Scientific Computing using MATLAB
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Lecture No. 52
Various Numerical Differentiation Formulas

Hello viewers. Welcome back to the course on scientific computing using Matlab. So, today we will also continue from the numerical differentiation we have started in the previous lecture. So, let us continue with the numerical differentiation.

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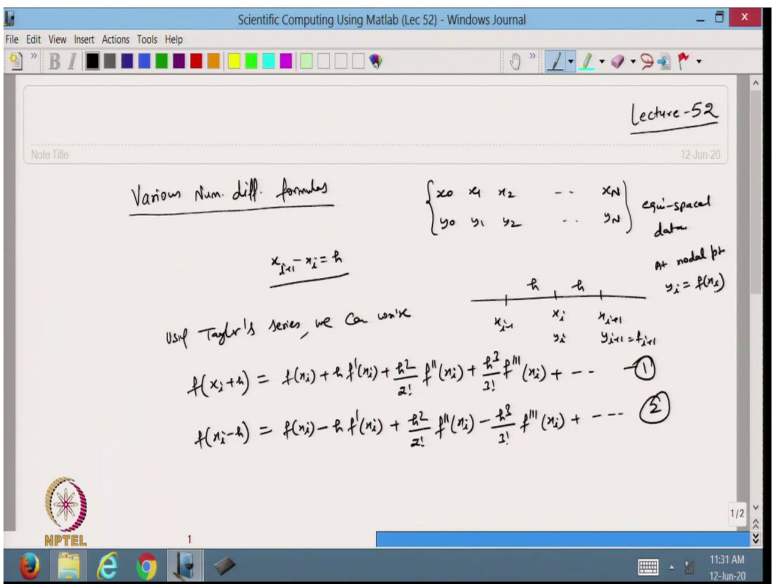

So, in the previous lecture we have discussed how we can use the Newton methods that we used in the interpolation to find out the derivative of the function at some value in between the given values. Now, we will continue with this one. Today I will discuss the various numerical differentiation formulas like we have the data points. So, this is my x0, x1, x2,...,xn. So, this is the n+1 number of points and the value of the function is given to me. Now suppose I have only this data and I want to find out the derivative of the function at any point in between. So, I will try to find out the various numerical differentiation formulas.

Now, this one we are doing, so this is I am talking about equispaced data. So, I have $x_{i+1} - x_i = h$. So let us take this as xi. This is x with the, now using Taylor series we can write f(xi+h). So, this one I can take any constant value plus h. So, this one I can write as $f(x_i + h) = f(x_i) + hf'(x_i) + \frac{h^2}{2!} f''(x_i) + \frac{h^3}{3!} f'''(x_i) + \ldots$. So, this one I take as a 1.

Similarly, I can find f(xi-h). So, this one I am doing as a forward and this is a backward. So, this one I can write

$$f(x_i - h) = f(x_i) - hf'(x_i) + \frac{h^2}{2!} f''(x_i) - \frac{h^3}{3!} f'''(x_i) + \ldots$$

this is equation number 2.

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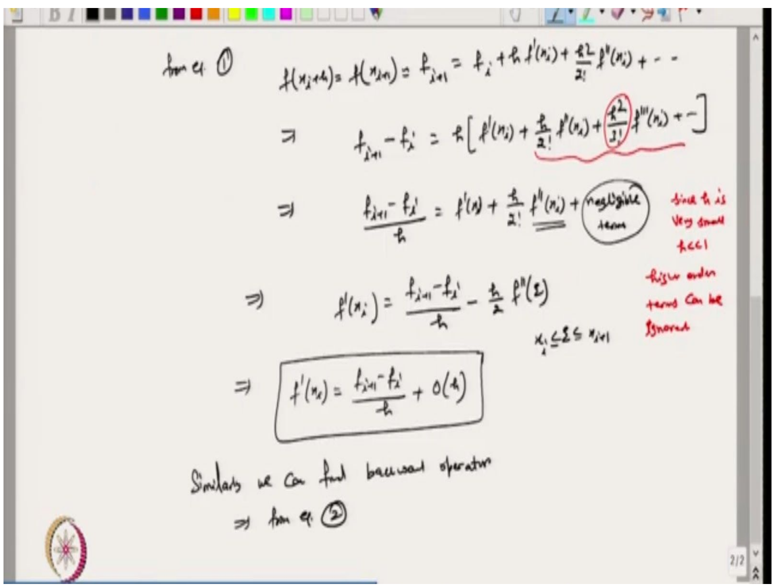

