Scientific Computing Using Matlab
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Lecture No. 57
Simpson's 1/3 Rule for Numerical Integration

Hello viewers, welcome back to the course on Scientific Computing Using Matlab. So, in the previous lecture we have started with Simpson's rule. Today, we will continue with that one.

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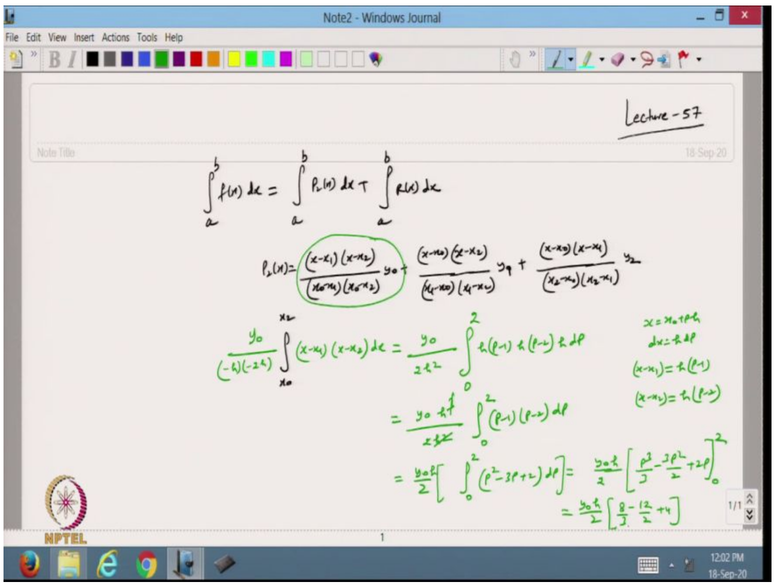

So, in the previous lecture, we have started doing the integration for the function from a to b f(x) dx and then we have approximated the function f(x) with P2(x) dx plus the I terms. So, this one we have started with. Now, then I know that the P2(x) was the second degree Lagrangian interpreting polynomial that was

$$p_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

. So, this is where we have started, then we wanted to do the integration of x0 to x2 of this factor.

So, this vector we started with the transformation that is x=x0+ph and then dx=hdp, x-x1 can be written as h(p-1) x-x2=h(p-2) and so on. So, this is what we have done in the previous one. So, from here I can write down this integration. So, y naught can be written like this, x0-x1=-h, x0-x2=-2h and inside I will get (x-x1)(x-x2)dx. So, this part can be written as y0. So, minus and minus so, it is 2h square and this one I can say that when x=x0, p will be 0, when x=x2, p=2.

So, from here I can write that this is equal to h(p-1)h(p-2)hdp. So, this value I am getting. So, from here I will get y0, h, h and h. This h I can take common. So, I can write

$$\frac{y_0 h^3}{2h^2} \int_0^2 (p-1)(p-2)dp$$

. So, this integration I need to find out. So, this h square will go with this. Now, from here I can write that this is equal to y0/2 and then integration of this one.

So, this is $\frac{y_0}{2} \int_0^2 (p^2 - 3p + 2)dp$.

So, this will be equal to (y0/2)*h. So, this h will come and then I will do the integration of this one. So, this will be $\frac{y_0 h}{2}\left[\frac{p^3}{3} - \frac{3p^2}{2} + 2p\right]_0^2$ and if I put the 2 here, so, this will be y0h/2. So, this would be 2 raised to all 3 that will be 8. So, 8/3-12/2+4. So, I will get this value and from here you will see that this is the 6 LCM.

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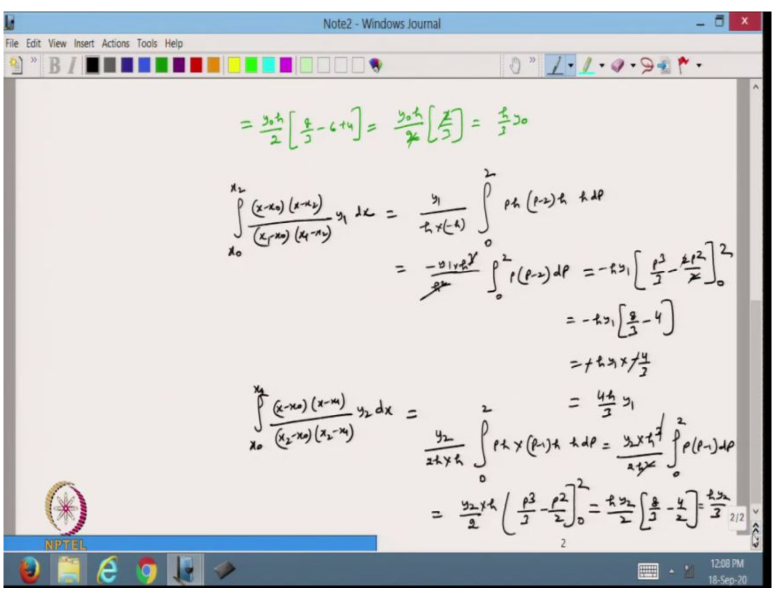

