Scientific Computing using MATLAB

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Lecture No. 53

Higher Order Accurate Numerical Differentiation Formula for First Order Derivative

Hello viewers. Welcome back to the course on Scientific Computing using MATLAB. So, now we will continue with the numerical differentiation as we have discussed in the previous lecture about the first order derivative. So, today we will continue from that.

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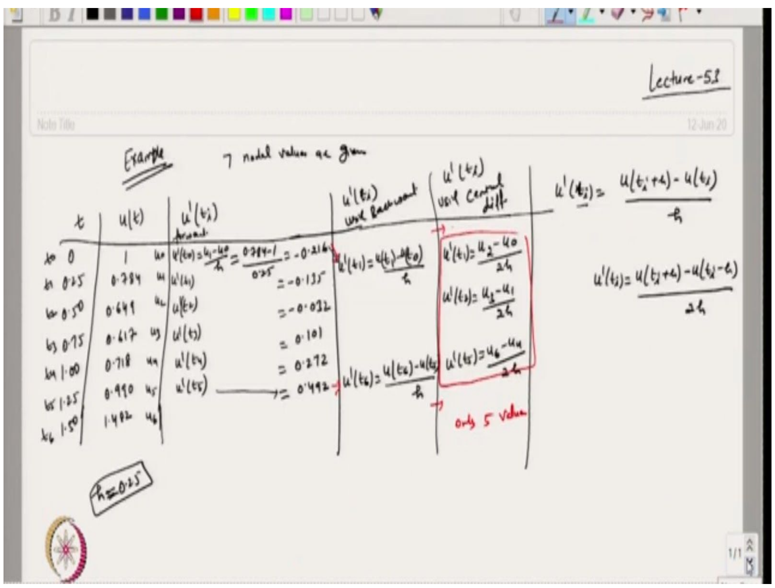

So, this is lecture 53. So, in the previous lecture, we have discussed forward operator, backward operator and the central operator. So, let us take an example of how we can use this one. Now suppose, I have the data suppose, I have some value of t and the value of u(t). So, this value is given to me that it is 0, 0.25, 0.50, 1.25 and 1.50. So, this is both my t0, t1, t2, t3, t4, t5 and t6. So, the points value, the function is given to me. So, that is it is 1, 0.784, 0.649, 0.617, 0.718, 0.990 and 1.482. So, this is the value given to me and this is my u0, u1, u2, u3, u4, u5 and u6. So, 7 nodal values are given.

Now suppose I want to apply the forward operator. So, forward operator I know that I want to find what is my u, the derivative, I want to find what is my u'(ti). So, this is what I want to find using forward. So, I know that for the forward $u'(t_i) = \frac{u(t_i + h) - u(t_i)}{h}$. So, it is equally spaced data. So, my h is 0.25 so I am finding my ti at this one, so I have to go one step forward. So in this case, I will get the value here. So, from here I can say that the u' at t0, this one I want to find. So, this will be $u'(t_0) = \frac{u_1 - u_0}{h}$.

So, if I solve this one, let us find out the value. So, this will be 0.784-1/h=0.25. Similarly, I can define the next one. So, this is my u'(t0), then u'(t1), u'(t2), u'(t3), t5. So, I can go up to t5 only because I am using the values after that. So, if I want to find out the derivative at u t6 that I cannot do with the help of this forward operator. So, this one I can find out. So, if I solve this one, this will be -0.216. Similarly, I can solve the next one. This will come -0.135. Then next is -0.032, the next is 0.101, 0.272 and 0.49. So, this is my value.

Now the same thing I can define, the u'(ti) using backward. So, if I use a backward, then I know that I will apply this formula. So, for the backward it will be, I will start from u dash t1

$u'(t_1) = \frac{u(t_1) - u(t_0)}{h}$. So, that is backward. So, in this case, I can go up to this. So, this would be $u'(t_6) = \frac{u(t_6) - u(t_5)}{h}$. So, now from here whatever the value I am taking so this will be the same as this value. This is the same as this value and this last value will be the same as this value.

