Scientific Computing Using Matlab

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Lecture 56

Trapezoidal Rule for Numerical Integration

Hello viewers, welcome back to the course on Scientific Computing Using Matlab. So, today we will continue with numerical integration. So, in the previous lecture we have introduced how the rectangular methods can be used for numerical integration. Now, we will continue with that one.

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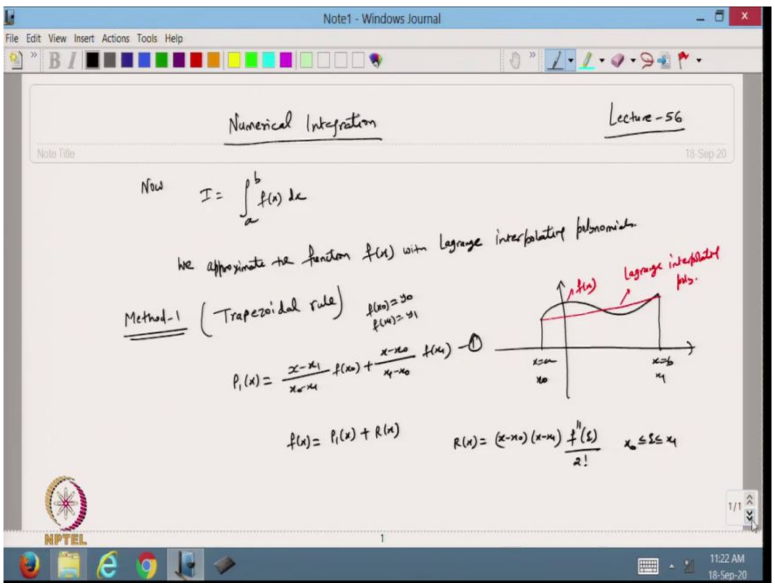

So, we take that, that now, this is a numerical integration. Now, I have an integral $\int_a^b f(x)dx$. So, what I do is now we, we approximate the function, that is f(x), so this function f(x) we approximate this one with, with the lagrange interpolating polynomial, with lagrange. I know that this Lagrange interpolating polynomial can be linear, it can be quadratic, the cubic, so based on this one we will get different, different types of methods.

So, I just take method 1 and we call it the Trapezoidal Rule. So, in this case what we are going to do is suppose I have a function and some function is given to me. So, this is my x=a and this is the value of x=b. Now what I do is that, I approximate this function, so this is my function f(x).

So, I approximate this function with a linear function, this value, so this is my Lagrange interpolating polynomial and this is my function f(x).

And I am considering that I know the value of the function, at this point and this point. So now, in this case we know that, if we go back and find the value of the Lagrange interpolating polynomial, then I know that I can find this polynomial P1(x) because I am represented by the linear function. So, this can be written as,

$$p_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1).$$

So, I call it, this value as x0 and this value I just call it x1. So, I have only two values there. Now, this $f(x_0) = y_0$ also, and $f(x_1) = y_1$. So, this is the linear interpolating polynomial, lagrange polynomials for the given function.

Now, we know that this polynomial comes with the errors, so I know that the function, my function f(x), can be written as P1(x). So, this is the interpolating polynomial plus the error term. So, that is the remainder term, so that is R(x). And I know that the R(x) in this case will be $R(x) = (x - x_0)(x - x_1)\frac{f''(\xi)}{2!} \quad x_0 \leq \xi \leq x_1$. So, this is the corresponding error we can find for this linear interpolating polynomial.

