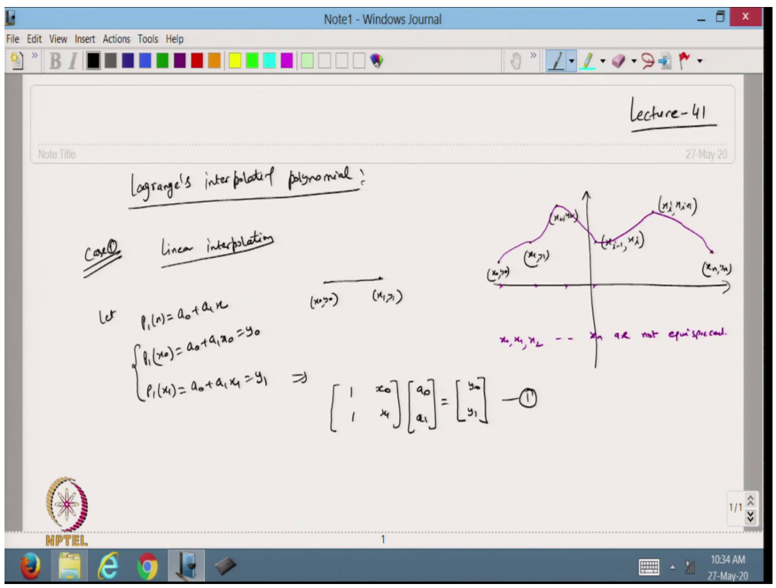
Scientific Computing Using Matlab
Professor Vivek Aggarwal & Professor. Mani Mehra
Department of Mathematics
Indian Institute of Technology, Delhi
Lecture No. 41
In Continuation of Lagrange's Interpolation Formula

Hello viewers, welcome back to the course on scientific computing using Matlab. So, in the previous lecture we have discussed that if we have a non uniform distributed nodal point, then how we can approximate unique interpolating polynomials based on those nodal values. So, today we will continue with this one and we will discuss the Lagrange interpolation polynomials.

(Refer Slide Time: 0:46)



So, today I am going to use that to find Lagrange's interpolating polynomial. So, I know that suppose I have some data points that are given to me. So, suppose this is my $(x_0, y_0), (x_1, y_1), \cdots, (x_{i-1}, y_{i-1}), (x_i, y_i), \cdots, (x_n, y_n)$, this is n+1 data points are given to me, and then based on this one. So, this point, if you see this is given here, this is here, then this is here, then this is here.

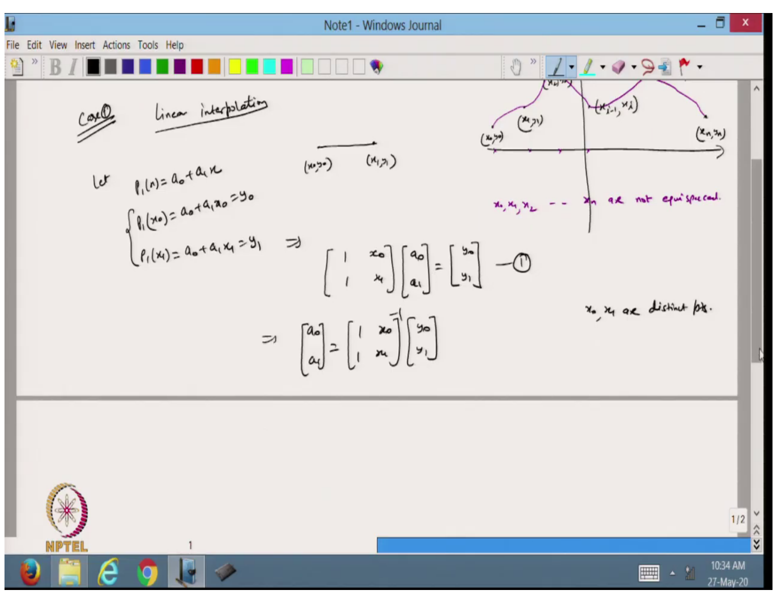
So, these nodal points $x_0, x_1, \cdots, x_n$ are not equi-spaced. So, in this case I want to find the interpreting polynomial that is passing through all these points. So, it is passing through all these points and that is unique interpolation polynomials I want to find.

