Scientific Computing Using Matlab
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Lecture - 54
Higher Order Accurate Numerical Differentiation Formula for Second Order Derivative

Hello viewers, welcome back to the course on Scientific Computing in Matlab. So, we will continue with the Numerical Differentiation, as we have discussed in the previous lecture. So, in the previous lecture, we have discussed how we can find out the first order derivative; whenever data is given to us that is not equispaced. And we are able to find this expression number 3 for that one.

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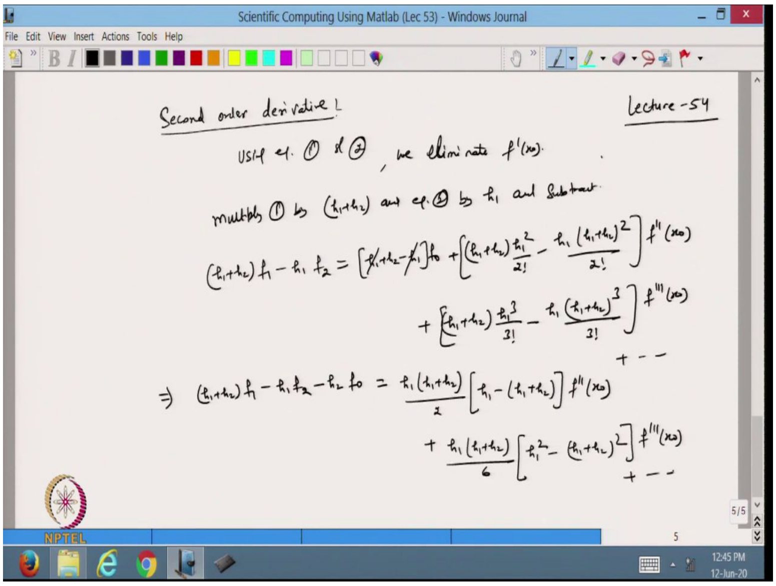

