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Scientific Computing Using Matlab
Professor. Vivek Aggarwal & Professor. Mani Mehra
Department of Mathematics
Indian Institute of Technology, Delhi
Lecture No. 38
Error Estimates and Polynomial Approximation

Hello viewers, welcome back to the course on Scientific Computing Using MATLAB. So, today we will discuss the error estimates in the polynomial approximation.

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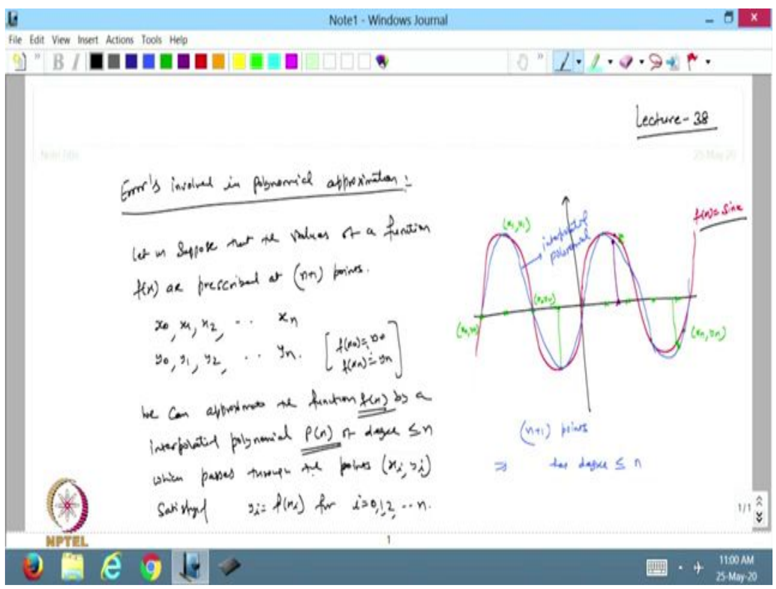

So, today's topic is errors involved in polynomial approximation. As we know that suppose this is my function, let us say I plot this function, so I take the sin function. So, let us suppose my $f(x) = \sin(x)$. Now, suppose the value, the function $\sin(x)$ is not given to me, but the value of the function at some points, that is given to me. So, I choose this point, then this point, then this point. So, I am taking the equispaced grid points, so at this point, this is the value of the function.

So, this is the value of the function, at this point the value of the function is 0, then I have this value, then this value, this value is given to me, this value is given to me, then this value is given to me, so like this value is given to me. So, I have these points, so that I call it (x_0, y_0), this is (x_1, y_1), this is I call it (x_2, y_2). And the same way I can go in the last one. So, suppose this is my (x_n, y_n). So, these points are given to me.

Now, in this case what I will do I will take the polynomial interpolation. So, the polynomial interpolation will be what, which is passing through all these points. So, I will start with from here suppose, it is passing through here, then the next point is here, so this is the point, the next point is here, it is going from here, then the next point is here, then the next point is here passing through this point, then passing through this point, then passing through this point and so on.

So, this is my interpolating polynomial. And I know that if it is points, so I have k+1 points implies that the interpolating polynomial has degree less than or equal to k, so this is not k, I should take it n+1 points and its degree would be less than or equal to n. So, now what is happening is that this is my interpolating polynomial and this red one is the actual function. Now, somebody asked me what is the value of the for this value of x?

So, if I take this value of x, this is my value I am going to get when I put this point in the interpolating polynomial. But the actual value for this value of x was this one. So, from here I can say that this is the error involved in this case. Similarly, I can choose different different points and from there I can find suppose I want to choose this point or this point. So, in that case, suppose I take this point from here, so, in that case you can see that, this is my actual value and this is my approximate value, so that this error is involved.

So, I want to find out what will happen about this error when I approximate the interpolating polynomial passing through all these points. So, this is what we want to discuss. So, let us write down that let us suppose the values of a function. So, in this case, this is my f(x), which I have chosen as a $\sin x$, are prescribed at n+1 points. So, that is given to me $x_0, x_1, \dots, x_n$, and the value of the function given to me at this point is $y_0, y_1, \dots, y_n$. So, basically from here I can say that f at x_0 is y_0, f at x_n is y_n, so this is given to me.

Then we can approximate the function. We can approximate the function f(x) by an interpolating polynomial, so I call it P(x) of degree less than equal to n. Because maximum degree is n, so it is degree less than equal to n which passes through the points (x_i, y_i) satisfying $y_i = f(x_i)$. So, this is for all $i = 0, 1, 2, \cdots, n$. So, we can approximate the function f(x) by interpolating polynomial P(x), which degree is less than or equal to n, which passes through the points satisfying this relation.