So, let us start with the expression. So, I will take case 1. So, in case 1, I will talk about first of all linear interpolation. So, in this case I have only two points that are given to me, so this is one point (x_0, y_0) and this is another point that is (x_1, y_1). So, this is my interpolating polynomial.

So, I define the let, I define the $P_1(x)$, that is interpolating polynomial as $P_1(x) = a_0 + a_1 x$. Now, this is interpolating polynomial, so it should pass through the points. So, if I put x_0, so $P_1(x_0) = a_0 + a_1 x_0 = y_0$ and then I define $P_1(x_1)$. So, that is $P_1(x_1) = a_0 + a_1 x_1 = y_1$. So, this is the corresponding two equations I get from here and this can be written as a system of equations.

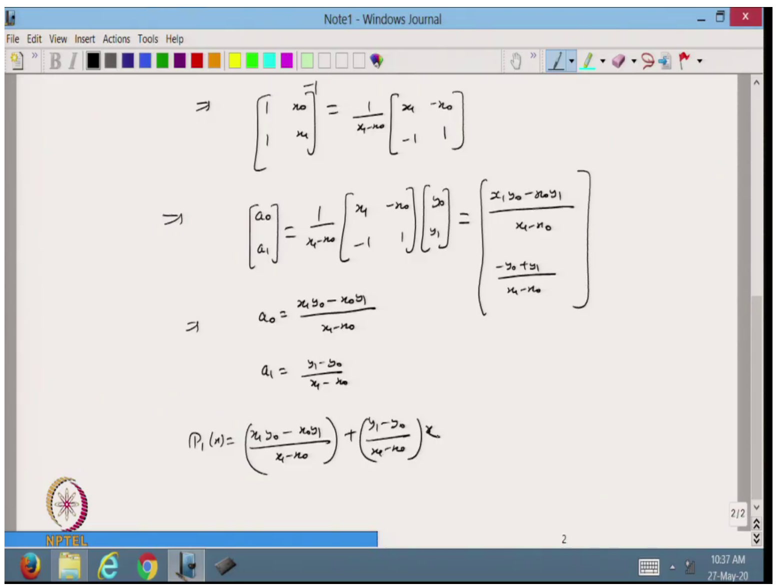
So, this expression I can write in the form of a matrix form. So, this is $\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$ So, this is a system I want to solve for linear interpolation.

(Refer Slide Time: 4:11)



So, in this case I know that this is my matrix a, so, and this the determinant of this matrix is $x_1 - x_0$. So, I know that x_0 and x_1 are distinct points. So, it means that this matrix is a non singular matrix. Now, I can solve this one with the help of an inverse of the given matrix and so, this can be written as $\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix}^{-1} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$

(Refer Slide Time: 4:49)



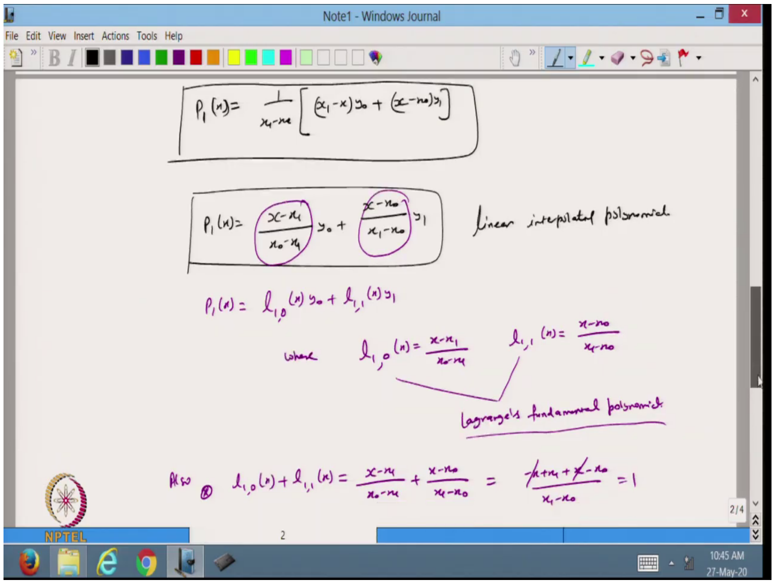
Now, from here I know that the inverse of this matrix is equivalent to,

$$\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix}^{-1} = \frac{1}{x_1 - x_0} \begin{pmatrix} x_1 & -x_0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{x_1 y_0 - x_0 y_1}{x_1 - x_0} \\ \frac{-y_0 + y_1}{x_1 - x_0} \end{pmatrix}$$