So, if you see this one, I will get the value (y0h)/2 and inside I will get (8/3)-6+4. So, it will be (y0h)/2 this is -2, so, -6, 8. So, it will be 2/3. So, this will cancel out and I will get (h/3)y0. so, I get this value. Similarly, I can go for the next integration

$$\int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1\, dx$$

. So, this I will get.

And from here in the same way we can find out, this will become y1 I can take common, this will be $\frac{y_1}{h \times (-h)} \int_0^2 ph(p-2)h\; hdp$. So, I will get this value. So, from here I will get $\frac{-y_1 h^3}{h^2} \int_0^2 p(p-2)dp$. So, this is the integration we are going to get.

So, from here you can say that this will be minus so, this will cancel out with this. So, it will be

$$-hy_1 \left[\frac{p^3}{3} - \frac{2p^2}{2}\right]_0^2 = -hy_1 \left[\frac{8}{3} - 4\right] = \frac{4h}{3} y_1$$

. Similarly, we can find out

$$\int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2\, dx$$

.

So, the same way we can find out by all this substitution, so, it will get we will get the value here this will be $\frac{y_2}{2h \times h} \int_0^2 ph \times (p-1)h\; hdp = \frac{y_2 h^3}{2h^2} \int_0^2 p(p-1)dp$. So, this will cancel out and if you do the calculation again so, it will be

$$\frac{y_2 h}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)_0^2 = \frac{hy_2}{2}\left(\frac{8}{3} - \frac{4}{2}\right) = \frac{hy_2}{3}$$

. So, that is the value we are going to get.

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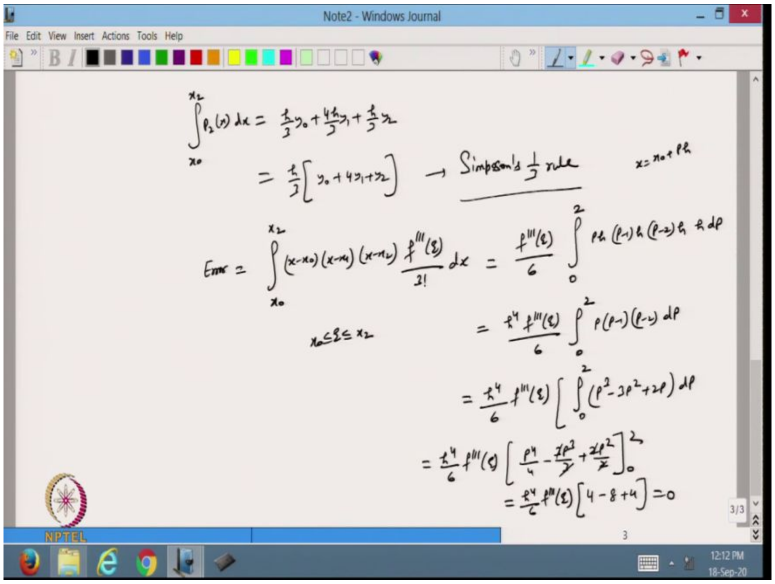

So, now, from here if I take the integration $\int_{x_0}^{x_2} p_2(x)dx$, then this will become $\frac{h}{3}y_0 + \frac{4h}{3}y_1 + \frac{h}{3}y_2$ and now from here I can take h/3 common So, I will get $\frac{h}{3}[y_0 + 4y_1 + y_2]$. So, this is the integration we are going to get when we approximate with the second order integrating a second degree interpolating polynomial. So, in this case I am getting h/3, so this is called Simpson one third rule, because you will be getting 1 by 3 so h by 3. So, this is the Simpson one third. So, now from here I am getting the solution of the integral.