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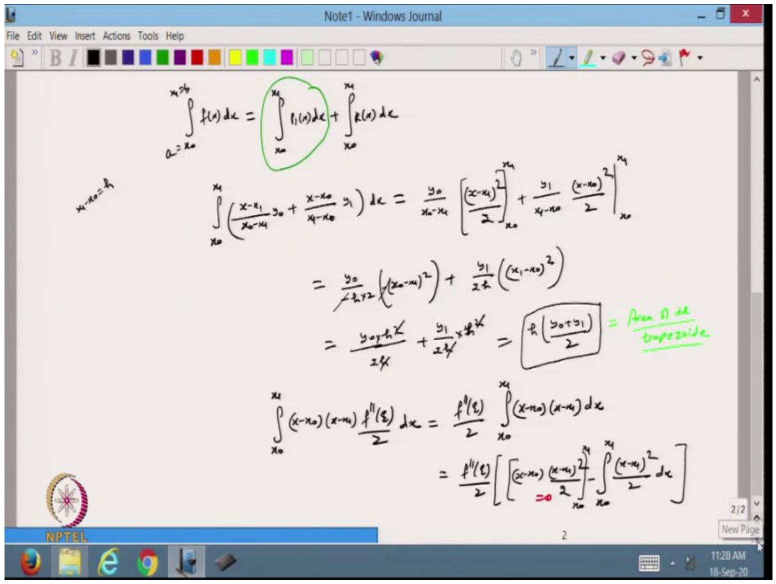

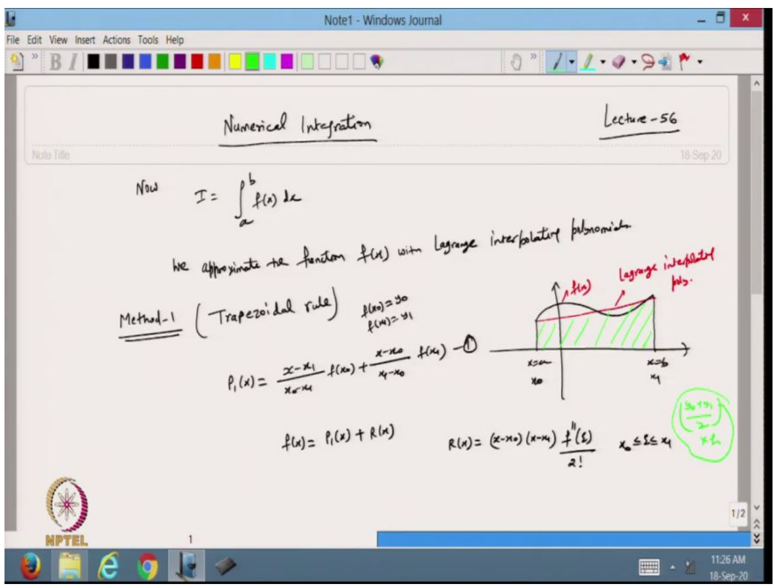

Now from here, now I want to do the integration, so I am integrating from $\int_{x_0=a}^{x_1=b} f(x)dx$. So, this one I am doing, so I have substituted the value here, so this is my

$$\int_{x_0=a}^{x_1=b} f(x)dx = \int_{x_0}^{x_1} p_1(x)dx + \int_{x_0}^{x_1} R(x)dx$$

. And R(x) is the error term we have introduced. Now, if I find, I want to find this one, so this integration I want to find out, so first, I will find out this integration. So, this integration is

$$\int_{x_0}^{x_1} \left(\frac{x - x_1}{x_0 - x_1} y_0 + \frac{x - x_0}{x_1 - x_0} y_1 \right) dx$$

.

So, this integration I want to do, now from here you can see that, this x0 and x1, so I can find that x1-x0 I represent by h, so this value. So, now this integration if you want to do, I will get y0 over x naught minus x1. So, I am doing the integration of this, so it will become

$$\int_{x_0}^{x_1} \frac{y_0}{x_0 - x_1} \left[\frac{(x - x_1)^2}{2} \right]_{x_0}^{x_1} + \frac{y_1}{x_1 - x_0} \frac{(x - x_0)^2}{2} \Big|_{x_0}^{x_1}.$$

$$= \frac{y_0}{-2h} - (x_0 - x_1)^2 + \frac{y_1}{2h}(x_1 - x_0)^2$$

$$= \frac{y_0 \times h^2}{2h} + \frac{y_1}{2h} \times h^2 = h\frac{(y_0 + y_1)}{2}$$

.

