Scientific Computing Using MATLAB
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Lecture - 55
Numerical Integration

Hello viewers. Welcome back to the course on scientific computing using MATLAB. So today we are going to start with another topic, that is numerical integration.

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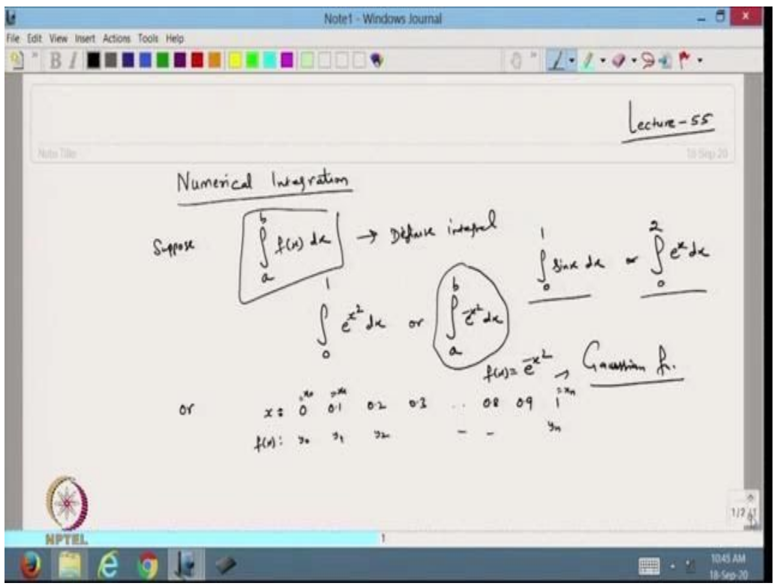

So in this case, suppose we have an integral $\int_a^b f(x)dx$. So this is a definite integral and I want to do the integration of this one. So if you want to do the integration of this, there are two ways. We know that there are two ways to find out the integration.

For example, suppose I want to find the integration of 0 to 1, maybe sine x dx or 0 to 2 may be an exponential function e x dx. So this type of integral we know that we can solve directly using the info using the knowledge of 11th or 12th standard and then we can find out the integral value.

But how the numerical integration is going to play the role is that sometimes, we are unable to find out the analytically solution of this integral. For example, suppose I want to find the integral $\int_0^1 e^{x^2}\, dx$ or I want to find the solution $\int_a^b e^{-x^2}\, dx$.

So is a, I know that in this case my function $f(x) = e^{-x^2}$ and this is a Gaussian function. And we know that this type of integral is used in the normal distribution whenever we deal with the distribution theory. So in that case, we have to find out this type of integral whenever we are dealing with the normal distribution and the integral value is found from the distribution tables. So this type of integrals we are unable to solve analytically. So, in that case, we can find the solution, we can solve this integral numerically.

So this is the motivation why numerical integration is needed. Or in that case, I can say that suppose I have a value of x that is given to me in the discrete form. So I have the value of x that is given at 0, 0.1, 0.2, 0.3 and suppose, it is 0.8, 0.9, and 1. Suppose this is the value of the x is given to me and the value of the function is given to me at this point.

So suppose, so this is I call it x0 x1 and this is my xn. So in that case, I have the value y0, y1, y2, and so on, yn. So in this case, I have the value of the function in the discrete form. Because this integral is in the continuous form, but here the function is given to me in the discrete form.

So in that case, now, I want to do the integration of this function. So how we can find out the integration of this function when it is given in the discrete form then we have to come across the numerical integration. So this is the way we can find the integration numerically for such a type of function.

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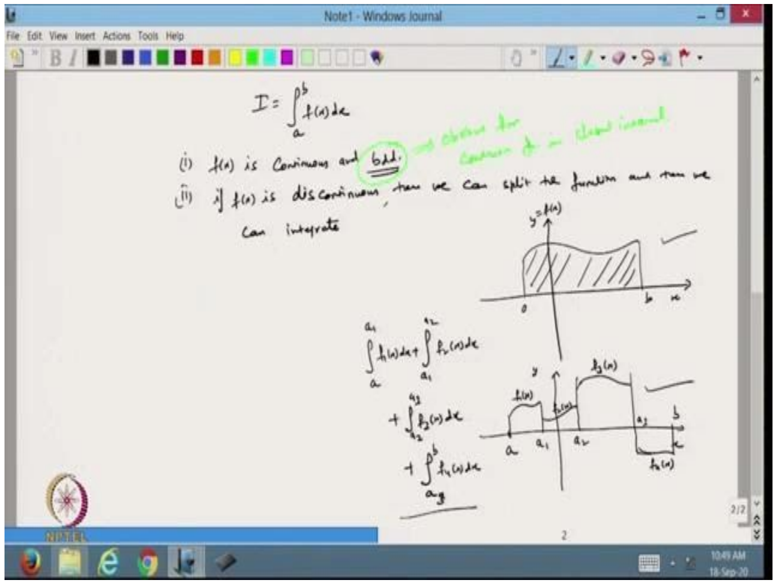

