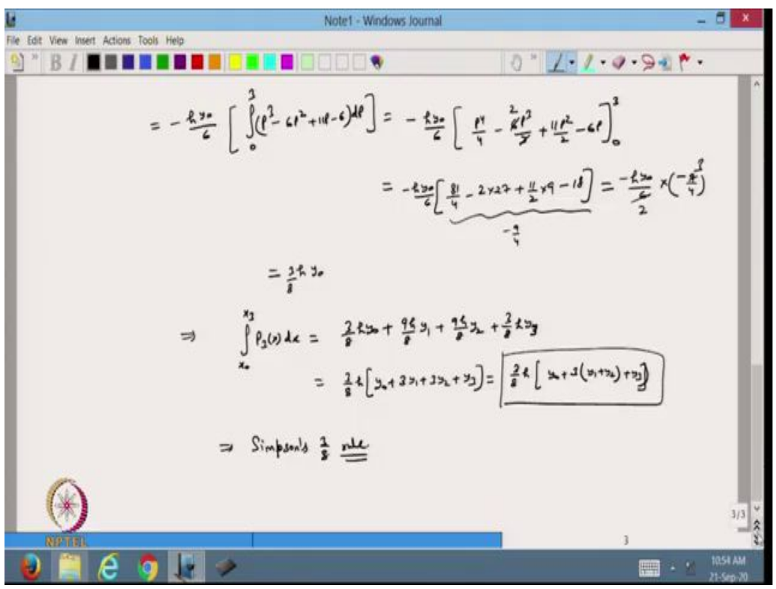

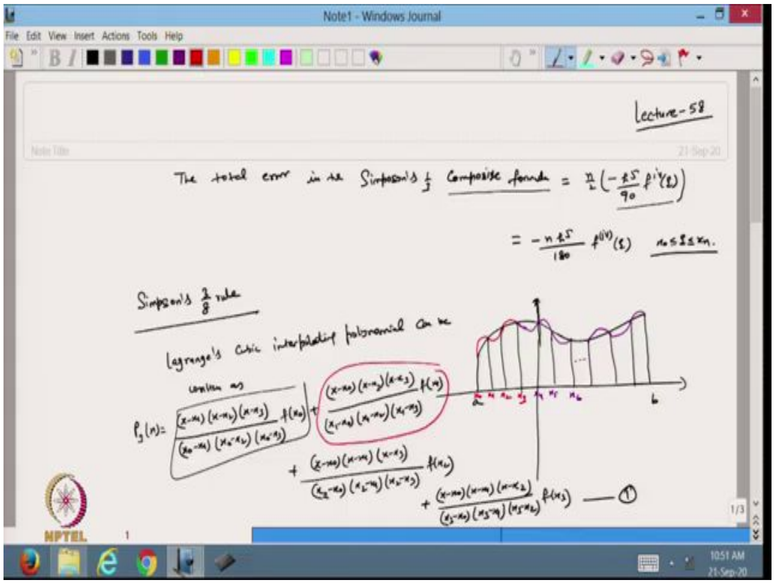

Now, in this case if I want to introduce the error, so this becomes my function f(x), I am interpolating this one with P3(x)+R(x). So this is what we are going to discuss. So, the R(x) we can define as $R(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3)\dfrac{f^{(iv)}(\xi)}{4!}$. So, this is the way we can define the corresponding error introduced in the interpolating polynomial.

So, now I can calculate the integral $\displaystyle\int_{x_0}^{x_3} f(x)dx$, so this value is given to me. So, first time, here I represented it in the composite form but you just look at this one that initially we have only this function, so initially I have only this function, maybe I can draw again. So, suppose my function is this one, from here to here and this is my x0, this is my x3 and I get this value so this is x1 and x2. And this function is approximated with this cubic polynomial.

So, this is my p3(x) and this is my function f(x). So, from here I can write this one as

$\displaystyle\int_{x_0}^{x_3} [p_3(x) + R(x)]dx$. Now the p3(x) I already defined then we can calculate this value. So, I will try to calculate just the first term and then the same way we can define the other terms. So, if I start doing this one so I have to do the integration

$$\int_{x_0}^{x_3} \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0)dx$$

.

Now, this value is h, h, this is h. So, I know that x0-x1=-h , this is -2 h, this is -3 h, so from here I can take the value and I know that f(x0), we generally represent by y0, so from here I can write

$$\frac{y_0}{-6h^3}\int_{x_0}^{x_3} (x-x_1)(x-x_2)(x-x_3)dx$$

. Now, from here again I will do the transformation x=x0+ph.