And you also from here you can see that at this point the nodal points whatever it is given to me, so at this point this function and the integrating polynomial are coinciding, only problem is to approximate the values in between these points. So, from here what I do is that, I define the function.

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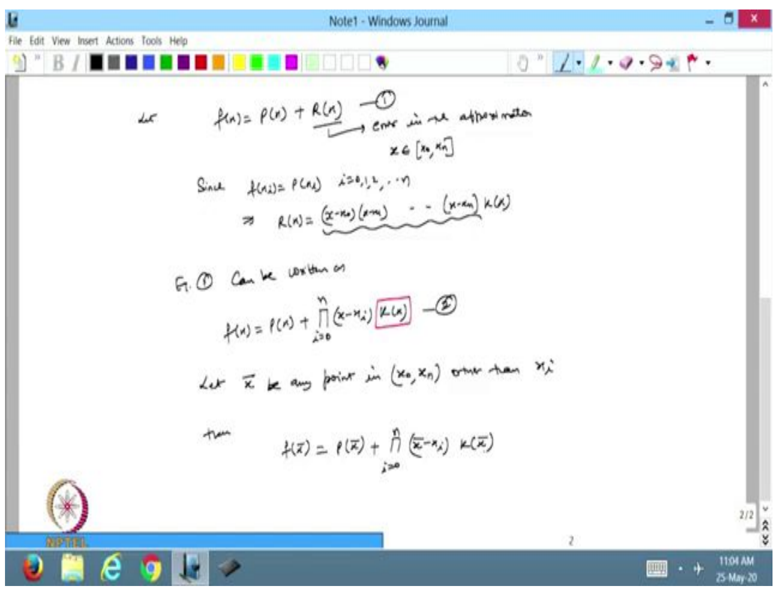

So, let we can write the function f(x) = P(x)+R(x) ….(1), where R(x) is the error in the approximation. And what is the x is, so x belongs to from x_0 to x_n. So, anywhere between the value x_0 and x_n. Now, we know that since $f(x_i) = P(x_i)$ that is already given to me.

From here I can say that this implies that I can choose my
$R(x) = (x - x_0)(x - x_1) \cdots (x - x_n)k(x)$.

So, because at this point when $x = x_i$, for all $i = 0, 1, 2, \cdots, n,$ this is the same value. So, whenever this is the same value the error will be 0 so that is why I choose the R(x) in this way. So, whenever I put any of the nodal points x_0, x_1, or x_n this the error term will be 0. So, now from here I know that the k(x) is the function I have chosen. So, from here my equation number 1 can be written as. So, equation 1 can be written as

$$f(x) = P(x) + \prod_{i=0}^{n}(x - x_i)k(x) \cdots (2)$$

Now, from here, now my what I want to find, I want to find what is this k(x). So, this k(x) I want to find. So, what I do is that I choose a point $\bar{x}$ be any point in the interval (x_0, x_1) other than x_i. So, I want to find any point $\bar{x}$, which is other than this x_i because at x_i this is the same, so I want to choose another point.

Then from here I can write that $$f(\bar{x}) = P(\bar{x}) + \prod_{i=0}^{n}(\bar{x} - x_i)k(\bar{x})$$. So, this $\bar{x}$ is any point I have chosen, so it can be any value. So, let us take this $\bar{x}$ is here. So, I will choose this value. So, suppose I take this, this is my $\bar{x}$.

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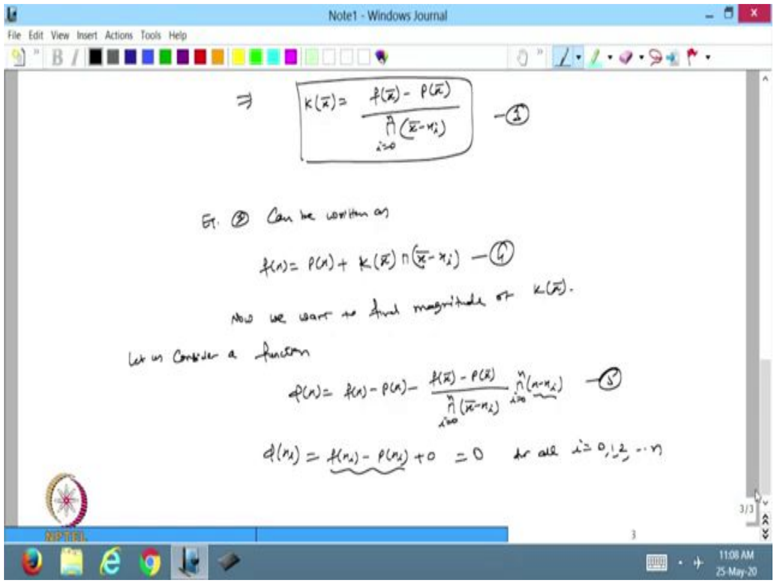