Now I want to write the forward so from equation 1, I know that this

$f(x_i + h) = f(x_{i+1}) = f_{i+1}$. So, this is basically if I write xi, I can write $y_{i+1} = f_{i+1}$. Similarly yi, so add the nodal points, yi=f(xi). This one, I know that one so from here this can be written as fi+h. Now this is the value f’(xi). Now from here I will take this term on the left-hand side. So, once I take these terms, I will get these values only and now I take the common value that is h.

So, it becomes $f_{i+1} - f_i = h\left[f'(x_i) + \frac{h}{2!}f''(x_i) + \frac{h^2}{3!}f'''(x_i) + \dots\right]$. From here I can divide by hi so I will get

$$\frac{f_{i+1} - f_i}{h} = f'(x_i) + \frac{h}{2!}f''(x_i) + \text{negligible terms}$$

. So, all these terms will contain the value of h, then h square. So, since h is very small we choose h very small, so h is very very small. Then, the higher order terms can be neglected, because the higher terms will be very small.

So, these terms will be very small as compared to this term. Next term will be very small. So from here, now this is a function, so I take the maximum of this one. So from here, I know that this can be written as my f'(x), so this is f'(xi), it can be written as

$$f'(x_i) = \frac{f_{i+1} - f_i}{h} - \frac{h}{2}f''(\xi), \quad x_i \leq \xi \leq x_{i+1}$$

.

So from here, I can say that my function f'(xi), the derivative of the function at xi,

$$f'(x_i) = \frac{f_{i+1} - f_i}{h} + \mathcal{O}(h)$$

. So, this is the same as we have done for the forward operator. So, I am taking the value of the function at i, i+1 minus fi divided by h and this is I know that the derivative is of order h. Similarly, we can find backward operators. So, I can use the previous one. So, using this one and the same way I will go, f(x-h). It will take on the left side. I will take the minus sign and so on.

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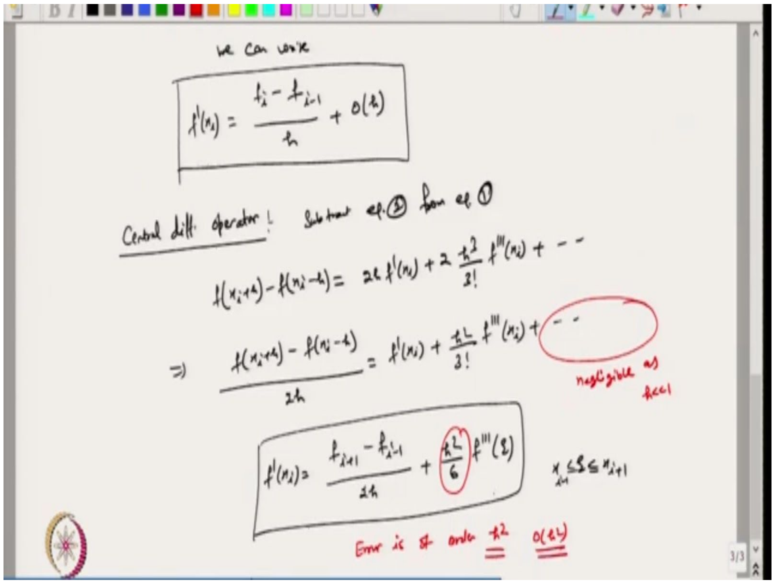

So from equation 2 I can, so from equation 2, we can write fi minus 1, fi minus fi minus 1, at xi plus the terms order of h because everything will be the same in this case. So, all the terms I will take on the left hand side. Then I will take the negative sign divided by h. So, the same thing will happen as we have done just now. So, from here I can write that my first order backward difference operator is of order h. So, this one we can write down. So, this is of order h. Now, I want to write in a higher order.