Now, let us find out the error. So, the error in this case will be

$$Error = \int_{x_0}^{x_2} (x - x_0)(x - x_1)(x - x_2)\frac{f'''(\xi)}{3!}dx, \quad x_0 \leq \xi \leq x_2$$

. Now, from here I can write this as 3! from here. So, this is basically 6. So, I can write this as 3!=6. Now, this integration I can again take the substitution that x=x0+ph. So, from here I know that this will be equal to $\frac{f'''(\xi)}{6}\int_0^2 ph(p-1)h(p-2)h\ hdp$. So, this will get.

So, from here I will get $\frac{h^4 f'''(\xi)}{6}\int_0^2 p(p-1)(p-2)dp$ and then from here if I do the integration I will get, so, I will get here just to make it simplified. So, this is p square, so it will be $\frac{h^4 f'''(\xi)}{6}\int_0^2 (p^3-3p^2+2p)dp$. So, this will I will get and now from here I will get $\frac{h^4 f'''(\xi)}{6}\left[\frac{p^4}{4}-\frac{3p^3}{3}+\frac{2p^2}{2}\right]_0^2$.

And if I further simplify, then it will be $\frac{h^4 f'''(\xi)}{6}[4-8+4]=0$. So, that becomes 0. So, in this case if you see the, the error is coming 0.

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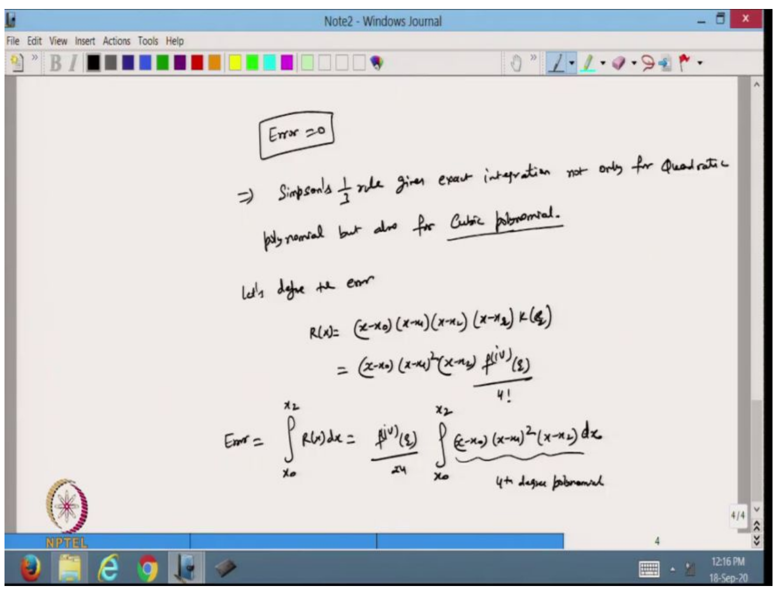

So, that shows that so, it means that if I choose the, this error and you know that this is a cubic polynomial if you multiply (x-x0) (x-x1) (x-x2), so, that is the cubic polynomial. So, in this case it is giving the error 0 so, which implies that, that Simpson one third rule gives exact solution, exact solution or integration, exact integration I should write, exact integration not only for quadratic polynomial but also for cubic polynomial. So, in this case it is also giving the exact integral for the cubic polynomial.

So, it means that it is going to give the error for the polynomial having the degree greater than 3. So, in this case would you do so, let us, say let us define the error. So, in this case I define the R(x)first. So, $R(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_r)k(\xi)$, that xr can be x0, x1, x2 into some function of, some function of xi. So, in this case we generally take that xr we take the middle point, so, from here we can choose my $(x - x_0)(x - x_1)^2(x - x_2)$. So, this we choose and then this is the fourth derivative by 4!.

So, that is the remainder term in the case of quadratic polynomials. So, now, we do the integration and find the error. So, the error in this case will be

$$Error = \int_{x_0}^{x_2} R(x)dx = \frac{f^{iv}(\xi)}{24}\int_{x_0}^{x_2}(x - x_0)(x - x_1)^2(x - x_2)dx$$.

So, let us see what will happen whether it is going to give the error for this, because this is a, if you see it is a fourth degree polynomial. So, now, I want to find out what is going to happen with this one.