Now, from here, I can write this equation, from here I will get my $k(\bar{x})$ and this is given by $k(\bar{x}) = \dfrac{f(\bar{x}) - P(\bar{x})}{\prod_{i=0}^{n}(\bar{x} - x_i)} \cdots (3)$. So, equation 2 can be written as

$$f(x) = P(x) + k(\bar{x}) \prod_{i=0}^{n} (\bar{x} - x_i) \cdots (4)$$

, now we want to find the magnitude of $k(\bar{x})$, so this is what I want to find. So, let us consider a function, so I call it $\Phi(x)$.

So, $\Phi(x)$ will represent by

$$\Phi(x) = f(x) - P(x) - \frac{f(\bar{x} - P(\bar{x})}{\prod_{i=0}^{n}(\bar{x} - x_i)} \prod_{i=0}^{n} (x - x_i) \cdots (5)$$

Now, from here you can see that Φ at x_i, so if putting x_i I am putting So, this will be 0 and then from here this will again, this so from here you can see that if I put x_i it will be $f(x_i) - P(x_i)$, then this part will be 0 because they are the same and this part will be 0 because here we put $x = x_i$, so this is also 0, so 0+0.

So, from here I can say that this $\Phi(x_i) = 0$ for all $i = 0, 1, 2, \cdots, n.$. So, this is there.

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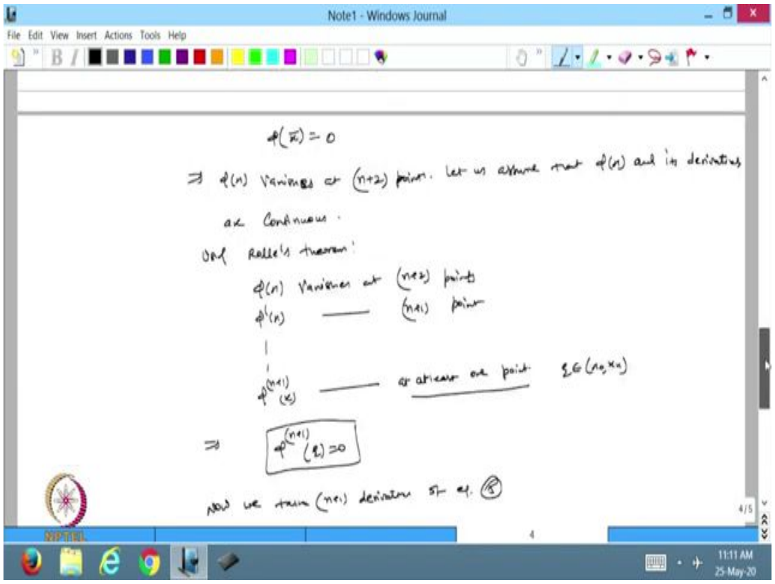

Now, from here also I can write $\Phi(\bar{x})$. So, this minus this and this part will be cancelled out so that this part minor this part will be 0. So, from here I can say that $\Phi(\bar{x}) = 0$. So, from here I can say that my $\Phi(x)$ is a function that vanishes at n+2 points, because n +1 points are this and one point is $\bar{x}$. So, apply, vanish at n+2 points, let us assume that that $\Phi(x)$ and its derivatives are continuous.

So, I can apply the using Rolle's Theorem, $\Phi(x)$ vanishes at n+2 points, $\Phi'(x)$ that vanishes at n+1 points, I am taking the derivative of this one, and so on. So, from here I take $\Phi^{n+1}(x)$. So this is the n+1 derivative that vanishes at one point or at least one point. So, let us call it $\xi \in (x_0, x_n)$. So, from here I can say that $\Phi^{n+1}(\xi) = 0$ Now, we have the equation number 5.

Now, we take n p+1 derivatives of equation number 5. So, when I take the n+1 derivative of equation number 5, I know that this polynomial is of degree n so, if I take the n+1 derivative, then its value would be 0.