So, that is the value of the integration we are going to get. And if you see from here, in the previous one, now if you look at this one carefully, then you will see that, this one is a trapezoid, and if I want to find the area of this trapezoid, so that is the, the average of the parallel side. So this will be y0, this is the y0+y1 that is the y1 and y2. So, this is the average of this value multiplied with the distance between them and distance between them is h.

So, that is the area of the trapezoid. So, the same value we are getting here for this one, so that is the same as the area of the trapezoid, trapezoid. So, that is why this is called the trapezoidal rule. So in this case, that is the area we are going to get. Now, we want to find out how the error will, so this error we want to find out, so this error is x-x0 dx.

So, this one we want to find out. So, if I want to find out from here then, I can, so this is the constant value, I can take this outside and then I can integrate this one. And now, this is a product of two functions, so I can simply do the integration by parts. And if you see this one, I can take this as a first function, so $\frac{f''(\xi)}{2}\left[\left\{(x-x_0)\frac{(x-x_1)^2}{2}\right\}_{x_0}^{x_1} - \int_{x_0}^{x_1}\frac{(x-x_1)^2}{2}dx\right]$. So this is, now if I substitute, so here also we have to substitute the limit. So, if I put x1, this will be 0, x0 it will be 0, so this part will be 0. So, this is equal to 0. Now from here, I left with this value.

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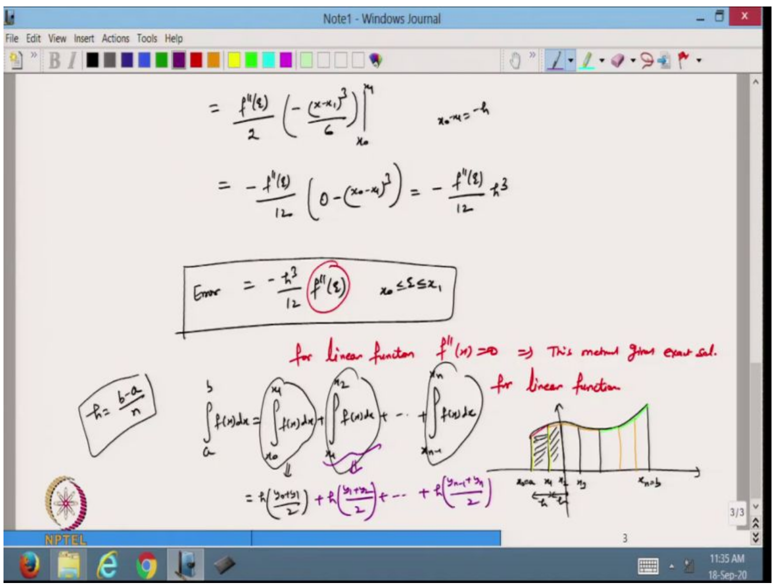