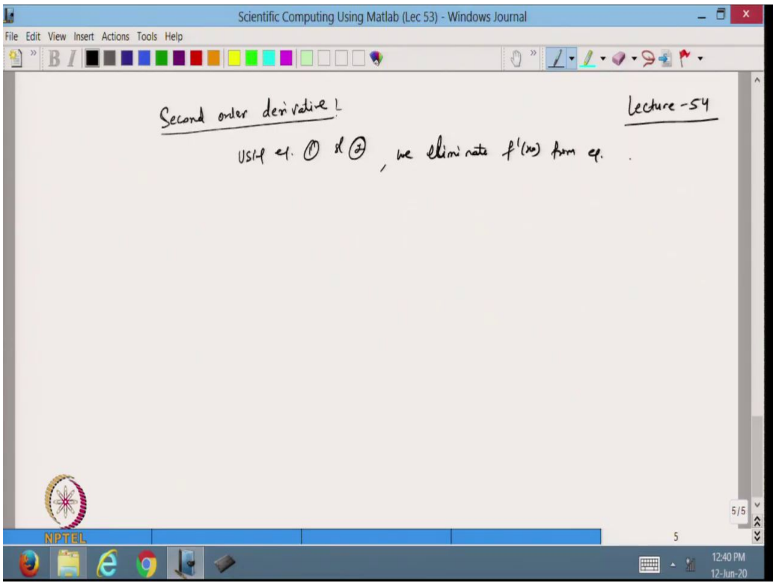

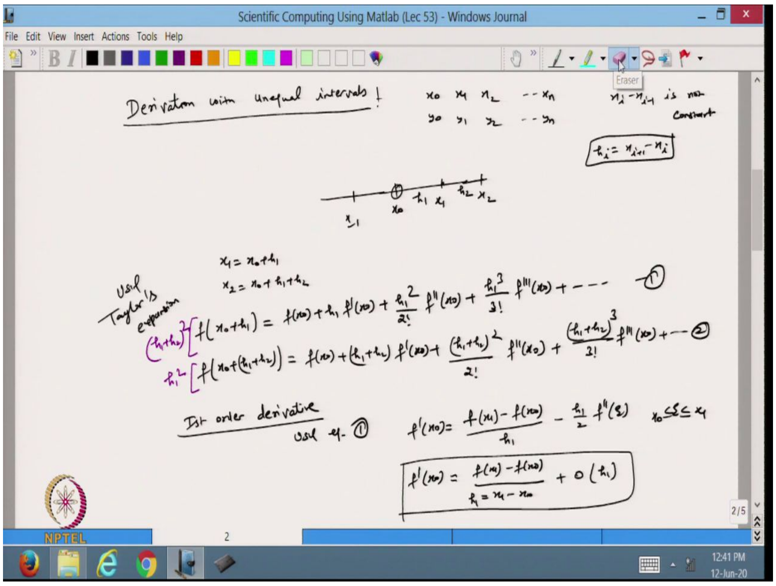

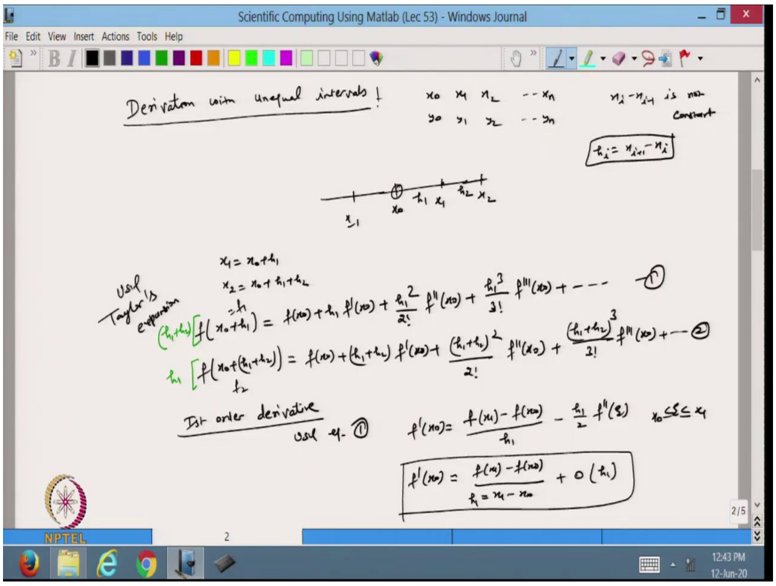

Now today, I want to find out that how we can approximate the second order directive. Now, we will use the expression 1 and 2 as we have discussed. So, using equation 1 and 2; so that is given to me. So, this is the equation number 1, and this is the equation number 2. We eliminate $f'(x_0)$ from equation. Using this one, we eliminate $f'(x_0)$. So, let us eliminate this one. So, how we can eliminate, multiply equation number by h1+h2 and equation 2 by h1 and subtract.

So, let us see what will happen. Now what I am doing, I am multiplying this one, because I want to remove this. So, I multiply this by h1+h2 and this one by h1 and then I subtract. So, let us see what will happen. So, I will get the expression (h1+h2)f1-h1f2.. So, this I am getting, because this is my f1, and this is my f2. So, I am getting (h1+h2) f1; h1, and this is h1+h2 minus h1 x naught. So, this is $(h_1 + h_2 - h_1)f_0$. So, that is what I am getting. Then this one, h1 multiplied by h1+h2 and this is a h1+h2 multiplied by h1. So, this will cancel out. And the next expression, I will get, is $(h_1 + h_2)\frac{h_1^2}{2!}$. So, this is I am getting $\frac{-h_1(h_1 + h_2)^2}{2!}$. And this I am getting a second derivative at x naught. So, this is the second derivative I am getting at x naught. Now that the third one is the next one is, it will be plus $(h_1 + h_2)\frac{h_1^3}{3!} - h_1\frac{(h_1 + h_2)^3}{3!}$. This is my third derivative and so on. So, this I am getting. Now from here, this will cancel out. Now from here, I can write that this expression becomes $(h_1 + h_2)f_1 - h_1 f_2 - h_2 f_0$. Now from here, let us see that, what the common term we can take. So, if you see from here, I can take $\frac{h_1(h_1 + h_2)}{2}$. So, this one 2 factorial is 2, so this is just 2 value. If I had taken common from here; so I will get here h1-(h1+h2), this one. Now from here, the same way, I will get the terms h1(h1+h2)/6. So, inside I will get $h_1^2 - (h_1^2 + h_2^2)$; because this will come out $f'''(x_0)$ and so on.