So, from here I know that my x-x1=(p-1)h, so from here I can write that this will become

$$\frac{y_0}{-6h^3}\int_{x_0}^{x_3} (p-1)h(p-2)h(p-3)h\ hdp$$

$$= \frac{-h^4 y_0}{6h^3}\int_0^3 (p-1)(p-2)(p-3)dp = \frac{-hy_0}{6}\int_0^3 (p^3-6p^2+11p-6)dp$$

$$= \frac{-hy_0}{6}\left[\frac{p^4}{4}-\frac{6p^3}{3}+\frac{11p^2}{2}-6p\right]_0^3 = \frac{-hy_0}{6}\times\left(\frac{-9}{4}\right) = \frac{3h}{8}y_0$$

. So that is the value I am going to get. Similarly, we can define the next value. So, the next value will be, again I am taking the integration of this part and then again the next and the next. So this one we can do our self and ultimately if I collect the all the terms then you start doing this one then I can have my

$$\int_{x_0}^{x_3} p_3(x)dx = \frac{3}{8}hy_0 + \frac{9h}{8}y_1 + \frac{9h}{8}y_2 + \frac{3h}{8}y_3$$

$$= \frac{3}{8}h[y_0+3y_1+3y_2+y_3] = \frac{3}{8}h[y_0+3(y_1+y_2)+y_3]$$

, so that is the formula for Simpson 3 by 8 rule. So, if the value is coming 3 by 8 so that is why it is called the Simpson 3 by 8. So, this is Simpson 3 by 8. So, this is what we have discussed.

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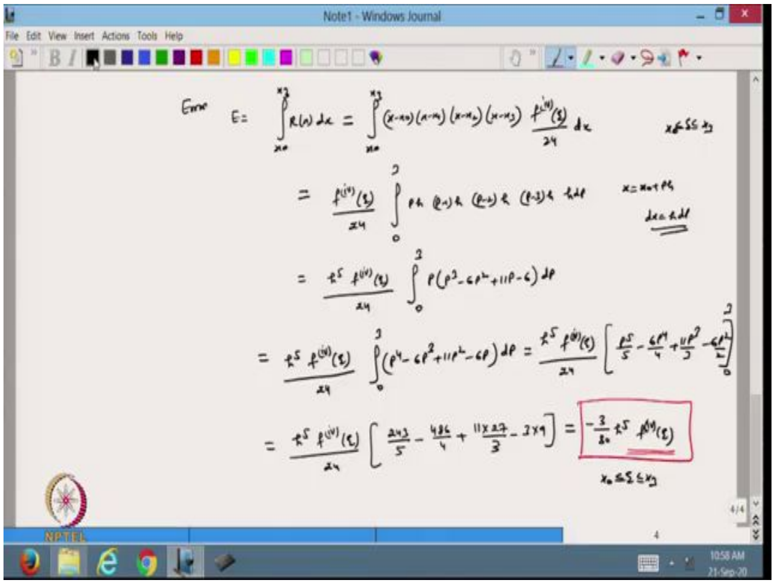

Now, I want to define the error in this case, so let us start doing the error. So the error term will be,

$$
\begin{aligned}
E &= \int_{x_0}^{x_3} R(x)dx = (x-x_0)(x-x_1)(x-x_2)(x-x_3)\frac{f^{(iv)}(\xi)}{24}dx \\
&= \frac{f^{(iv)}(\xi)}{24}\int_0^3 ph(p-1)h(p-2)h(p-3)h\, hdp, \quad x = x_0 + ph,\ dx = hdp \\
&= \frac{h^5 f^{(iv)}(\xi)}{24}\int_0^3 p(p^3-6p^2+11p-6)dp \\
&= \frac{h^5 f^{(iv)}(\xi)}{24}\int_0^3 p(p^4-6p^3+11p^2-6p)dp = \frac{h^5 f^{(iv)}(\xi)}{24}\left[\frac{p^5}{5}-\frac{6p^4}{4}+\frac{11p^3}{3}-\frac{6p^2}{2}\right]_0^3 \\
&= \frac{h^5 f^{(iv)}(\xi)}{24}\left[\frac{243}{5}-\frac{486}{4}+\frac{11\times 27}{3}-3\times 9\right] \\
&= -\frac{3}{80}h^5 f^{(iv)}(\xi), \quad x_0 \le \xi \le x_3.
\end{aligned}
$$

. So, this is the error term we are getting here. Now if you compare it with the Simpson one third rule, so in the Simpson one third rule also that was giving me the exact value for linear function, for quadratic and as well as cubic.