Now I want to define the central, central difference operator. So, subtract equation 2 from equation 1. So I can write $f(x_i + h) - f(x_i - h)$. So, this I have written because I am just subtracting 2 from 1. So, this will cancel out and I will get this value. So, from here I will get $f(x_i + h) - f(x_i - h) = 2hf'(x_i) + 2\frac{h^3}{3!}f'''(x_i) + \ldots$. Now again, this will cancel out and this will be 2 times for this one and so on. So, from here I can write this as f xi plus h minus f xi minus h, and now I take the 2h common from here. So, that will be

$$\frac{f(x_i + h) - f(x_i - h)}{2h} = f'(x_i) + \frac{h^2}{3!}f'''(x_i) + \ldots$$

, all other terms.

So, next term will contain h cube h4 and all the terms. Now from here the same way I can define all the terms after this, these terms are, as h is very very small. So, I can ignore all the terms. So, from here, I can write that my f'(xi), thus the first derivative and the point xi can be written as

$$f'(x_i) = \frac{f_{i+1} - f_{i-1}}{2h} + \frac{h^2}{6} f'''(\xi), \quad x_{i-1} \le \xi \le x_{i+1}$$

. So, from here I will get the central difference operator for the first derivative.

Now from here, you can see that in this case, my error is of order h square. So, from here I can say that in this case, my error is of order h square, order of h square. So, when h is small, so in that case the error is very small. So, I can say that the central difference method will be better for finding out the first derivative for the given data. So, this is a we can derive the first derivative, even we can derive the second derivative also. So, this is the 3 first derivatives we have defined.

Now I have used the Taylor series methods to find the first derivative at the value xi using the forward operator, using the backward and in the center. So now the question comes that can we use other points also, neighbourhood points to find out the approximation for the first order derivative.

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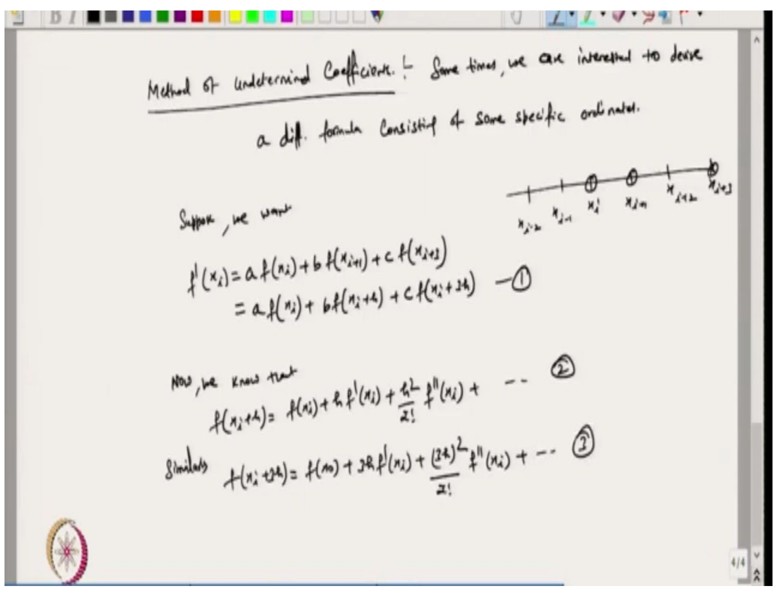

So, for that one, we have a method that is called undetermined coefficient. So in this case, this is a method of undetermined coefficients. So, sometimes we are interested to devise a derivative or differential formula consisting of some specific coordinates. Like suppose I have my data given to me and suppose, this is my xi this is xi+1, xi+2, xi+3, xi-1, xi-2. So now, suppose we want to use the values corresponding to suppose this value and this value and I want to use this value.