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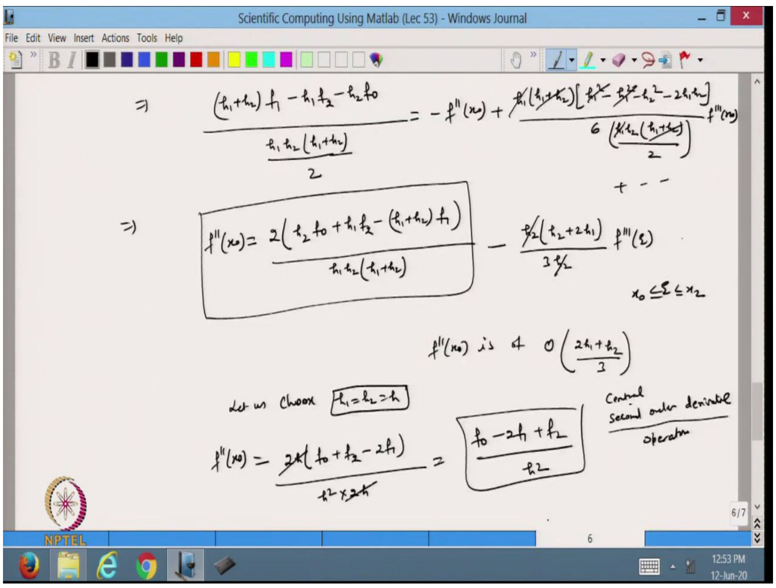

Now, from here I can write this expression, and this will cancel out and I will get -h2. So from here, I can write this expression as

$$\frac{(h_1+h_2)f_1 - h_1 f_2 - h_2 f_0}{\frac{h_1 h_2(h_1+h_2)}{2}} = -f''(x_0) + \frac{h_1(h_1+h_2)(h_1^2 - h_1^2 - h_2^2 - 2h_1 h_2)}{6\frac{h_1 h_2(h_1+h_2)}{2}} f'''(x_0) + \dots$$

; because I am taking the common terms, so I will divide the term, all the terms after this. So from here, my second order derivative,

$$f''(x_0) = \frac{2(h_2 f_0 + h_1 f_2 - (h_1+h_2)f_1)}{h_1 h_2(h_1+h_2)} - \frac{h_2(h_2+2h_1)}{3h_2} f'''(\xi) \quad x_0 \le \xi \le x_2$$

. So, this will cancel out and from here, I get this expression. So now from here, I can say that my second order derivative at x naught is of this form. That contain the value at x0 at x2 and x1 and this is of order this. So, this formula is of order h2, using the third order derivative, Now let us see, what will happen. So, let us choose h1=h2= h.

So let us see, what will happen, in that case, it will be second derivative at x naught. This is h, this is h, it is 2h. So, I can take this common; so I will get 2h so inside I will get

$$\frac{f_0 - 2f_1 + f_2}{h^2}$$

. And this is my second order derivative. So, using the central scheme.

So, this is a central second order derivative. That is $\frac{f_0 - 2f_1 + f_2}{h^2}$. So, this is the central second order derivative, difference scheme, operator. So, that we can define. And now if we put h and h here, so this will be of order h in this case. But generally, it is of order h square, so that is when we do the expression that will come out. So, this is the way we can, we are able to write the second order derivative for un-equispaced data.

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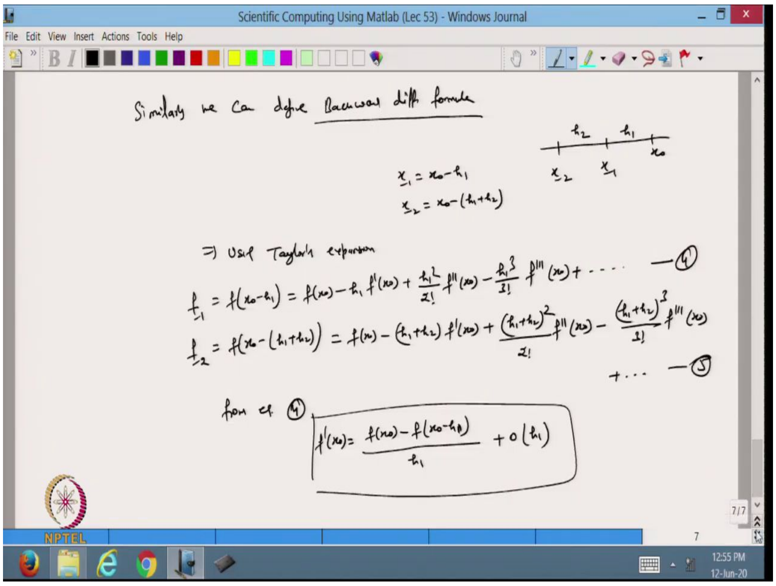

Now, this is, we have used for the forward; similarly, we can define backward difference formula. So, in the backward difference formula; suppose, we have the un-equispace data. And let it is my x naught, this is my x1 and this is my x2. This is my h1 and this my h2. Or you can take as, x i, x i-1; x i-2. Does not matter. So, I can write from here, this is my

$x_{-1} = x_0 - h_1$ and this is $x_{-2} = x_0 - (h_1 + h_2)$.