And here also we found that this is giving the exact result for a maximum up to cubic and after that it starts giving the error. So if you see from here then there is no much difference between the errors in the terms of Simpson one third or 3 by 8 rule. So in terms of accuracy Simpson one third and 3 by 8 are the same. So this is the way we have defined Simpson 3 by 8.

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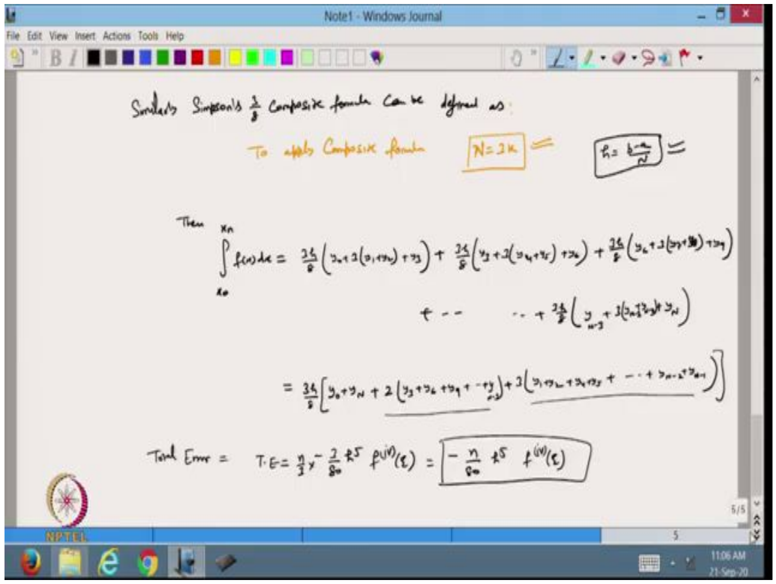