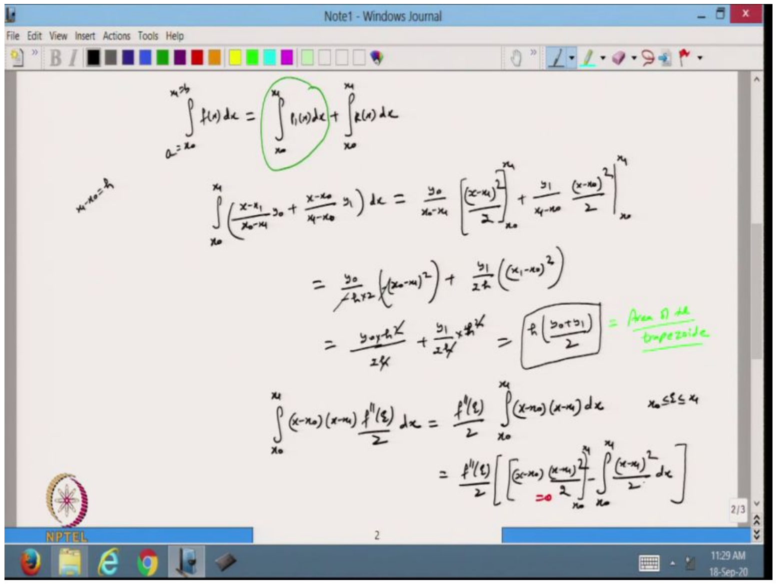

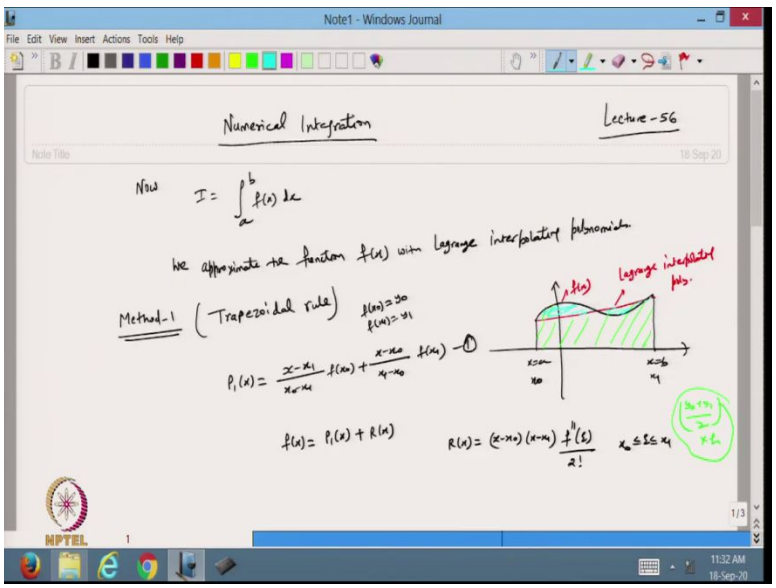

So, from here you can say, that we have $\frac{f''(\xi)}{2}$ and then I will, to the integration this xi I know, that this xi belongs to x0 to x1. So, this negative sign and I will take the integration of this part, so this will be q by 6, so it will be 6 there. So from here, this will remain minus, so I can write $\frac{f''(\xi)}{2}\left[-\frac{(x-x_1)^3}{6}\right]_{x_0}^{x_1}$. So, when I put the value x1 here it will be 0 and x0 it will be h with the negative sign.

So, this will be $-\frac{f''(\xi)}{12}\left(0-(x_0-x_1)^3\right)=-\frac{f''(\xi)}{12}h^3$.

So, I get this value. So, from here I can write that the error term in this integration will be

$Error=-\frac{h^3}{12}f''(\xi) \quad x_0 \leq \xi \leq x_1$. So, that is the error in this integration. Now from here, you can see that this error is the second order, second derivative with the function of x. So, I can say that for the linear function if the second derivative is equal to 0 which implies that this method gives an exact solution for the linear function.

So, for the linear function if we do the integration then we will get the exact value of using the trapezoidal rule. So, that is the way we can find out the error. Now this is, we have done now. Now what is happening in this case, in the previous if you see that the whole function I am approximating with a linear interpolating polynomial. And we are able to find the solution, but if you see from here the error is quite large.

So, this is the error we have introduced, this is the error, this is the error. So, the error is very large in this case. Now, what do you do that instead of this one, what we can do, we can split the function into n+1 nodal values as we are doing, and this is my xn=b, this is my x0=a, this is x1, x2, x3, and so on. Now, what we can do, this function in this first sub interval, I will approximate this one, with the lagrangian interpolating polynomial.

And then I have, so this is the first trapezoidal I am going to get, and then I will find out this value. Next one is that this function, the next is this function, this function, and this value I can this, this, this, and so on. So, based on this one, I can, for my integration from a to b my f(x) dx, I can split this function the whole integral in this form from x0 to x1, so this, the area of this one.