So, I have integral I; this is a definite integral. So and I want to evaluate this one. So in this case, the first thing is that f(x) is continuous because you know, I know that the function is continuous then it is integrable. And second one is that if f x is discontinuous, then we can split the function. So discontinuous, it is a piecewise discontinuous, then we can split the function, and then we can integrate. For example, suppose I have a function like this one. So this is my x-axis and this is my y-axis. So y=f(x).

Now, suppose my function is given to me as a continuous function and I want this to be my a and this is my b. So I want to integrate this function, so, and I know that I can integrate and I will find the area under this curve, so that is the integration I am going to have. This is the first case. Now, what will happen next?

So in the next case, I have a x-axis and y-axis. Suppose my function is a piecewise continuous so that it has a discontinuity. So suppose a function like this one here, then it is as well as coming like this. Then suppose it is this, then suppose it is this. So in that case, what will we do? This is my supposed, and this is supposed some, I can say, a1, then this is my a2, and then this is my a3.

So in this case, if I want to integrate this function, then I will split this into from a to a1 and then I choose these functions. Suppose this is my f1(x), this is my f2(x) , this is my f3(x), and this is supposed to be the last it is f4(x). So my function f(x) is, has a piecewise continuous function and that is my f1, f2, f3, and f4. So I will integrate this one

$$\int_{a}^{a_1} f_1(x)dx + \int_{a_1}^{a_2} f_2(x)dx + \int_{a_2}^{a_3} f_3(x)dx + \int_{a_3}^{b} f_4(x)dx$$

. So in this case, my function is continuous and of course, it is from a to b, so it is also bounded. So only then we can say that this function is integrable but it is continuous in the closed interval, so it is automatically bounded. So maybe you can skip this condition.

So maybe I can choose that this is obvious for continuous function in a closed interval. So my function is continuous, in the closed interval, it is always bounded. And this is this discontinuity also. So in this, in both the cases, I can say that a function is integrable. So my function can be this way or this way, it is integrable. Now, what are the conditions we have to choose for numerical integration? So that we have to do.

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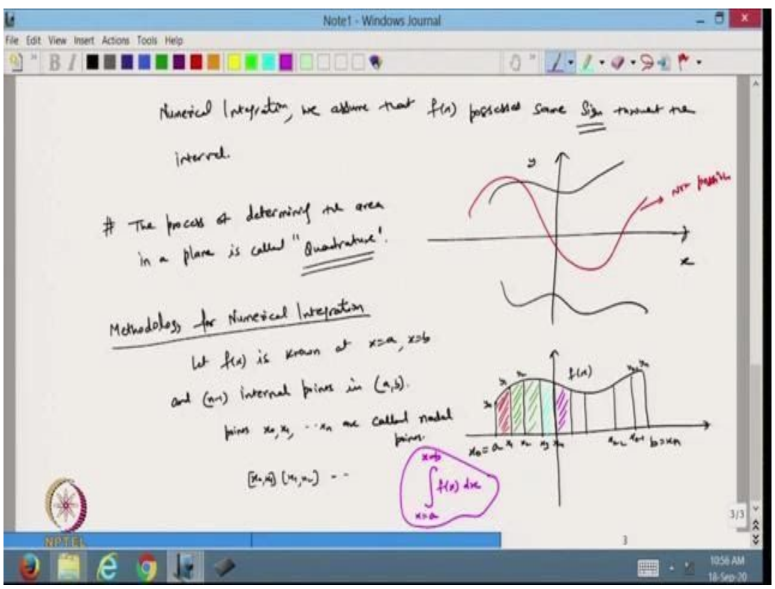

Now, we assume that, so for numerical integration, we assume that the function f(x) possesses the same sign throughout the interval. It means, same sign means if the function is positive like this one, so I can do the numerical integration of the function; for this function, I can do or maybe this function is there.

So this is a positive function I can do the integration, this is a completely negative function, I can do the integration. But what if I have a function like this one then not possible. So we cannot do the numerical integration for this type of function. We are dealing only with the function which has the same sign. So that is our condition for numerical integration

So now, for numerical integration, I know the process of determining the area in a plane is called quadrature in the engineering sense; so we call it the quadrature. So this is the way we can deal with numerical integration.