So from here, now using Taylor's expansion, we can define the function at, so this I define as f at minus 1.

$$f_{-1} = f(x_0 - h_1) = f(x_0) - h_1 f'(x_0) + \frac{h_1^2}{2!} f''(x_0) - \frac{h^3}{3!} f'''(x_0) + \dots$$

So, this terms we can take, so I just define this as equation number 3, so it is 4. Let us take it 4. Similarly,

$$f_{-2} = f(x_0 - (h_1 + h_2)) = f(x_0) - (h_1 + h_2) f'(x_0) + \frac{(h_1 + h_2)^2}{2!} f''(x_0) - \frac{(h_1 + h_2)^3}{3!} f'''(x_0) + \dots$$

. So, this is equation number 5. Now from here, if I take the first order derivative. So, from equation, from equation number 4, we can find the simple first order derivative. So, I can define this as $f'(x_0) = \frac{f(x_0) - f(x_0 - h_1)}{h_1} + O(h_1)$. So, this is the backward difference operator for the first order derivative. So, that we already know, if h1=h then this is just the backward operator for the equispaced data. So, this one we can define very easily then.

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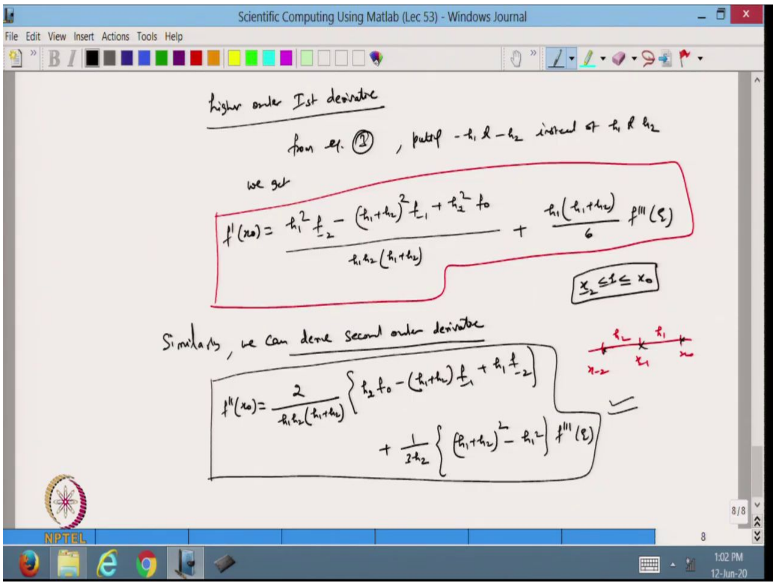