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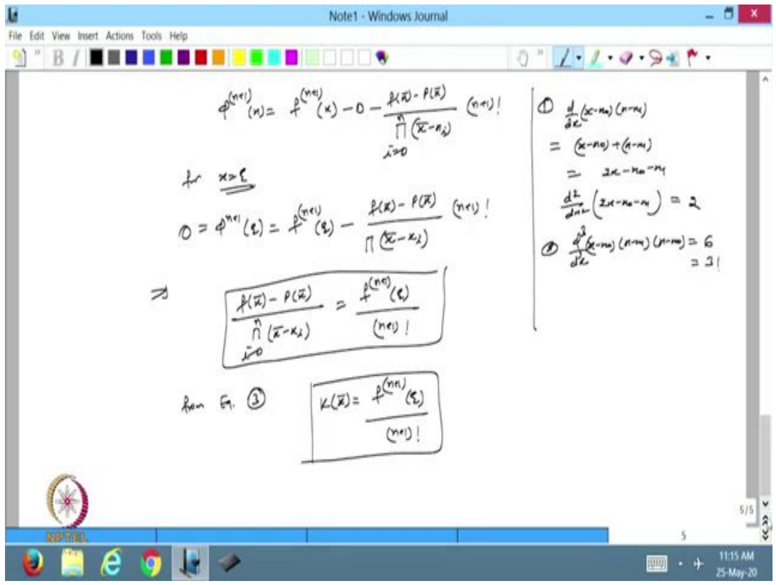

So, from here I can say that this will be $\Phi^{n+1}(x) = f^{n+1}(x) - 0..$.. And from here, you can see that this is just the constant value, but here it is a factor. So, if I take n+1 time derivative of this 1, so I can, I will get $(n+1)!$ from this. So that you can verify from taking 2 factors. So suppose I have a factor like this one, $(x - x_0)(x - x_1)$ taking the derivative. So I just take the derivative, I will get it, I will apply the product rule.

So this $(x - x_0) + (x - x_1)$. So here, I can get that this will be called to $2x - x_0 - x_1$. So this is the second quadratic polynomial, I am taking the first derivative, if I take the second derivative of this, so that will be $2x - x_0 - x_1$. So that would be 2, Similarly, if I take 3 factors, like $(x - x_0)(x - x_1)(x - x_2)$ and taking the derivative. So, this is the quadratic, this is the cubic.

So, in this case, if I take the cubic derivative of this, then it would be 6 and this will be equal to

$3!$. So, I am taking n+1 times derivatives of these factors. So, from here I will get,

$$\Phi^{n+1}(x) = f^{n+1}(x) - 0 - \frac{f(\bar{x}) - P(\bar{x})}{\prod_{i=0}^{n}(\bar{x} - x_i)}(n+1)!$$

For $x = \xi$, $$\Phi^{n+1}(\xi) = f^{n+1}(\xi) - \frac{f(\bar{x}) - P(\bar{x})}{\prod_{i=0}^{n}(\bar{x} - x_i)}(n+1)! = 0$$ From equation (5). $$\frac{f(\bar{x}) - P(\bar{x})}{\prod_{i=0}^{n}(\bar{x} - x_i)} = \frac{f^{n+1}(\xi)}{(n+1)!}$$ So from equation (3) my

$$k(\bar{x}) = \frac{f^{n+1}(\xi)}{(n+1)!}$$

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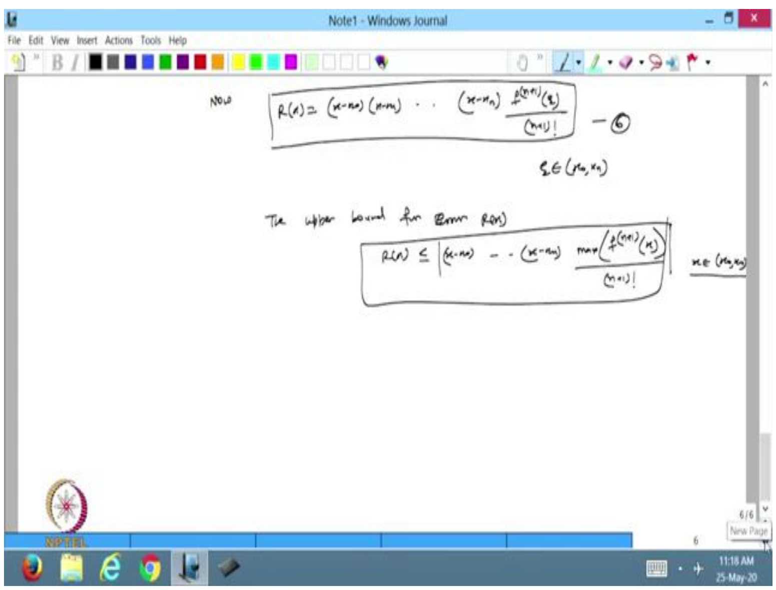