So, u'(t5) will be the same as u'(t6) when I take the backward. So, the total number of 6 values we can find using the first order derivative. The same way I can define for central. So, this is u'(ti) using the central difference. So, in the central difference I know that I was taking

$u'(t_i) = \frac{u(t_i + h) - u(t_i - h)}{2h}$. So, in this case, I am going one step forward one step backward divided by 2h. So, if I want to find this value I can take the value of u'(t1) so this will be $u'(t_1) = \frac{u_2 - u_0}{2h}$.

Then I can take u'(t2) so it will be $u'(t_2) = \frac{u_3 - u_1}{2h}$ so I can go up to u'(t5), it will be equal to $\frac{u_6 - u_5}{2h}$ and that is it. So, in this case when I was applying the forward method, then I was able to take the 6 derivative, the value of this function at 6 places, the derivative. With the backward also I had the 6 values but using the center values we have only 5 values.

So, I do not know what is the value at u'(t0) and u'(t6). So, whenever we have the data and we want to use our center difference operator to approximate the derivative, then we are able to find the value of the derivative only at the interior nodals. So, these are the interior nodal values t1, t2 up to t5 and the boundary values at this place and this place we cannot approximate the value of the derivative using the central difference. So, if I want to calculate the value of the derivative at the initial point and the last point, we may use the forward operator here and in the end I can use the backward operator here. So, that way we can approximate the derivative of the function for each value of the data point.

So, this all things will be used when we solve the differential equation. So, in solving the differential equation boundary loop problem or the initial loop problem, we have to take care whether we want to apply the forward operator, backward operator or the central difference operator to approximate the derivative involved in the differential equation. So, this is a use of how we can approximate. I can use that different different type of difference operator.

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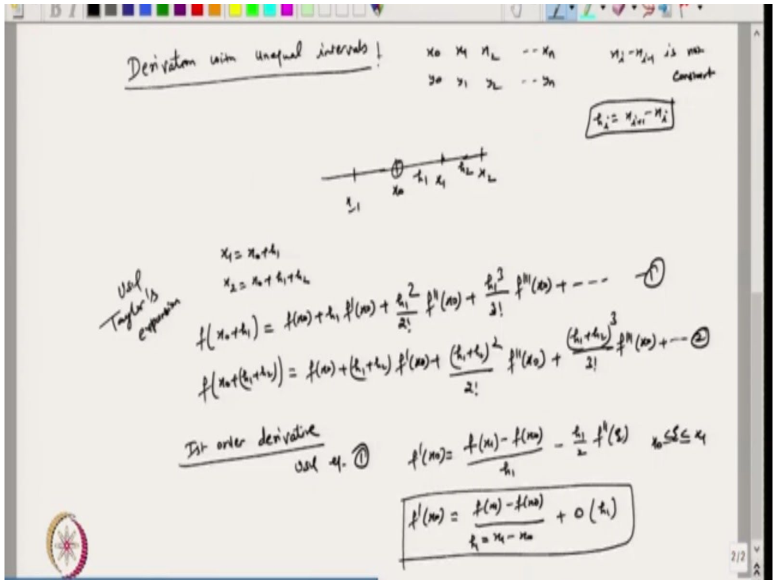

Now, this is what we have done for the equi space derivatives, equispaced data. Now, I want to define the derivation with an unequal interval. So in this case, I have the x0, x1, x2,...,xn, y0, y1, y2,...,yn and $x_i - x_{i-1}$ is not constant. So, in this case I can define my $h_i = x_{i+1} - x_i$ and this is different. So, we have the data that is not equispaced. Now suppose I want to use the derivative. So, suppose my x0 is this and I take x1 and I take x_{-1}, this is my x2. Now my x0, suppose this is my h1 and this is my h2. So, let us write, my x1 can be written as x0+h1. My x2 can be written as x0+h1+h2.

Now, suppose I want to find the derivative of my function at this point with the help of suppose I want to take the forward operator. So, either using this one or this one. So, let us write what is the Taylor expansion for x0+h1. This can be written as

$$f(x_0 + h_1) = f(x_0) + h_1 f'(x_0) + \frac{h_1^2}{2!} f''(x_0) + \frac{h_1^3}{3!} f'''(x_0) + \ldots$$

. So, this is equation number 1 and I am expanding with the help of Taylor expansion because I know the value of the function only at this point and this point so that is why I am finding the value at x0+h1 with the help of Taylor expansion.