So, I can write this value, this area, this area, so then I can write

$\int_a^b f(x)dx=\int_{x_0}^{x_1} f(x)dx+\int_{x_1}^{x_2} f(x)dx+\ldots+\int_{x_{n-1}}^{x_n} f(x)dx$. So, I split the function into all these sub intervals and now, this one can be separately can be solved as we have done in the previous one using the lagrangian interpolating polynomial. Similarly for this one, similarly for this one, and so on. So, if you see from here, then, if I find the value of this integral with the trapezoid rule, then I will get the value here.

So I, what is the area of this trapezoidal, I will get the average of this value, this is the y_0 and this is my y_1 and then distance between these two. So, from here I can write that using the trapezoidal rule, the area of this trapezoid will be h, so h I am taking now, so my $h=(b-a)/n$.

So, n is the number of subintervals I am finding. So, this is my h basically, this is the h. So, from here I will get, $h(y_0+y_1)/2$. The next one I want to find out this value, so this will be again $h(y_1+y_2)/2$ and so on. The last I will get, $h(y_{n-1}+y_n)/2$. So, this is the composite form we are getting.

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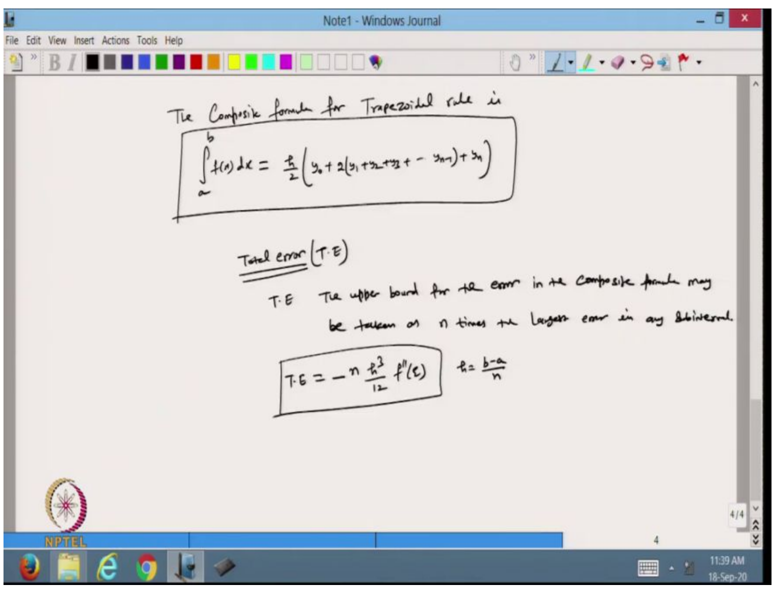

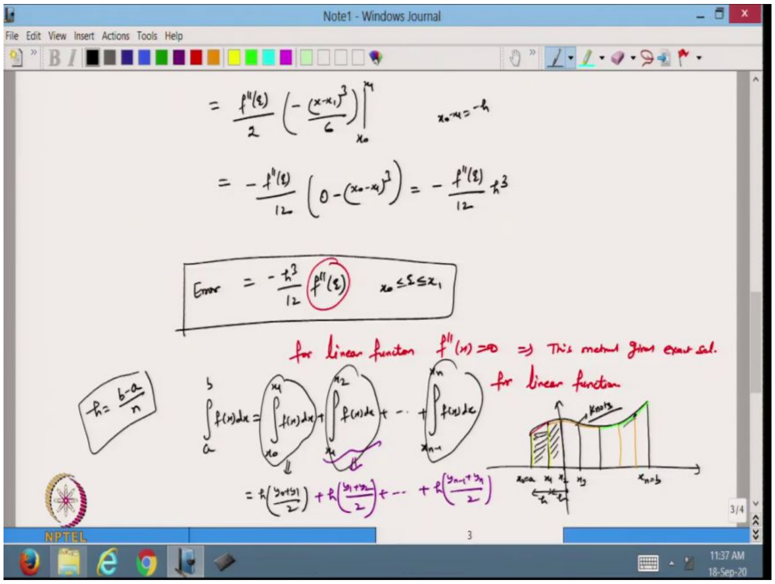

So from here, the composite formula for the trapezoidal rule is, so from here if I add all together then my integration from f(x) dx that becomes,

$$\int_a^b f(x)dx = \frac{h}{2}(y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n)$$