So, I want to devise a formula for the first order derivative which contains only these values means value of y at xi, value of y at xi+1 and value y at xi+3, and I do not want to use the value of this one, this one and this one. So basically, I want to find the value of

$$f'(x_i) = af(x_i) + bf(x_{i+1}) + cf(x_{i+2})$$
$$= af(x_i) + bf(x_{i+h}) + cf(x_i + 2h)$$

. So, let us take it 1.

Now, I want to find the value of this constant that is called the undetermined coefficients. So, this constant I want to find out and then I will be able to get the value of the first order derivative for the given nodal points. Now we know that f(xi+h) can be written as using Taylor series. So let us take it 2. Similarly, this can be written as 3h f'(xi).

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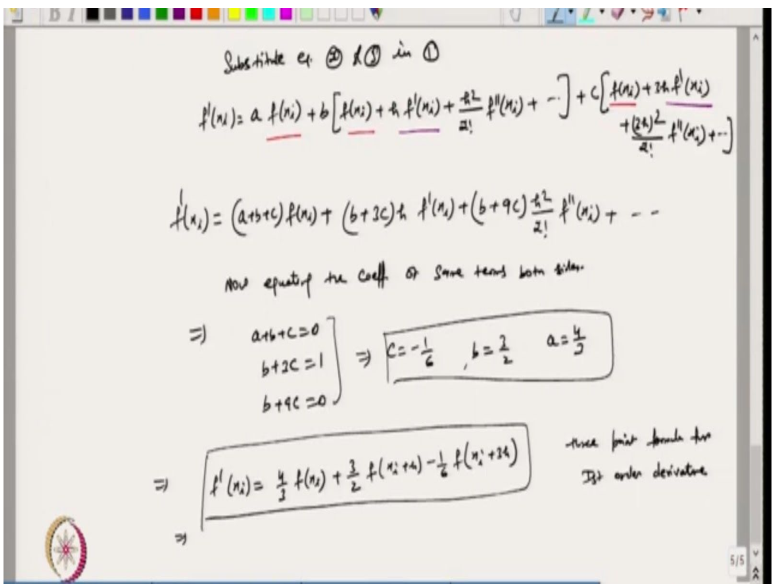

This one, now substitute equation 2 and 3 in 1. So, I can write from here, if a f(xi)+b and I can write this way plus c/2!f''(xi) and so on. So now, this is the expression, final expression we get. Now from here, I can collect the terms corresponding to f(xi). Then I can collect the terms corresponding to f'(xi) and so on. Now from here I can write that, I can write this function as f xi is equal to so I am collecting the terms corresponding to f(xi). So, that will come from all. So this willbe

$$f'(x_i) = (a+b+c)f(x_i) + (b+3c)hf'(x_i) + (b+9c)\frac{h^2}{2!}f''(x_i) + \ldots$$

.

Now, equating the coefficients of the same terms on both sides. So, from here I can say that I have the left hand side and that contains only f' and the right hand side this one and this is true for all xi. So from here, I can say that a+b+c=0, and b+3c=1,and b+9c=0, corresponding to the terms this h square f''. So, then the coefficient will be 0. This is the coefficient that will be 1.

So I need the 3 equations, so from these 3 equations, if you solve, then from these 2 equations I will subtract, so I will get c=-1/6. If I put c=-⅙ here, I will get my b=3/2. So this will be 3/2 and then my a=-b+c=4/3. So, from here, I am able to find this value of a, b and c. So, that gives me that my

$$f'(x_i) = \frac{4}{3} f(x_i) + \frac{3}{2} f(x_i + h) - \frac{1}{6} f(x_i + 3h).$$

So, that is the expression for the first order derivative that contains 3 points. That is f at xi, xi+h and xi+3h. So, this is also called a 3 point formula for first order derivatives. So, in this case I can define the same way we can define the first order derivative using the different points.