So, I can write here using Taylor's expansion. Now the same way I can define f(x0+h1+h2). So, this one is again

$$f(x_0+h_1+h_2)=f(x_0)+(h_1+h_2)f'(x_0)+\frac{(h_1+h_2)^2}{2!}f''(x_0)+\frac{(h_1+h_2)^3}{3!}f'''(x_0)+\ldots$$

. So, that I take as question number 2. Now from here if I want to find out the first order derivative. So, the first order derivative, the simplest form is that using equation.

So, using equation 1 what I do, I take this on the left hand side. This is my value of f at x1. This is f(x0) and this h1 I can take common, so from here I can write that

$$f'(x_0)=\frac{f(x_1)-f(x_0)}{h}-\frac{h_1}{2}f''(\xi),\quad x_0\le\xi\le x_1$$

. The same way we have done as for the equispaced data and from here I can say that this is of order h1.

So, this is I can say that, in this case

$$f'(x_0)\approx\frac{f(x_1)-f(x_0)}{h=x_1-x_0}+\mathcal{O}(h_1)$$

. So, that is my forward operator for unequal data. So, this one we have defined.

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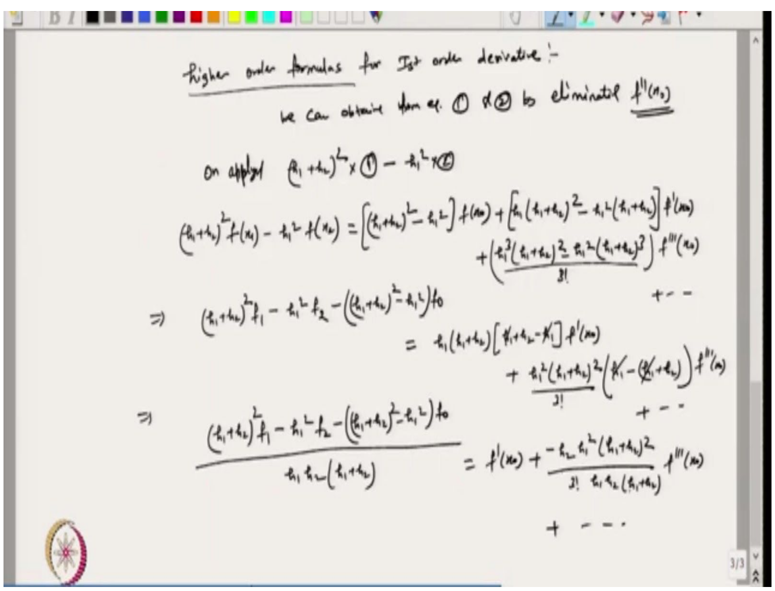

Now similarly, I can define the higher order formula for first order derivatives. So, first order derivatives, you see that in this case I have used this one and I am able to get the derivative of the function at x naught with the order of accuracy, the order of h1. Now, I want to increase the order of accuracy. So, what should I do? I want to find the higher order formula for the first order derivative. So, this I can write down. So, we can obtain from equation 1 & 2 by eliminating f''(x0). So, we can eliminate this one. How can we eliminate f''(x0)? What I do is, so I multiplied this equation with (h1+h2) square.

So, this is there and I multiply this equation with h1 square and then I subtract. So, I can write from here $(h_1 + h_2)^2 \times$equation 1-$h_1^2 \times$ equation 2. So, this one we can do. On applying now what I do? I will get it from here,

$$(h_1 + h_2)^f (x_1) - h_1^2 f(x_2) = [(h_1 + h_2)^2 - h_1^2] f(x_0) + [h_1(h_1 + h_2)^2 - h_1^2(h_1 + h_2)] f'(x_0) + \frac{[h_1^3(h_1 + h_2)^2 - h_1^2(h_1 + h_2)^3]}{3!} f'''(x_0) + \ldots$$

. So, this one we can write. Now from here, we are able to eliminate my second derivative fx. Now from here I can write so this one can be written as, so let us write this. So, this can be written as

$$(h_1 + h_2)_1^f - h_1^2 f_2 - [(h_1 + h_2)^2 - h_1^2] f_0 = h_1(h_1 + h_2)[h_1 + h_2 - h_1] f'(x_0) + \frac{h_1^2(h_1 + h_2)^2}{3!}(h_1 - (h_1 + h_2)) f'''(x_0) + \ldots$$

. So, this one I can write from here. So, from here also this will cancel out.