Now, the question comes about how we can proceed for numerical integration. So numerical integration, so I can define the methodology for numerical integration. So this is the methodology we are going to discuss. So let's suppose this is my function. So suppose I have my function is this, f x, and it is given from a to b. So there are two ways to solve the problem: find out the numerical integration.

First one is that you have the function f x, it is, the value of the function is given to you at two points, suppose here and here, this one. So this value and this value, then what I can do? At these two points, I can interpolate this function or approximate this function with a linear function and then I can integrate very easily.

Or the other ways that I can split this one into the larger number of subintervals and then in each subinterval I can find the integration and then add up all together that will give you the numerical integration. So this is the methodology.

So let my function f x is known at x equal to a and x equal to b and n minus 1, n minus 1 internal points in open interval a b. So what I do that, so suppose this in my I call it x0 and this is, call it xn. Then I can split this one into n subintervals. So I call it x1, x2, x3, x4, and this is x n-1, this is x minus 2, and at this point this value the function this is y0, then I can find this one as a y1. By substituting the value of x1 in the function I get y1, this is my y2, this is my yn. and this is my yn minus 1.

So in this case, what am I going to do? I have subdivided, I have divided the whole interval into the n number of subintervals and then, so these points, the points x0, x1, up to xn are called nodal values. So these are the nodal points.

Now, in each of the subintervals, suppose I have the subinterval now, x0 to x1, this is my1 of the subinterval, then x1 to x2 and this one. So I can find out the solution because basically, for integration, I need to find this area.

So I can find this area here, then I can find this area, this area, then I can find this area, then I can find this area, and altogether, I will add all this one and I will get the solution for this integration.

So basically, what I am doing here is solving this integration $\int_{x=a}^{x=b} f(x)dx$. So this is we are going to find out or this type of.

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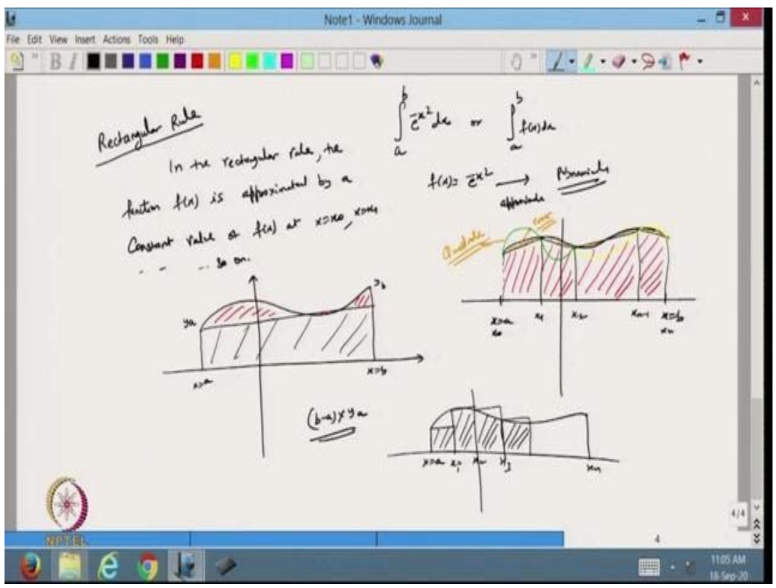

Now, suppose; so how are we going to do that one? Suppose I, somebody asked me how we are going to find out this interval, this integral, the Gaussian function, and I want to find the integral of this one or a general function.

Now, what we can do is that as we know that this is a function f(x), so my $f(x) = e^{-x^2}$ and then, I can interpolate. This one I can interpolate, approximate maybe I can should write approximate. So I can approximate this function with polynomials and then I can integrate because I know that the polynomials are very easy to integrate, so I can approximate this function with a polynomial, and then we can find.

So that is, we know that we can fit a polynomial or a quadratic function for any of the functions f x and then we can integrate it for that one. So that way, we can do it. So, for example, I have a (function) from x=a and this is x=b, and suppose, I have a function like this one. So in this case, I can do that I can subdivide this one into n subinterval So this is my x0 this is my x1, this is suppose I take x2, and this is my xn, this is suppose xn minus 1.

So what I can do is that this is my values given to me. So between these two values, I can approximate this function by a polynomial or by a linear function like this one. Or I can also approximate this function by a linear function, this also by a linear function, this also by a linear function.

And then this area I can find, I can integrate and I will get this area; very easily I can find out because then it becomes a trapezoid and then, I know how to find the area of the trapezoid. So this way I can find out the area under this curve and that is the integration. And if you see from here, then this part if you see, so this will be the error and this because in this this is approximation. So in this case we are also introducing some errors.