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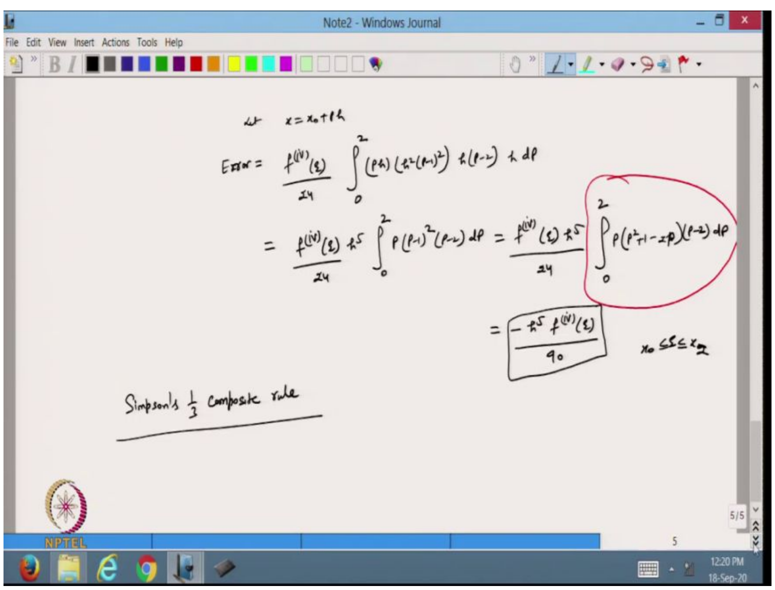

Now the same thing, I will choose that let x=x0+ph, then I will get my error. So, this will be $Error = \frac{f^{iv}(\xi)}{24} \int_0^2 (ph)(h^2(p-1)^2)\ h(p-2)\ hdp$. So, this is what I am going to get. Now from here I will get

$$Error = \frac{f^{(iv)}(\xi)}{24} h^5 \int_0^2 p(p-1)^2(p-2)dp$$

. So, this value we are going to get.

Now if you further solve this one, I am getting

$$Error = \frac{f^{(iv)}(\xi)}{24} h^5 \int_0^2 p(p^2+1-2p)(p-2)dp$$

. I will calculate. So, if you do the calculation for this integration you can do the calculation for this the same way, you will get the value, this is this will come out as $Error = \frac{-h^5 f^{(iv)}(\xi)}{90}$, where $x_0 \le \xi \le x_2$. So, this calculation you can do then will get the 90. So, from here I can say that this is my error.

So, from here I can say that, that the error will be, so, this method is going to give the error whenever we are dealing with the function whose fourth derivative is nonzero. So, it is up to the cubic polynomial the error was 0, but if you, if you have a function with a fourth degree, then some error will be introduced and this is the error in the case of Simpson one third rule. So, let us go for composite rules. Simpson one third composite formula. So, in the composite formula what we are going to do is so let us do on the next page.

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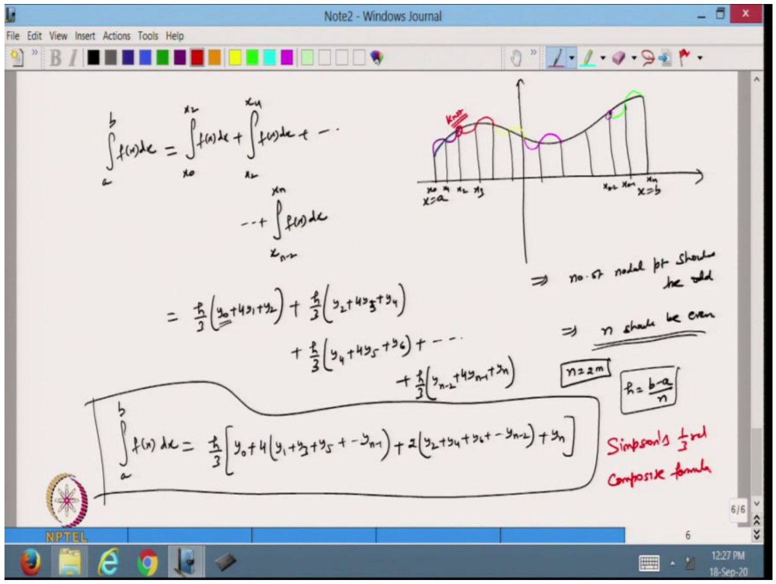

I have the function, so suppose this function in this, this is x=a, this is x=b. Now, I split this into the sub intervals with the nodal values. So, this is my x0, x1, x2, x3, xn-1, xn, xn-2 and so on. Now, what I do is that instead of taking the, the one quadratic polynomial for the whole function, I will split this one into the sub intervals and then what I do is that 1, 2, 3; so, this 3 value I choose. So, this is a function I have. So I approximate this function with a quadratic. This one, the next function is from here to here. So, I will approximate this function with this quadratic.