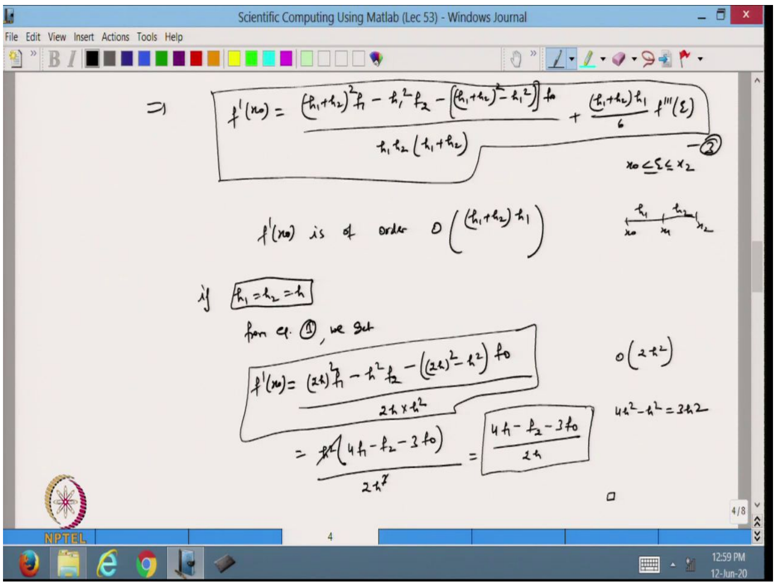

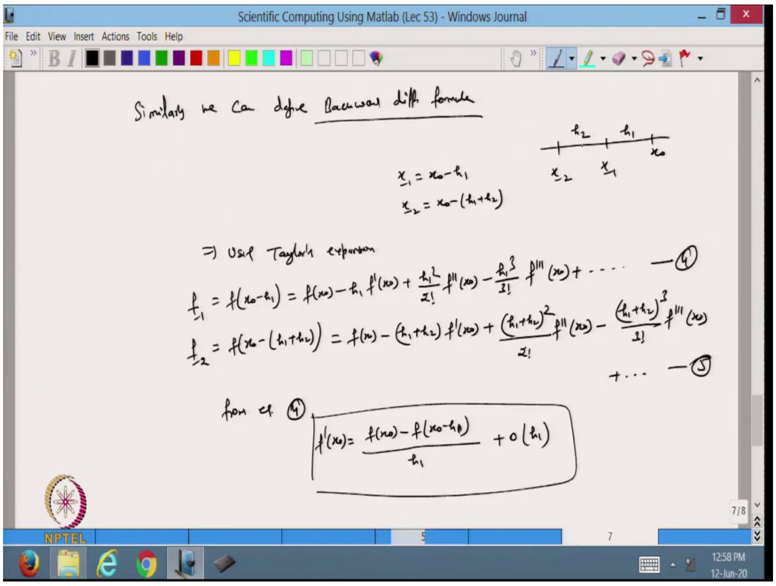

Now suppose, I want to define this for the higher order derivative. So, higher order first derivative. So, the same way we have done in the previous lectures, this one. So, if I take this expression, for the first derivative from equation number 3 and instead of h1 and h2 I put -h1 and -h2. So, let us take a, from equation 3 putting -h1 and -h2, instead of h1 and h2. So, we get, so, if I do that one, so this will be the same, square term is there. This is also square, this is also square, but here it will be -h -h2 and this is a minus sign here.

So now, if I just write that one, then our f dash at x naught can be written as,

$$f'(x_0) = \frac{h_1^2 f_{-2} - (h_1 + h_2)^2 f_{-1} + h_2^2 f_0}{h_1 h_2 (h_1 + h_2)} + \frac{h_1 (h_1 + h_2)}{6} f'''(\xi) \quad x_{-2} \le \xi \le x_0$$

. Actually, if you see in this case, this h1 and h2 was this one; it was my x0, x1 and x2. So, it was my h1 and this my h2. Now, I have x0, x-1 and x-2. So, this is, I am writing as h1 and h2; just to write the same expression, but otherwise you can also write this as a h0 or h1, so does not matter. But we are going forward.

So, that is the expression for the first order derivative, using the backward. Similarly, because the same expression I have to do again and again, similarly we can derive second order derivative. So, in the second order derivative, I do the same thing. I have the 3 values; 1, 2, and 3. So 3 points stencil is there, and if I do this one, I can eliminate the value of f dash 1 f dash at x naught. So, I can write this directly. So, this will come out,

$$f''(x_0) = \frac{2}{h_1 h_2 (h_1 + h_2)} \{h_2 f_0 - (h_1 + h_2) f_{-1} + h_1 f_{-2}\} + \frac{1}{3h_2} \{(h_1 + h_2)^2 - h_1^2\} f'''(\xi)$$

, where xi is belonging from this one.

So, from here I can write directly, but this expression you can find by eliminating the f dash x naught from this expression. So I want to, if I want to eliminate this f dash x naught from here and from here, I will multiply by h1 and plus h2 here, and by h1 and then I subtract. So, this will cancel out and then we can derive this formula. So, we will get this formula for the second derivative. So, this is the formula for the second derivative.

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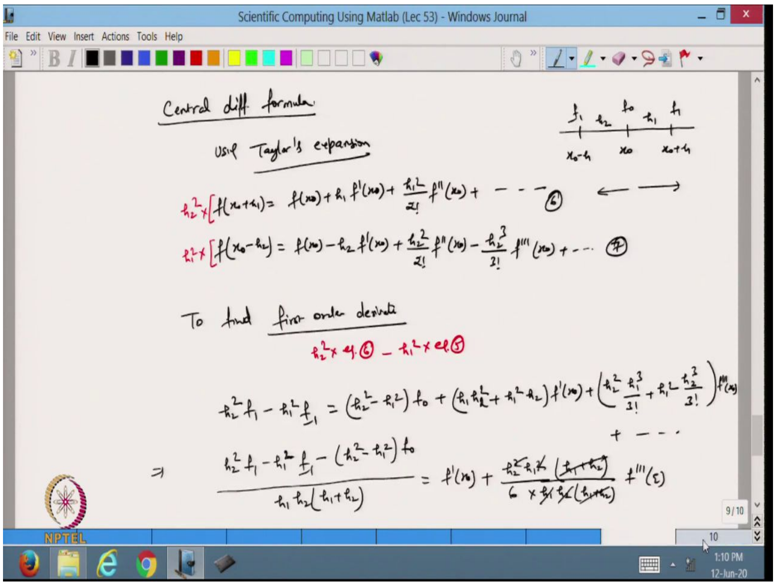