So this will be the error or somebody says that, okay, instead of approximating this function with a linear function, I can approximate this with a quadratic function. So in the quadratic function, I can, for the quadratic function I need three points. So suppose, I choose these three points and I represent this function with a quadratic function or these three values I can represent by this value.

So basically, what we are doing, we are splitting the whole domain into subdomains which have three points here, then three points next, the next three points, in this way and then, we can approximate the function with a quadratic function. So this is my quadratic function.

So in that case, we can find out this quadratic function, then I can integrate and if I integrate this one, I will get the area under this curve and that will be the integration. Similarly, I can find the area under this curve and in this sense there is some error also there. So this way, we can find out.

So the given function can be approximated with a larger number of functions, which are very easy as compared to the function we are easy to evaluate or we are able to find the integration. So this way, we can find out. So we can go with the very basic rule. So that is called the rectangular rule. And we already know that when this definite integral is taught to us in plus-2 level, then we use this rectangle rule to find out the definite integral.

So what is this rectangle rule? So in the rectangular rule, the function f(x) is approximated by a constant value of f(x) at x=x0, the nodal value x=x1, and so on. So in this case, because this nodal value, I know the value of the function, so we can approximate the function by constant value of the function in the first interval, the second interval, in the third interval, and so on. So this way we can define.

For example, like suppose I have the function. So this is what my function is given to me. This is my x=a and this is my x=b. Now, I will split this one. So the first thing is that I can approximate this function by the value of the function here that is given to me. So this is my y0 and this is supposed my y, I can write a, and this is yb if I have the two values. Then what I can do, I can approximate this function with a straight line. This is, so this is the value of the function. So here what I am doing? I am approximating the function by the constant value given at x=a.

So this is a rectangle, you know, if you see this, this is a rectangle. And I can find the area of this rectangle. So the area of this rectangle will be length into breadth. So from here, I can say that the length will be (b-a)y. So that is my area so that we can do. But in this case, if you see, the error is quite large. So to reduce the error, what we can do is that instead of this I even split this one.

So again, I am finding, splitting this one. Let us plot again and then, I am taking it equal to xa then this is x1, x2, x3, and suppose xn. Now, in this case, I will approximate this function from here to here; in this subinterval, I will represent this function with this one. Then, I will represent this function in this way.

The third one is I will represent the function by this. Then I will represent this function by maybe next one, and so on. So I will get the rectangles for this one. And then, I can find the area of this rectangle because the area of the rectangle is very easy to find. And then, altogether, I will get the approximate value of the integral for the given function. So, this is the rectangular rule

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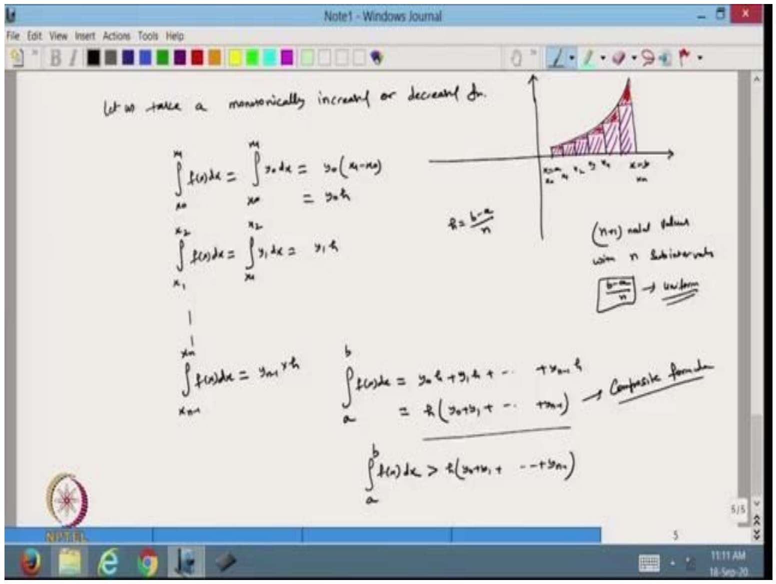

So let us take the, let us apply; take a monotonically increasing or decreasing function. So, for example, suppose I take the first one. Let us take a monotonically increasing function like this one. So suppose this function is $x=a$ and this is given to me at $x=b$.

So in this case, I will split this one into the n subinterval like this one and then you can see that, so this is my x, so I will call it x_0, I will call it x_1, x_2, x_3, x_4, and the last one I call it x_n. So I have a $n+1$ number of points; $n+1$ nodal values with n subintervals. And how can we find this one? Just take b minus a and divide by n for the uniform. So that will give you the $n+1$ nodal values.