So, from here I can get the value of a_0, so that is $a_0 = \dfrac{x_1 y_0 - x_0 y_1}{x_1 - x_0}$ and $a_1 = \dfrac{y_1 - y_0}{x_1 - x_0}$. So, this is the value I am getting. Now, if I substitute this value in the given expression, then my linear polynomial $P_1(x)$ P1 x can be written as

$$P_1(x) = \frac{x_1 y_0 - x_0 y_1}{x_1 - x_0} + \frac{y_1 - y_0}{x_1 - x_0} x$$

(Refer Slide Time: 7:08)



Now, if I want to verify that whether it is satisfying the interpolating polynomial properties or not , that at the nodal points, its value should be the same. So, if I put x_0 in $P_1(x)$ so, in this case, this expression actually can be written in this form also

$$P_1(x) = \frac{1}{x_1 - x_0}\left[(x_1 - x)y_0 + (x - x_0)y_1\right]$$ so this can be written like this also.

So, from here I can see very clearly that if I put $x = x_0$, then this will cancel out and this will be x_0x naught and this x, $x_1 - x_0$ will cancel out and get y_0y naught. And, if I put $x = x_1$, in that case this will be 0, this value will be 0 and then from here $x_1 - x_0$ this will cancel it, I will get y_1, so that is the corresponding interpolating polynomial.

So, this interpolating polynomial generally we will write, directly looking at the points we can write this polynomial as $$P_1(x) = \frac{x - x_1}{x_0 - x_1}y_0 + \frac{x - x_0}{x_1 - x_0}y_1$$ So, this expression is the same, only difference is that if I put $x = x_0$ in the coefficient of y_0, then this will cancel out and become 1.

Similarly, I am putting the coefficient of y_1, so, it will be $\frac{x - x_0}{x_1 - x_0}$. So, that is the expression for linear Lagrange's linear interpolation. Now, the coefficient, this coefficient and this coefficient, they have some specific names So, I can write my expression

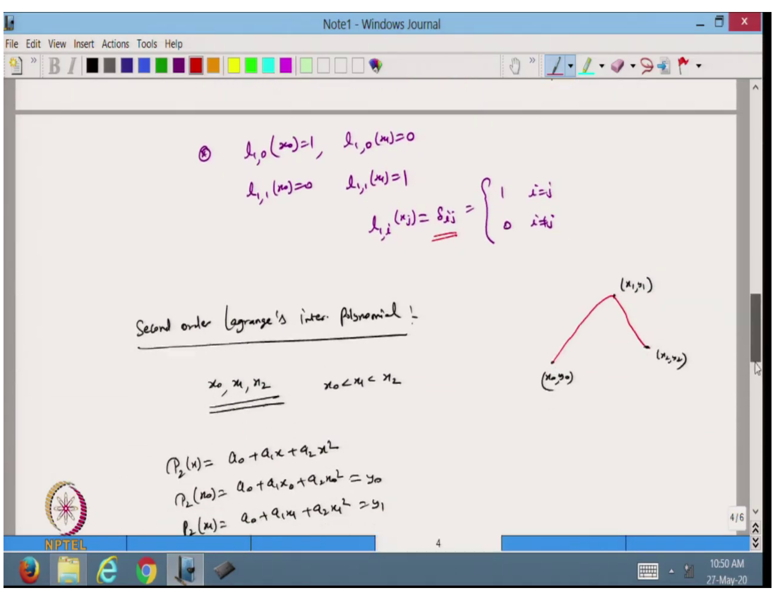
$P_1(x) = l_{1,0}(x)y_0 + l_{1,1}(x)y_1$ where, $l_{1,0}(x) = \frac{x - x_1}{x_0 - x_1}$, and $l_{1,1}(x) = \frac{x - x_0}{x_1 - x_0}$. So, these two factors ($l_{1,0}(x)$,$l_{1,1}(x)$) are called Lagrange's fundamental polynomials.