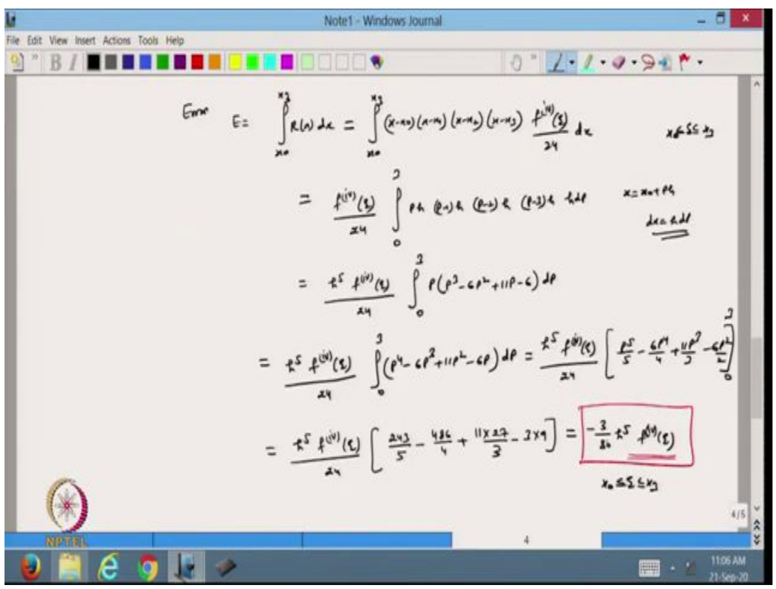

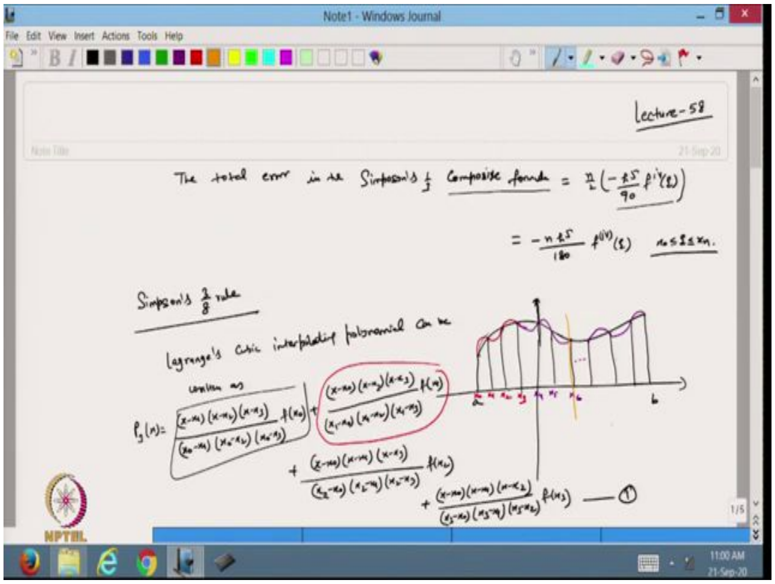

So, similarly Simpson 3 by 8 composite formula can be defined as, so in the composite formula one thing we have to keep in mind that whenever we are dealing with this one, we have 3 points in the, so we have 3 points in the first formula, where we can apply, then next we are adding 3 points more, so first are the 4 points, then 3 points, then 3 points again and again. So, if you see from here, then you can see that if I need, suppose I have only, I have my function up to this point, then there are only 4+3, 7 points, then if I have the next one, I have 7+3, 10 points.

So, like this one, from here you can see that for a composite formula, so to apply composite formula my n, the number of sub-interval should be a multiple of 3 only then we can find. So that is the condition for applying the Simpson 3 by 8 composite formula that the number of sub-intervals should be a multiple of 3. So, suppose I have k=1, then n=3, then I get the 4 points, when n=2, I get the 6 points, 6 sub-interval I mean 7 points and so on. So that is the condition for applying the composite Simpson 3 by 8 rule.

So from here, then the composite rule is

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}(y_0 + 3(y_1 + y_2) + y_3) + \frac{3h}{8}(y_3 + 3(y_4 + y_5) + y_6) + \frac{3h}{8}(y_6 + 3(y_7 + y_8) + y_9) + \ldots + \frac{3h}{8}(y_{N-3} + 3(y_{N-2} + y_{N-1}) + y_N), \quad h = \frac{b-a}{N}$$

. So, the last part of this composite will be this one.

So if I combine altogether, then it will be 3h/8, I can take common, so y0 is coming here, y3, y6, but y6 is also coming here, y3 is also coming here, so I have to take this one that this will be equal to y0+y n, first and the last that I am taking here. Then it will be 2 times the common node so this is y3 then the y 6, so this is y3, y6, then y and so on. Then plus the internal one, so this is I can write as

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}[(y_0 + y_N + 2(y_3 + y_6 +_9 + \ldots +_{N-3}) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{N-1} + y_N)]$$

. So, this is the composite formula for Simpson 3 by 8, so we have to take care about these points and these values. Now, we are doing this one all these things 4n by 3 times, so in that case if I have to apply the composite formula for this one, then error total error, so that is the total error.

So, the total error whatever we are doing here is so whenever I take formula for one set of value so I am getting this error, when I consider the whole domain as only 4 points, so in that case I have this -(3/80) h 5 this value, so this value will be there -(3/80) h 5, so this value will be same. Now, we are dividing this method by 3 times, so from here I can write that this will be equal to $-\frac{n}{80}h^5 f^{(iv)}(\xi)$. So, that will be my total error.

But again we can say that this method and Simpson method have the same errors because they are exact for up to the cubic, so it does not matter if you apply Simpson one third or 3 by 8, both are going to give you the same, almost same result for the given function. So, this is the result for the Simpson 3 by 8.

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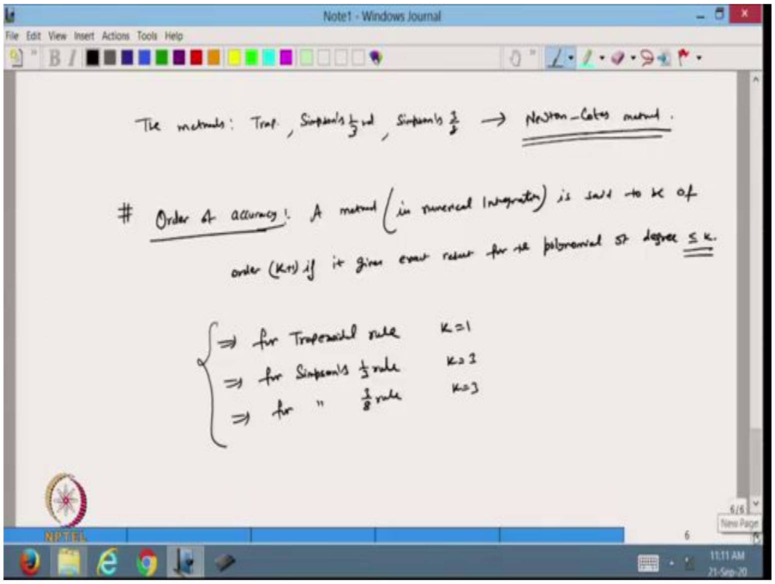

Now, the question so after doing this one, I just want to define that this type of methods, the Trapezoidal, so whatever the method we have discussed so far, they are also called the methods like Trapezoidal, Simpsons one third or Simpson 3 by 8 or they are also called Newton Codes methods, so they are also called Newton cotes methods. This is a cotes. So, in this type of method we know the value, nodal values that are x0, x1,...,xn.