So, from equation 4, I will get that my or I can say from here that, because my error is defined now. So, this is my R(x), now from, so using this now the R(x) can be defined as

$$R(x) = (x - x_0)(x - x_1)\cdots,(x - x_n)\frac{f^{n+1}(\xi)}{(n+1)!}\cdots(6)$$, where this $\xi \in (x_0, x_n)$. So, any value between x_0x 0 to x_n, so this is my error. So in that case, you can see that, first of all I should know the value of the function and its n+1 derivative.

So, only then I can get this one. So, from here I can find the upper bound also the upper bound for error R(x) will be, so in this case my R(x) I can say that this will be less than or equal to, if I choose this value $(x - x_0)$ and if I choose this one the maximum of this. So, if I take maximum of all values, x belongs to x_0 to x. So, what I do I take any x which is the maximum of this value, so, all the maximum I choose. So, this R(x) will be always less than or equal to this.

So, that way we can define the maximum, the upper bound for the error. It can be negative also, in that case I will take the modulus value. So, no problem I can choose the modulus value or I can from here itself I can say that this will always be less than x, the modulus value. So, that it gives me the upper bound for that error.

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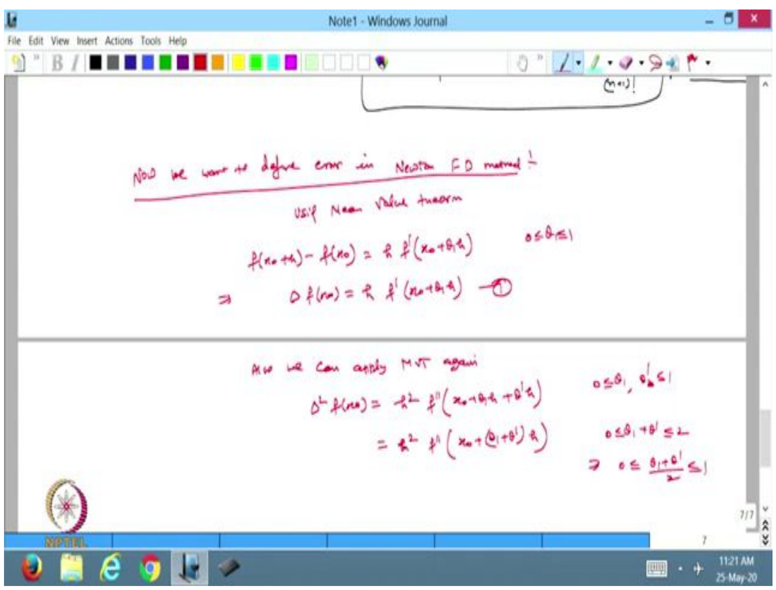

Now, I want to define the, now we want to define error in Newton Forward Difference method. As we have done in the previous lecture, we have discussed the Newton Forward Method, so I want to define what will be the error in that case. So, the error in that case I can or the same thing will be there only thing is that I could change my function fn plus 1 with the finite difference operator. So, I will apply for this one.

So, using Mean Value Theorem, I can write $f(x_0+h)-f(x_0)=hf'(x_0+\theta_1 h), 0 \le \theta \le 1.$. So, that is I can have from the Mean Value Theorem. So, from here I can write this as a $\Delta f(x_0)=hf'(x_0+\theta_1 h), 0 \le \theta \le 1.$.So, this is the function I have defined. Now, I can apply again.