So, these fundamental polynomials have some properties. So, from here I can write that this fundamental polynomial has some property. So, from here I can write that also, if I put $l_{1,0}(x) + l_{1,1}(x)$, so if I am adding all these together, I will get from here

$$l_{1,0}(x) + l_{1,1}(x) = \frac{x - x_1}{x_0 - x_1} + \frac{x - x_0}{x_1 - x_0} = \frac{-x + x_1 + x - x_0}{x_1 - x_0} = 1.$$

So, from here that the sum of these Lagrangian polynomials that its value is equal to 1 also, so this is the first property.

(Refer Slide Time: 12:01)



The second property is that $l_{1,0}(x_0) = 1, l_{1,0}(x_1) = 0.$ Similarly, $l_{1,1}(x_0) = 0, l_{1,1}(x_1) = 1.$ For any l,

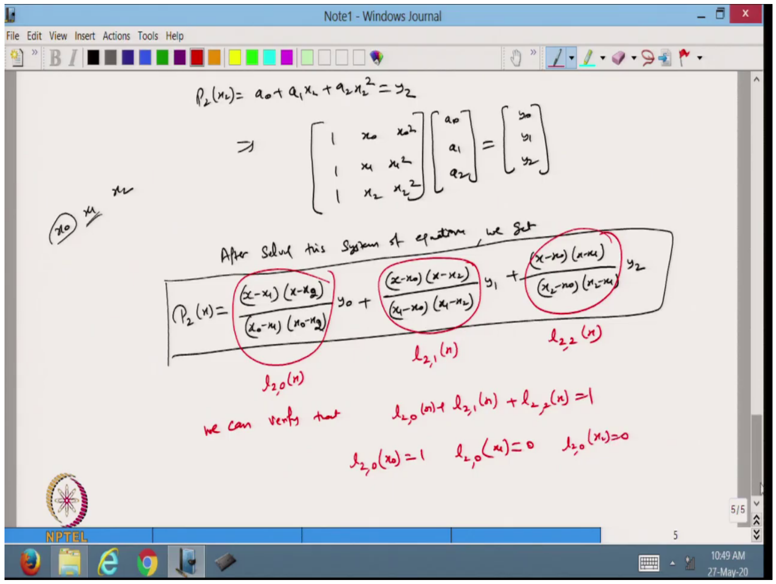
$l_{1,i}(x_j) = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ So, that is the expression for Lagrange's fundamental polynomials.

So, this is the property we have to keep in mind for the Lagrangian fundamental polynomials. So, based on this one, so this is the, we have defined only two points similarly, suppose I have 3 points that are given to me one point is here, one point is here and one point is here. So, this is my $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ so, only 3 points are given to me. So, in this case if I want to find an interpolating polynomial, so this will be like this one.

So, in this case I will define the second order Lagrange's interpolating polynomial. So, in this case we have 3 points. So, I will try to write down the equations, the quadratic polynomial directly, like here from the taking that, here I am defining the point coefficient of y_0, so I am taking the other, all other points except x_0. Here I am taking all the points except x_1. So, now from here that we have the points x_0, x_1, x_2 such that I define that $x_0 < x_1 < x_2$, so we are taking in the ascending order.

Now, from here, I want to write the quadratic polynomial, because this will pass into 3 points, so we can define the quadratic polynomial. So, I will write that $P_2(x)$, so in the form of $P_2(x) = a_0 + a_1 x +_2 x^2$. And then I am defining the properties of this one. So, $P_2(x_0) = a_0 + a_1 x_0 + a_2 x_0^2 = y_0$. Similarly, $P_2(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 = y_1$.

(Refer Slide Time: 15:34)



And, $P_2(x_2) = a_0 + a_1 x_2 + a_2 x_2^2 = y_2$. And now from here, I will write in matrix form
$\begin{pmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$. So, this is the corresponding system of equations we are getting. And now, if we solve this one, then I can write, so after solving the same way we can solve. So, after solving this system of equations we get, now from here I can write my $P_2(x)$, so this should be equal to,