So, that is the formula with the composite for trapezoid rule. And what would be the error in this case, so of course, the error will be added. So, from here I can write the total error. So, the Total Error, TE. So, how we can find the total error, I can add this one. So, this one I can add that the, the upper bound, for the error in the, in the composite formula, may be taken as n times, the largest error in any sub interval.

So, if you see from here, I can write then the total error will be

$Error = -n\frac{h^3}{12}f''(\xi) \quad h = \frac{b-a}{n}$. So, from here I can write this one. So, this is the total error.

Where xi I am choosing, that this value is the largest error, so I will find out the largest error in any sub interval and then I multiply by n, so that is the truncation error we are going to, so this is the upper bound for that one. So, truncation error we are going to get in the composite formula.

So, that is the way we can find out the error in the composite formula. Now, this is the simplest one we have taken.

Now, the next method we are going to introduce is, because in this case we are approximating the given function with the linear lagrangian interpolating polynomial, but we can go further and how we can go further.

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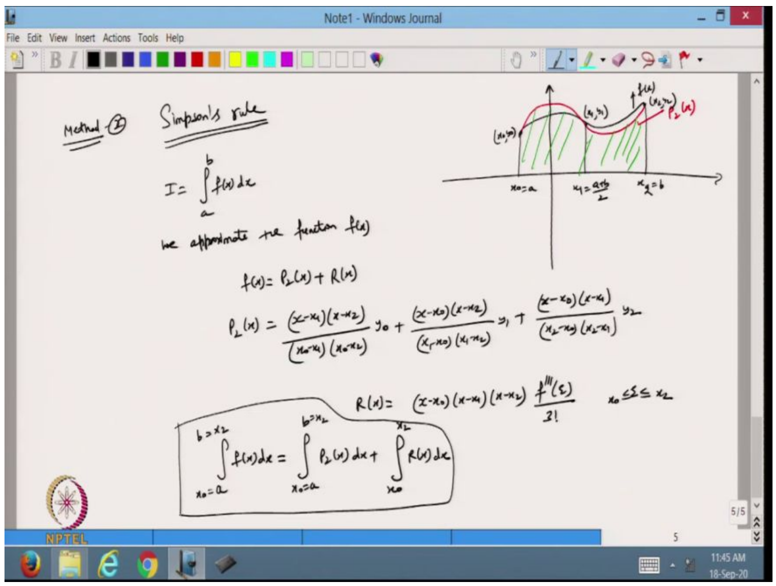