The value of the function is known to us at this point and then we calculate the value of the integral. So these types of methods are called the Newton-cotes methods. Also, for the integration the order of convergence we have not defined, order of or maybe in this case I can say that the order of accuracy. So, what is the order of accuracy here?

So, a method, so we are talking about the numerical integration, a method in numerical integration is said to be of order k if it is said to be a method of order k or maybe k+1, I should write order k+1 if it gives exact result for the polynomial of degree less than equal to k. So, that is the way, so it is giving the exact results for the polynomial of degree less than equal to k, for example, if I take the Trapezoidal rule.

So, in the Trapezoidal rule we go to the accuracy, so in the Trapezoidal rule our result was only for second order, so it was second derivative by all there. So, in this case I can say that for the Trapezoidal we are going to have the order of accuracy, in this case I will say k+1=2, so k will be 1. So, that is giving the result for k plus, k=1.

So, I can say that from here, so from here I can say that for Trapezoidal rule k=1. For Simpson one third rule k=3 and also for Simpson 3 by 8 rule k=3, so that is the way we can define the order of accuracy in the case of numerical integration. So, this is all about the Newton cotes methods.

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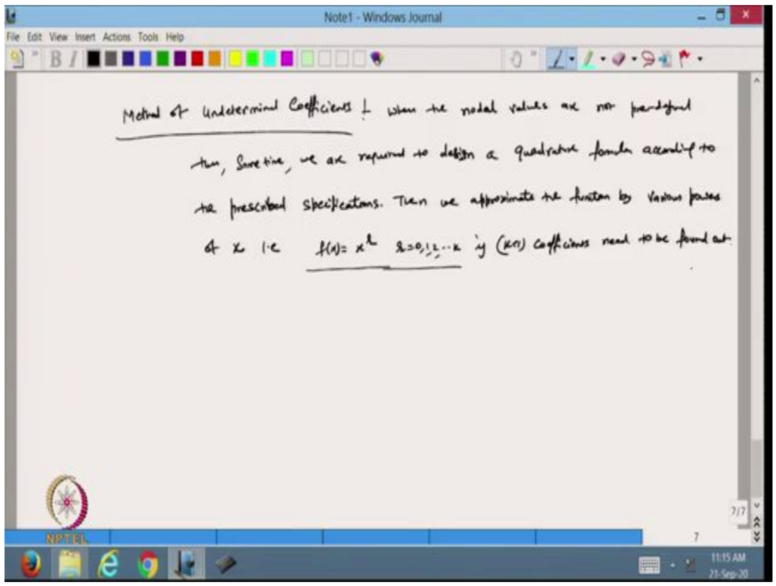

Now, after doing this one we define another method, which is not based on this fixed value of x0, x1, x2, so that is called the method of undetermined coefficient. So it is, so in this case we do not have the fixed value of x0, x1,...,xn. So, in this case when the nodal values are not predefined, then sometimes we are required to design a quadrature formula according to the prescribed specification.

Then we approximate the function by various powers of x, that is, I will take the

$f(x) = x^r, \quad r = 0, 1, 2 \ldots, k$, i.e., k+1 coefficients need to be found out. So, in that case we will take the function, the value of the function is equal to x raised to power r because in this case we need k+1 coefficient, so for to find the value of the k+1 coefficient, we consider that the method is exact for up to x raised to power r where r is from 0, 1 up to k, so in that case will get the system of equation and then will be able to solve that one.

So, this type of method is called the method of undetermined coefficient. So, we will discuss it in the next lecture. So let me stop here today. So today we have discussed the Simpson 3 by 8 rule, and it also a total error in the case of composite formula and also, we have discussed the meaning of order of accuracy in the numerical methods for integration and then we have started with the method of undetermined coefficient. So, we will continue with this one in the next lecture. So, we will stop here and thank you for watching. Thanks very much.