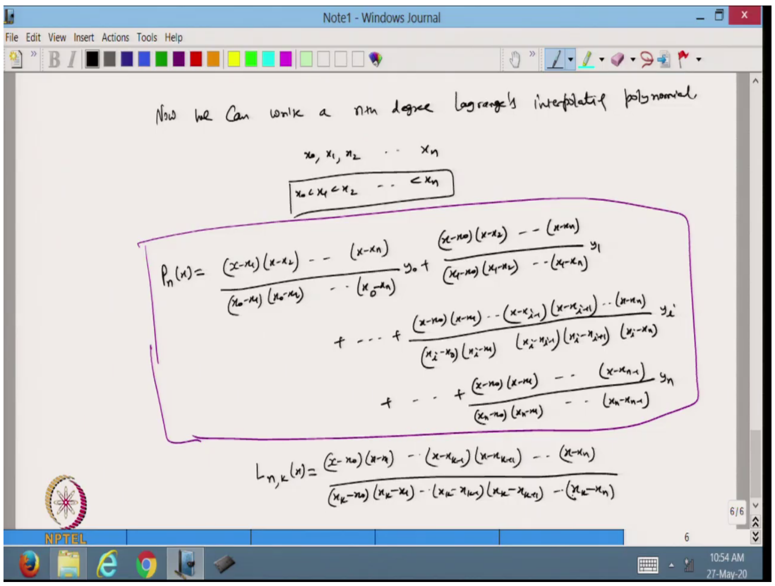
$$P_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

So, that is the quadratic polynomial interpolating polynomial, we can write directly with the help of the previous example, the linear interpolation.

So, this will be the quadratic polynomial interpolating polynomial and from here I can write that this will be $l_{2,0}(x)$. So, that is a fundamental interpolating polynomial, this is $l_{2,1}(x)$ and this expression is $l_{2,2}(x)$. So, from here we can verify that $l_{2,0}(x) + l_{2,1}(x) + l_{2,2}(x) = 1$, if I take the sum of this one, and also from here I can see that $l_{2,0}(x_0) = 1$ $l_{2,0}(x_1) = 0, l_{2,0}(x_2) = 0$.

So, from here, we can say that this is also satisfying the second property of the δ function we have defined. So, based on this one, we can write the quadratic interpolating polynomial which satisfies whose coefficient as a fundamental Lagrangian polynomial satisfies the corresponding conditions for the coefficients. So, from here now, I will try to write down the general Lagrange's interpolation.

(Refer Slide Time: 19:52)



So, now we can write a n^{th} degree Lagrange's interpolating polynomial. So, now we have a $x_0, x_1, \cdots, x_n$. So, these are the nodal values that are given to me. Now, I can rearrange this form in ascending order, because this is non uniformly distributed. So, I can rearrange that one in the, in this ascending order form.

Now, based on this one, I want to define n^{th} order polynomial, so $P_n(x)$ will be

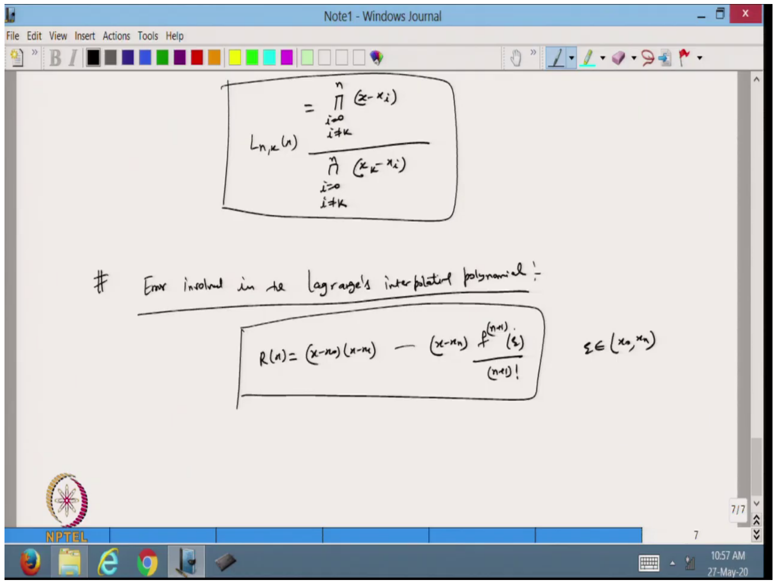
$$P_n(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1 + \cdots$$

$$+\frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}y_i + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n$$