So I can say that this is uniform nodal values. The distance between two nodal values is constant in this case, so that is a uniform. Now, what I can do is that, so I will approximate this function with the left value. So suppose I take this value, then this, then this, then this, this, this. So this value, I just take the left value and if I find the integral, so I will get these areas. So in this case, I will get from x_0 to x_1 f x dx.

Now, I have approximated the function by value at x_0. So this integral becomes x_0 to x_1 and f x is just y_0 dx. So it will be y_0 and this will be x_1-x_0. So if it is constant then we call it h, the interval subinterval n. So this will be $h=(b-a)/n$. So that is the value, so I call it y naught h.

Then I can find out the integral in the next subinterval $f(x)$ dx and in that case, I am choosing by y_1. So if you see from here, this will be x_1 to x_2 y_1 dx, so this will be y_1 h. And the same way, we can go, so in the end, if you see, this is $n-1$ to x_n f x dx, so this will become y_{n-1} into h.

So if I want to find the integral from a to b of this function, then we can add together. So this will be equal to y0 h y1 h up to yi minus 1 h. I can take the common h, so this will y0 y1 to yi minus 1. So this is what I am doing, so this is called a composite. Because we are dividing the domain and then adding together, so this is the composite formula. So if I find out the integration, this is the value of the integration.

And I know that in this case, because the error is there. So this is the error we are introduced to at each of the subinterval. So this is the error we are adding. So from here, I can say that from a to b f x dx. So this is the area under this curve and it is the area we are getting. So this area definitely will be lesser than this one. So I can say that this area is greater than y0 y1 up to yn minus 1. So that is there. So definitely and the difference between these two values with the error we have introduced here.

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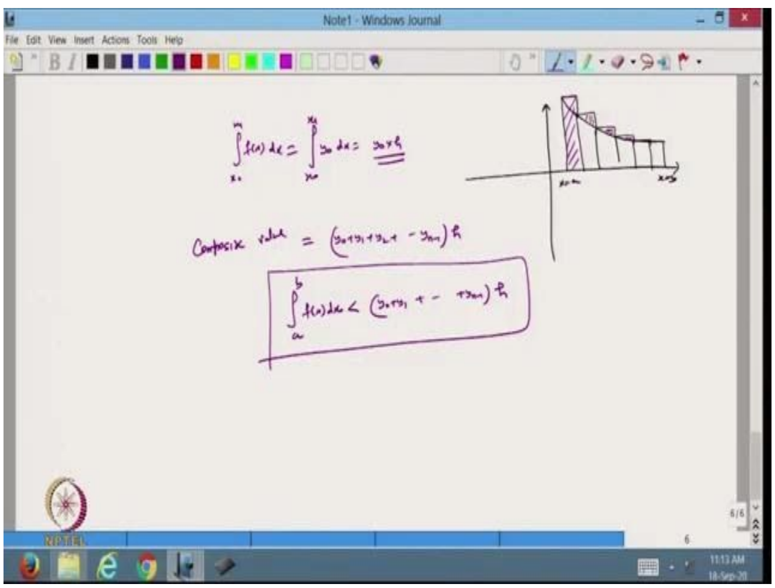

Similarly, I can define the monotonically decreasing function. So in the decreasing function, this is my x=a, this is my x=b, then again I split this one into the n subintervals. So now, what am I going to do? Again, this function I approximate with this value.

So this is the value I am going to get, next value I am going to get is this one, next is this one, next is this one, and so on. So in this case also, we are approximating the function with this rectangle. Now, this function is approximate by this value. Now, if I want to find the integration from x0 that is a to x1 f x dx. So this value will be the area under this rectangle. So this is my x0 to x1 and to y0 dx. So this is y0 into h. So this is my y0 value, then next is y1.

So in this case also, the composite value of the composite value will be again (y0+y1+y2+yn-1)h. But here, now, this area is definitely lesser than the areas we are taking for this rectangle, so this is the extra area we are adding for this one. So from here, I can say that from a to b f x dx, this is the integration we are getting. So it definitely will be less than y0 y1 up to yi-1 h.

So this is the way we can define the rectangular ways to find out the numerical integration for a monotonically increasing or decreasing function. So this is just a preliminary work we can do or a basic work we can do for the numerical integration for a given function. So we will stop here today.

So today, we have started with the basic concept of numerical integration and we have discussed how we can approximate the given function with the rectangular values. And in the next lecture, we will continue with this one. Thanks for watching. Thanks very much.