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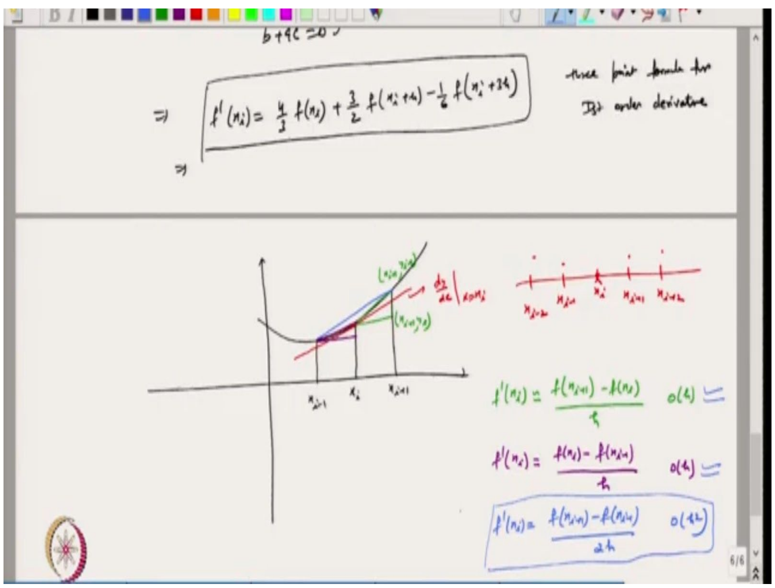

Somebody maybe somebody is looking for the point so maybe it happened that somebody is interested in finding the derivative at xi using this point, this point, this point and this point. So, that is the 5 points we have. So, that also we can find out or maybe some other terms. So, with the help of Taylor series, we can find out any value of the derivative of a function at any point xi using the neighbourhood points that are given to us as a terms of data. So, that we can do with the help of the method of undetermined coefficients. Now if you want to see how this different

different type of expression for the derivatives are helpful. So, this one I can represent with the help of the figure.

So, suppose I have this function. So, this function is given to me like this one and the value of the function is given to me at xi, next value is given to me as this at xi+1, next is previous one was given to me at xi-1, suppose I take this value. Now if you see at this point, the derivative of this, so this is my derivative, the real derivative. So, this is my $\left.\frac{dy}{dx}\right|_{x=x_i}$. Now what I am doing. First I am approximating this derivative. Suppose I am doing the forward operator, so this value and this value I am taking. So, I take this value and this value. So, that is the triangle we are going to have.

So, in this case at this point, I have xi+1 at yi+1, this is a point we have xi+1 and this is yi. So, if I take this value then I will represent f'(xi), I am going to represent the value of the function as $f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$. So, if you see this one that is the basically tangent of this angle, this is divided by this. So, this is of order h. Now the same if I am doing this one with the backward formula, so I am taking this value and this value. So from here, you can see that considering this value, now using the backward, I am having

$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$ and that is of order h also. So, that is the, if you see this is a triangle. So, f(xi)-f(xi-1) and divide by this value.

Now if I take the center scheme, so in the center scheme if I want to so I have this point and this point so I am taking the line connecting this point and from here, I will get

$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$ and this is of order h square. So, you can see that this chord seems parallel to this tangent. So, that is why we can say that this center scheme or the central finite difference is order of h square and is going to give you a more accurate value as compared to the forward finite difference or the backward finite difference. So, that is the way we can approximate the derivative of the function at any nodal value xi with the help of forward, backward or the central. So, this is a way we can have the approximation.

So, let me stop here today. So, today we have discussed how we can define the forward operator, the backward operator or the central difference operator for the first order derivative, and then we also discussed that how we can use the method of undetermined coefficient to derive a formula for the first order derivative using the other nodal points, the neighbourhood nodal points. So, we will continue with this one in the next lecture. So, thanks for watching. Thanks very much.