Now what will happen, if I want to define the central difference formula. Till now, we were heading either on the right-hand side, means forward or the backward. Now, what will happen if I have x naught here and this is my x0+h and this is my x0-h. And I will call it this h1 and I call it this h2. So basically, this is my f1, this is my f0 and this is my f-1.

Now in this case, I want to go forward also and backward also. So, we will go one point forward and one point backward. So that is the way, we can define the central difference formula. So in this case, I know that using Taylor's expansion, I have to take x0+h1 plus so on. So this I, 4, 5, 6, I will take it as equation number 6. And then I will take x0-h2, going backward. So, if you see this one, then it will be

$$f(x_0 - h_2) = f(x_0) - h_2 f'(x_0) + \frac{h_2^2}{2!} f''(x_0) - \frac{h_2^3}{3!} f'''(x_0) + \dots$$

. So, I call it equation number 7. So, in this case I am going forward and backward also.

Now, to find out the first order derivative, what I will do, now to approximate, to find the first order derivative. The finite difference formula for first order derivative. So, what I will do is that; in that case, I want to find these values. So, what I do, I will eliminate, suppose I will eliminate f dash, second derivative. So, what I will do that, I will multiply this by h2 square, and I multiply this by h1 square, and then subtract. So, h2 square multiply equation 6 minus h1 square multiply equation number 5. So, what we will get, we will get

$$h_2^2 f_1 - h_1^2 f_{-1} = (h_2^2 - h_1^2) f_0 + (h_1 h_2^2 + h_1^2 h_2) f'(x_0) + \left(h_2^2 \frac{h_1^3}{3!} + h_1^2 \frac{h_2^3}{3!} \right) f'''(x_0) + \dots$$

. Now, from here I can write this expression as

$$\frac{h_2^2 f_1 - h_1^2 f_{-1} - (h_2^2 - h_1^2) f_0}{h_1 h_2 (h_1 + h_2)} = f'(x_0) + \frac{h_2^2 h_1^2 (h_1 + h_2)}{6 h_1 h_2 (h_1 + h_2)} f'''(\xi)$$

; all other terms are ignored.

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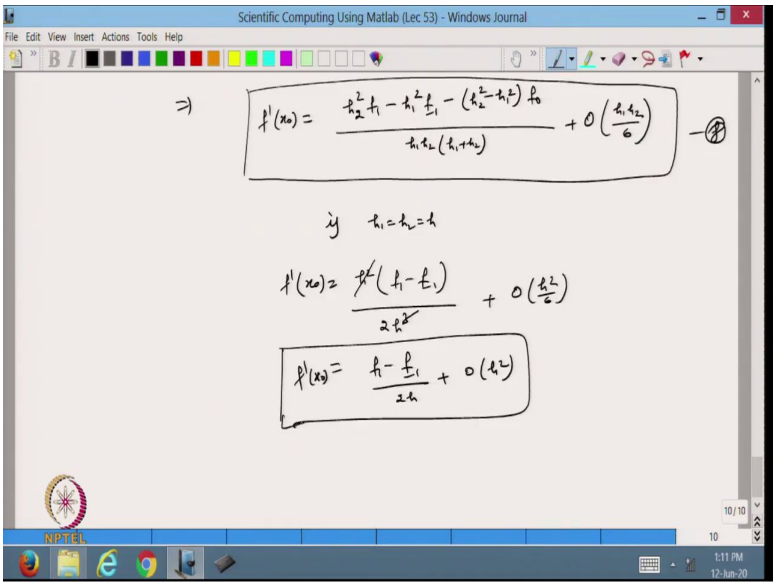