So, also we can apply the Mean Value Theorem again and in that case I will, I can write that $\Delta^2 f(x_0)=h^2 f''(x_0+\theta_1 h+\theta' h), 0 \le \theta, \theta' \le 1.$

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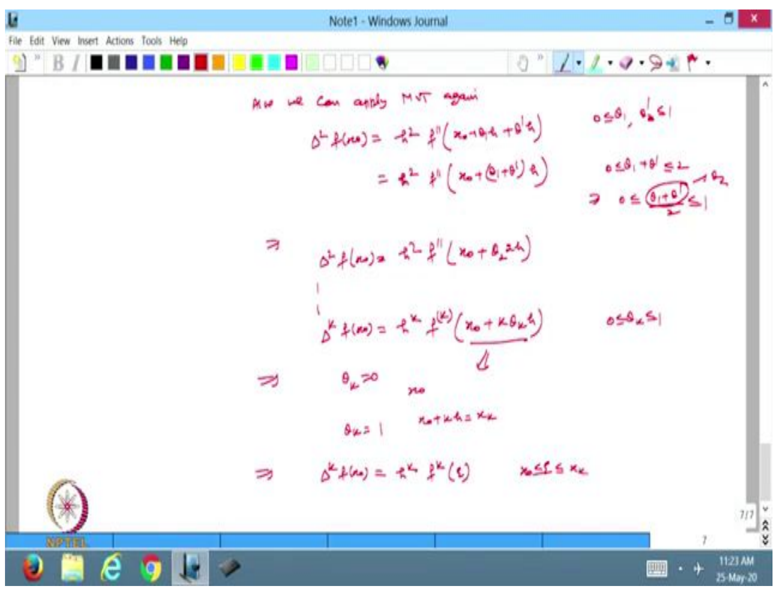

So, from here I can write my second order finite difference, forward finite difference operator as $\Delta^2 f(x_0)=h^2 f''(x_0+\theta_2 2h)$. Where $\theta_2=\theta_1+\theta'$ So, this way we can define, the similar way I can define k. So, let us take, kth forward finite difference operator. So, it can be written as $\Delta^k f(x_0)=h^k f^k(x_0+k\theta_k h)$.

So, using this one from here, I can say that. Now, if I take, so this θ_k I know that lies between 0 and 1. When θ_k is 0, we will get this value will be x_0 naught when θ_k is 1, in that case this value will be $x_0 + kh$. So, this will be k times I am forwarding, so this is the k+1. So, that is I can call it x_k. So, now from here I can say that the k times forward difference operator can be written as $\Delta^k f(x_0) = h^k f^k(\xi), x_0 \leq \xi \leq x_k$.

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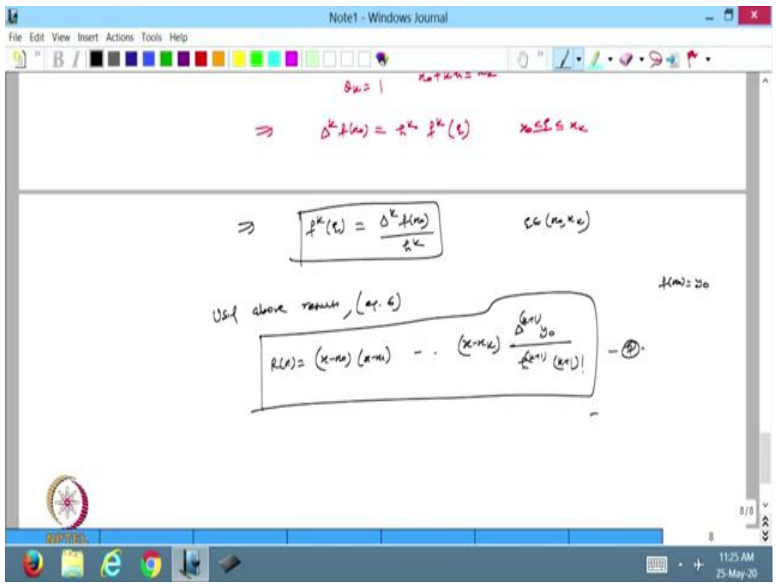

So, now we are able to define this one. So, from here I can write my $f^k(\xi) = \dfrac{\Delta^k f(x_0)}{h^k}, \xi \in (x_0, x_k)$ Now from the previous results, so this is my equation number 6. So from the above results, using above results equation 6, I can write my error function

$$R(x) = (x - x_0)(x - x_1)\cdots(x - x_k)\frac{\Delta^{k+1} y_0}{h^{k+1}(k+1)!} \cdots (7)$$

Now, from here I can see that the error involved in the Newton Forward Difference operator is this one. So, that is the equation number 7 I can define, and that is the error involved in the finite difference, Newton forward finite difference operator.

So, now we should stop here. So, today we have discussed the errors involved in the interpolation when we have n+1 number of points in the given, in the data and we want to interpolate the interpolating polynomials. So, in the next lecture, we will continue with this one. So, thanks for watching. Thanks very much.