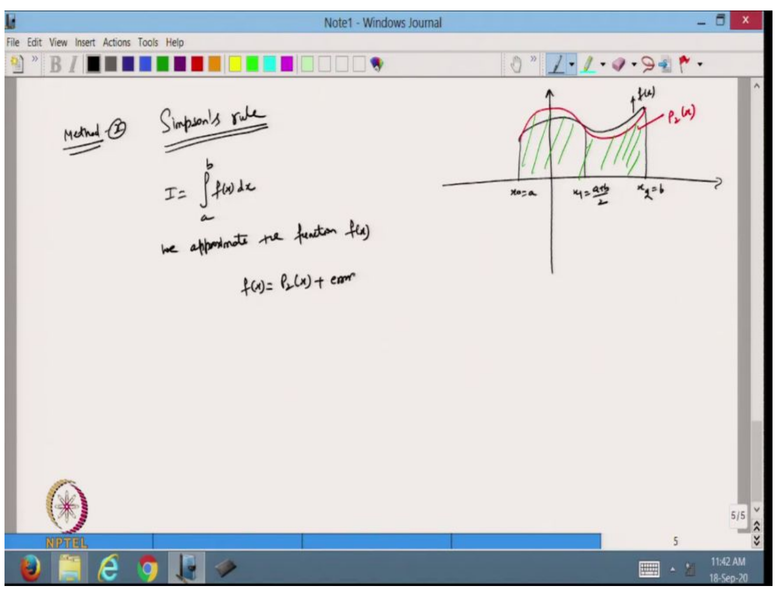

Now, suppose I have this function, so I know the value of the function at x0 and maybe x1, I just take or I just take it x2. So, this is my a and this is my b. Now, I choose a point in between. So, that is my, I call it x1, so x1=(a+b)/2. Now in this case, this is my function f(x), so I can approximate this function with a quadratic polynomial, interpolating polynomial.

So, what is the quadratic interpolating polynomial, that it is passing, it is a quadratic and passing through the points. So this is my, I call it P2(x). Now, this is a quadratic interpolating polynomial, so now we are approximating a function with P2(x) and some error will be introduced. And then what we do, then we will solve this one and we will find out the integration under this P2. So, that is the way we can define.

So, this is the method 2 and this is called the Simpson rule, Simpson rule. So, in this case what we are going to start with, we have a function, so this is my integration I want to find out. Now, we approximate, we approximate the function f(x). So, f(x) I approximate with P2(x) plus the error term all the remainder, error we call it remainder term plus R(x). Now, what do you mean P 2, so P2 is the polynomial, second degree polynomial, that is we call it P2.

So, in this case it will be x, so x0 is there, so it will be

$$p_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

, and the error terms, R(x) is the, the remaining term, so I know that this will be equal to

$$R(x) = (x-x_0)(x-x_1)(x-x_2)\frac{f'''(\xi)}{3!} \quad x_0 \le \xi \le x_2$$

. So, this is the corresponding error in the approximating function with the second degree lagrangian interpolating polynomial. So, from here, now we have to introduce this one, so from here, I want to find the integration $\int_{x_0=a}^{x_2=b} f(x)dx$. So, this can be written as

$$\int_{x_0=a}^{x_2=b} p_2(x)dx + \int_{x_0}^{x_2} R(x)dx$$

. So, this integration we have to find out.

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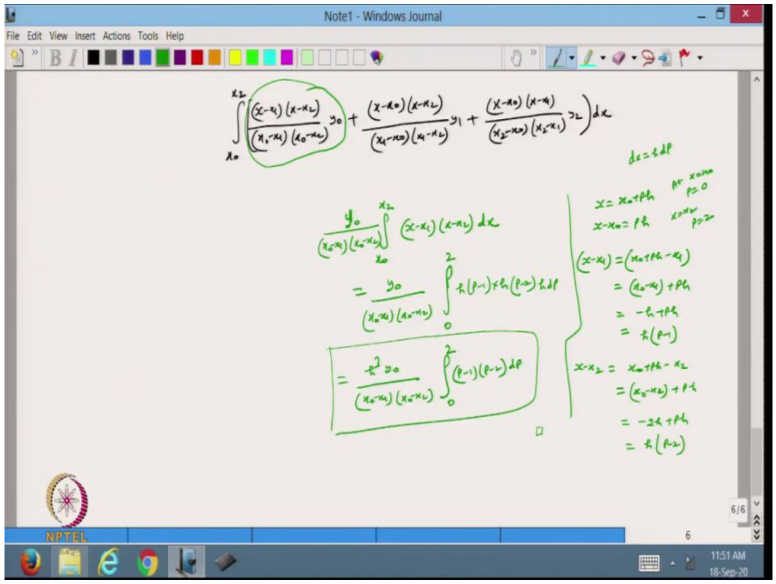