Then I will choose this one and we will write this one. Then I will find out this value and approximate this function and in the end I will approximate this function by this quadratic polynomial. So, by this way, we have a quadratic polynomial in the pieces. So, from here I can say that the integration $\int_a^b f(x)dx$. So, this is my integration I am going to do. So, this will be equal to $\int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \ldots + \int_{x_{n-2}}^{x_n} f(x)dx$.

So, in this case one thing is that, if I am to choose 3, 3 points, so 3 points, this 1 then next 1 and 2, 2 points more so, 5, then 2 points more 7. So, from here I can say that the number of points, nodal points should be odd. So, it should be 3, 5, 7, 9 greater than 1 definitely, of course greater than 1. So, from here I can say that n should be even because when the n is even, the n+1 points will be odd. So, that is the criteria. So, n should be 2m always an even value. Only then we can apply the composite formula for Simpson one third rule.

And from here I can say that this is represented by this polynomial, interpolating polynomial. So, from here I can say that now, I will use the formula so, this is the value we are going to get. So, it is h by 3 y0, y1, y2 multiplied by 4. So, this one I am going to use. So, h/3 so h=(b-a)/n. So, this is my h. Now, this is the formula y0+4y1+y2. Now the next interval it will be h by 3. Now, y2 is coming here because this is the connecting node. So, you know this is a node so that is the connecting node.

So, here I am getting y2+4y3+4. So, 2, 2, 3, 4, then plus h/3 , then y4+45+y6 and so on. And in the end, I will get h/3 yn-2+4yn-1+yn. So, this is the composite formula for the Simpson. Now, if I add collect all the terms, I can take my h by 3 common now, you can see that

$$\int_a^b f(x)dx = \frac{h}{3}[y_0 + 4(y_1 + y_3 + y_5 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2}) + y_n]$$

. So, this is the integration of the function. So, this is a Simpson one third rule composite rule for the given function.

So, in this case what we will do, I will find out the h that h we can find from here, then y naught plus 4 times we have to do this odd value plus 2 times we have to do with the even value that is the interior values plus yn. So, y naught and yn that is the boundary point we know that the value of the function there only we have to take the internal values odd and even and multiply by 4 or 2. So, that is the solution for so, this is the Simpson one third composite rule.

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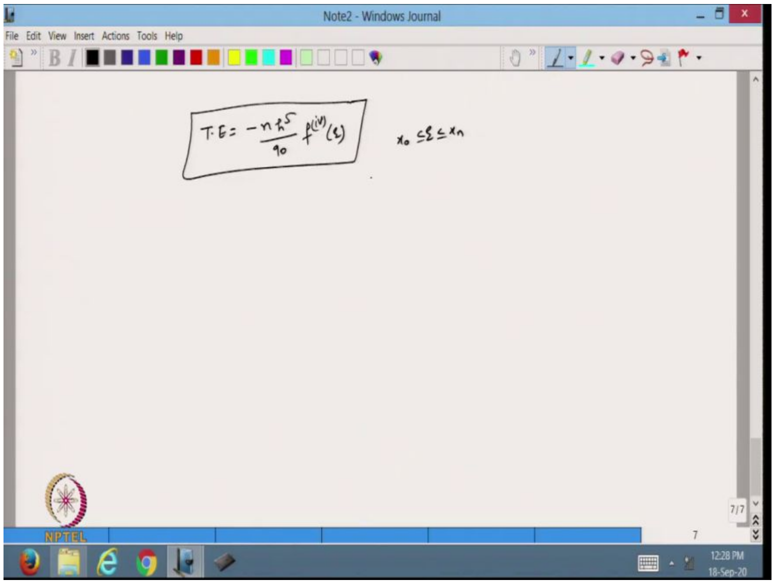

So, from here I can define the same way we can define the error. So, I can define the total error. So total error, I know that we can multiply by n. So the same way,

$$T.E = \frac{-nh^5}{90} f^{(iv)}(\xi), \quad x_0 \le \xi \le x_n$$

. So, the same way, we can choose the maximum. So, first we choose the maximum error in any of the intervals, and then we just multiply by n. So, that will be a total error. So, this is basically upper bound for the total error in the Simpson one third rule. So, this is the formula for the total error.

So we will stop here. So, today we have discussed the Simpson one third rule, we derive the formula for that, and then we have discussed the composite one third rule and the errors corresponding to this. So, in the next lecture, we will continue with this one. Thanks for watching. Thanks very much.