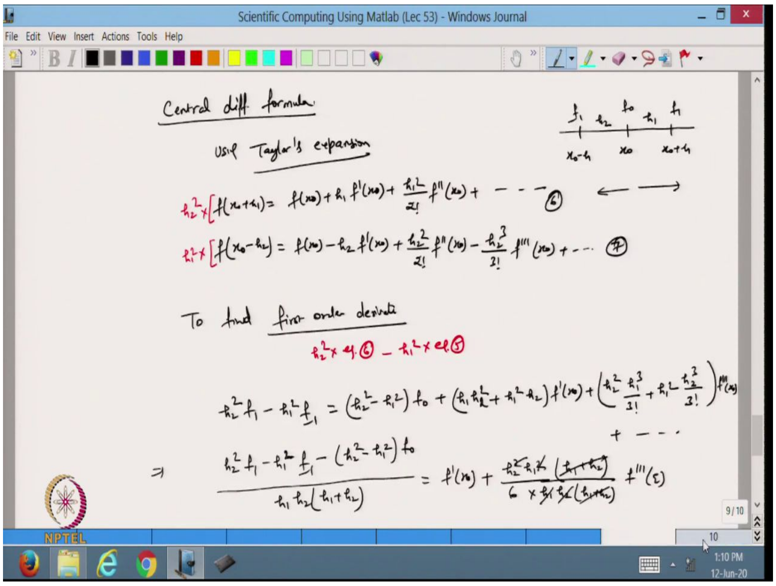

So, from here this will cancel out, this will also cancel out. So, I can write my central finite difference for first order derivative, for the first order derivative. So, this becomes

$$f'(x_0) = \frac{h_2^2 f_1 - h_1^2 f_{-1} - (h_2^2 - h_1^2) f_0}{h_1 h_2 (h_1 + h_2)} + \mathcal{O}(\frac{h_1 h_2}{6})$$

. So, that is the expression, central difference expression for the first order derivative for un-equispaced data.

Now this one I can write, if h1=h2=h; then for equispaced data I can write this as, so h h and h I can take common. So, this will be 0. In that case, I can write from here,

$$f'(x_0) = \frac{h^2 (f_1 - f_{-1})}{2h^3} + \mathcal{O}(\frac{h^2}{6})$$

. Now this will, I cancel out, and from here I will get

$$f'(x_0) = \frac{(f_1 - f_{-1})}{2h} + \mathcal{O}(h^2)$$

. And this is I know, that this is the first order centre scheme for the first order derivative. So, that is the way we can verify this one.

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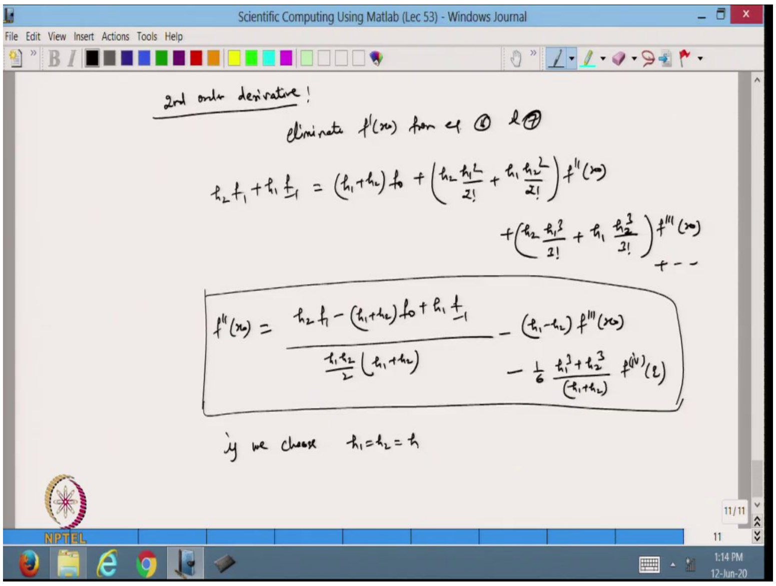