Now from this, whatever the expression I get, I will divide this by this factor. So, from here, I will, I can write that I can write from here

$$\frac{(h_1 + h_2)^2 f_1 - h_1^2 f_2 - [(h_1 + h_2)^2 - h_1^2] f_0}{h_1 h_2 (h_1 + h_2)} = f'(x_0) + \frac{-h_2 h_1^2 (h_1 + h_2)^2}{3! h_1 h_2 (h_1 + h_2)} f'''(x_0) + \ldots$$

. So, all the terms we will divide by this factor.

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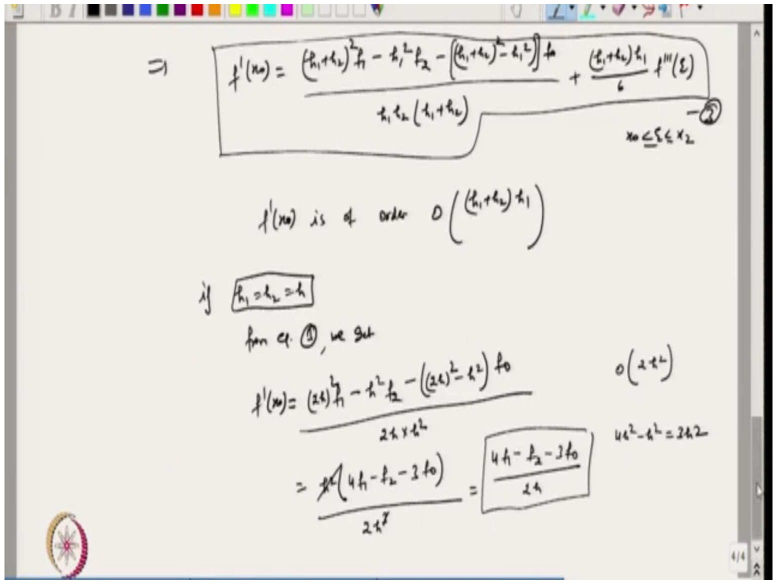

So from here if you see that, this will cancel out. So, this factor will cancel out this one, h2 will cancel out and this will cancel out with this h. So, from here, I can write directly that my f dash at x naught can be written as now

$$f'(x_0) = \frac{(h_1+h_2)^2 f_1 - h_1^2 f_2 - [(h_1+h_2)^2 - h_1^2] f_0}{h_1 h_2 (h_1+h_2)} + \frac{(h_1+h_2)h_1}{6} f'''(\xi), \quad x_0 \le \xi \le x_2$$

So, this one I can take because all other terms will contain the higher order of these terms so that we can ignore and then this. So from there, this is my expression for the first order derivative that contains 3 points values, point value at f0, then f1 and f2. So, this is of order (h1+h2)h1. Now, let us see. So, this is my equation number 3 because the previous one has only 2 equations so I can have the value 3. So, this is of order this value. So, I can write that f dash x naught is of order (h1+h2) h1 by 6 or 6 we can ignore. This is of order. That is it.

Now, let us take if h1=h2=h, what will happen if that my data is equispaced, then in that case what you will get. So. from equation number 3, we get f'(x0). Now, this will be

$$f'(x_0) = \frac{(2h)^2 f_1 - h^2 f_2 - ((2h)^2 - h^2) f_0}{2h \times h^2}$$

and the order becomes 2h square.

Now I can take my terms common from here. So, I write my h square common, so I can write from here, this will be $\frac{h^2(4f_1 - f_2 - 3f_0)}{2h^3}$. So, I will get this value because I have taken at h square common and this will cancel out. So, from here, I will get that this is equal to

$\frac{(4f_1 - f_2 - 3f_0)}{2h}$. So, I will get this value for the first order derivative and this is of order h square. So in this case, I am using the 2 values f1 and f2. So, that expression becomes the very easy expression when the data is equispaced.

Now, in this case now we will find out what about so that we will do in the next chapter. So, let me stop it. So, today we have discussed how we can drive the first order derivative whenever we have the data, which is not equispaced. So, in that case, we get the expression using Taylor's expansion, and then in the next lecture we will continue with how we can find the second order derivative. So, thanks for watching. Thanks very much.