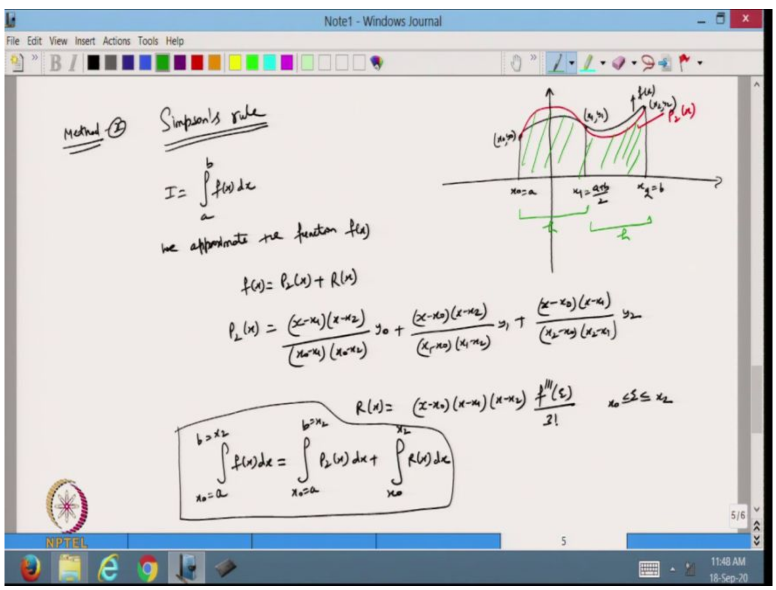

Now, the question is how we can integrate this one. So in this case, we are going to start the integration. So, now I need to find the integration

$$\int_{x_0}^{x_2} \left(\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2 \right) dx$$

. So, this one we need to find out. So, let us do the integration for this. Now, we know that our, so the first integration I want to solve, so let us solve this part first.

So in this case, so let us start this one. So, now I want to find out this x0 to x2 and this is the part I can take outside because this is a constant value, so x0-x1 and x0-x2. So, this part I can take common no problem, inside I will get,

$$\frac{y_0}{(x_0 - x_1)(x_0 - x_2)} \int_{x_0}^{x_2} (x - x_1)(x - x_2)dx$$

, so this one value I want to find out, now find out.

So now I will, the one thing is that we can do the integration directly, but then it will sometimes become cumbersome to do this one. Now, just take the transformation, now I know that my value x can be written as x=x0+ph. This is what we have done for the, when we were doing the interpolating polynomial that we can represent the x=x0+ph, where p is the parameter we introduce, generally lying between 0 and 1.

So, from here I can write my x-x0=ph. Now from here, if I put x-x1, then this will be x0+ph-x1, this one I can write. So, from here I can write (x0-x1)+Ph. Now, I know that x0- x1, so if you see from the previous one, I can say that this is equal to h, and this is also equal to h, the distance between x1 and x0 and x2-x1. So, from here I can write that this is, -h+ph.

So, that can be written as h(p-1). Similarly, x-x2=x0+ph-x2. So again, I can take this (x0-x2)+ph, so this is x2 and x2 minus, this is -2h+ph and from here I can take h(p-2). Now, from here, if we then, also that if you write dx, so it will be h dP. And when x=x0 p is 0. So, at x=x0 p is 0, x=x2 p=2 because x=x2 so it will be 2h so P will be 2.

So from here, I can that this integration can be written as

$$\frac{y_0}{(x_0 - x_1)(x_0 - x_2)} \int_{0}^{2} h(p - 1) \times h(p - 2)hdp$$

, so this integration becomes this. So, from here I can write,

$$\frac{h^3 y_0}{(x_0 - x_1)(x_0 - x_2)} \int_{0}^{2} (p - 1)(p - 2)dp$$

.

So, this is an integration we need to find out. Similarly, we can define integration for this, and this. So, we will stop here. So, in this lecture we have started with the method that is called the trapezoidal rule. And then we have started with the Simpson rule. So, in the next class we will carry out with the Simpson rule. Thanks for watching. Thanks very much.