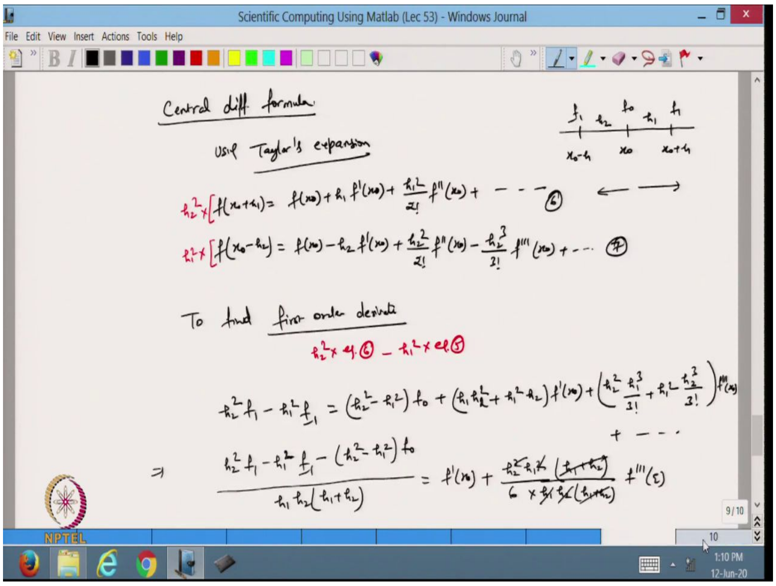

Now, I can give one more expression for a second order derivative. So, second order derivative. So, I can eliminate, so from 6 and 7 eliminate f dash x naught, from equation 6 and 7. So, we can eliminate that one. So how can we eliminate it? I want to eliminate the first order this one. So, I will multiply this by h2 and this by h1, and then we will add. So in that case, if I do that one, then I can write

$$h_2 f_1 + h_1 f_{-1} = (h_1 + h_2) f_0 + \left(h_2 \frac{h_1^2}{2!} + h_1 \frac{h_2^2}{2!} \right) f''(x_0) + \left(h_2 \frac{h_1^3}{3!} + h_1 \frac{h_2^3}{3!} \right) f'''(x_0) + \ldots$$

. Now, I can take my h1 and h2 on the left-hand side, taking this term together. So from here, I can write directly that my

$$f''(x_0) = \frac{h_2 f_1 - (h_1 + h_2) f_0 + h_1 f_{-1}}{\frac{h_1 h_2}{2}(h_1 + h_2)} - (h_1 - h_2) f'''(x_0) - \frac{1}{6} \frac{h_1^3 + h_2^3}{(h_1 + h_2)} f^{(iv)}(\xi)$$

. So, this will be the expression for second order derivative using the centre scheme. And now, from here you can verify that if we choose h1=h2=h.

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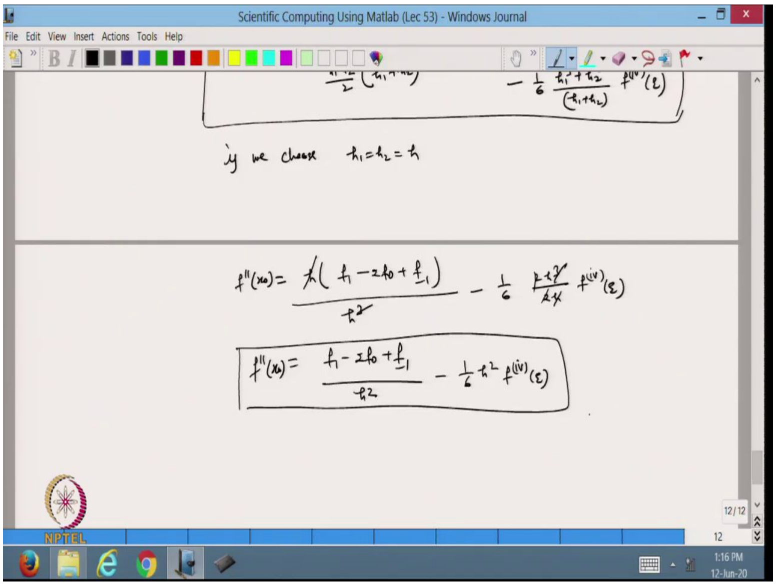

Then my f double dash x naught can be written as. So this will be h, this will be 2h, this will be h. So, I can write my h common. So, this will be

$$f''(x_0) = \frac{h(f_1 - 2f_0 + f_{-1})}{h^3} - \frac{1}{6}\frac{2h^3}{2h} f^{(iv)}(\xi)$$

. So this will cancel out, and this will cancel out. So from here, I can write that this is equal to

$$f''(x_0) = \frac{(f_1 - 2f_0 + f_{-1})}{h^2} - \frac{1}{6} h^2 f^{(iv)}(\xi)$$

. And that we already know, that if we have the equispaced data and we want to find out the second order finite difference, using the central scheme. So, this is the expression for that one and that is of order h square. So, this one we can verify by putting h1=h2=h.

So let me stop here. So, today we have discussed that how we can derive the formulas for the first order derivative, for the second order derivative, of higher accuracy that is of order h or order h square using for the data that is not equispaced. And with this one, we will want to close this unit that is a numerical differentiation. And from the next lecture, we will continue with the next unit that is numerical integration. So, I hope that you have enjoyed this one. Thanks for watching, thanks very much.