So, based on this one, I am able to write the nth degree Lagrange's interpolating polynomial where I can define my expression nth degree polynomial for any kth coefficient. So, this is

$$l_{n,k}(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)} = \frac{\prod_{i=0, i\neq k}^{n}(x - x_i)}{\prod_{i=0, i\neq k}^{n}(x_k - x_i)}$$

(Refer Slide Time: 23:52)



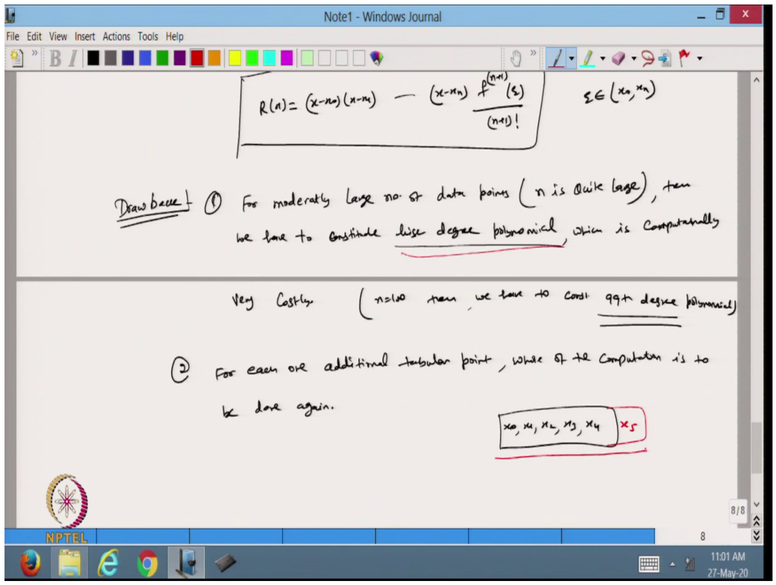
So, this is the way we can define the fundamental Lagrangian interpolating polynomial.

Now we want to define form here, the error involved. So, error is involved in the Lagrange's interpolating polynomial. So, we know that in the previous cases also the error involved I have written the R(x), so R(x) can be written as

$$R(x) = (x - x_0)(x - x_1)\cdots,(x - x_2)\frac{f^{(n+1)}(\xi)}{(n+1)!}, \xi \in (x_0, x_n)$$

So, in this case also, if I know the value of the function, I take its $(n+1)^{th}$ derivative and then there is some ξ exist in between, such that this value is given. So, that is the error we can find and if I take the maximum value of this, then this is the maximum involved in this one. So, this is the way we can find the error involved in the Lagrange's interpolating polynomial. So, after doing all this expression, we want to define the drawback of this one.

(Refer Slide Time: 26:17)



So, what is the drawback, here? The first one is that for a moderately large number of data points that is when n is quite large, then we have to approximate or we have to constitute higher degree polynomials, which is computationally very costly. So, which is computationally very costly. It means that suppose I have n = 100 then, we have to constitute, so n is the number of points we have suppose 100, then we have to constitute 99^{th} degree polynomial, and finding the 99^{th} degree polynomial is quite expensive in the terms of computations.

So, that is the drawback that we have if a large volume of data is given to us, then it is very difficult to write the higher degree polynomials. The second one is that for each one additional tabular value the whole of the computation is to be done again. Like we have the 5 points, as my 5 points are given to me x x_0, x_1, x_2, x_3, x_4. So, these 5 points are given to me and I will based on this one, I will constitute the interpolating polynomial 4^{th} degree polynomial, then somebody asked me that, we add one more point and let us add this one.

So, now we have 6 points, so in this case, I have to constitute the whole process again. So that I am I should be able to write the fifth interpolating polynomial, now. So, if I add just one point, then the whole process we have to repeat. So that is why we add one point, then the same process will be repeated and again the whole computation, we have to do again and again. So, that cost me a lot, so that it can be said that it is computationally costly.

So, that is another drawback of this one. So, the first drawback was that higher degree polynomial and second drawback was this one that adding 1 more point needs the whole process to repeat again. So, let us stop today, here. So, today we have started with the linear Lagrange's interpolating polynomial. And then based on this one we have discussed the quadratic and the nth degree Lagrange interpolating polynomial.

So, in the next class also we will continue with this one. So, thanks for watching this lecture and thanks very much.