Scientific Computing Using Matlab
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Lecture 37
Interpolating Polynomial Using Newton's Forward Difference Formula

Hello viewers welcome back to the course on Scientific Computing in Matlab. So, today we will use the finite difference table to approximate the value at any x in between the given values. So, let us continue with the finite difference table.

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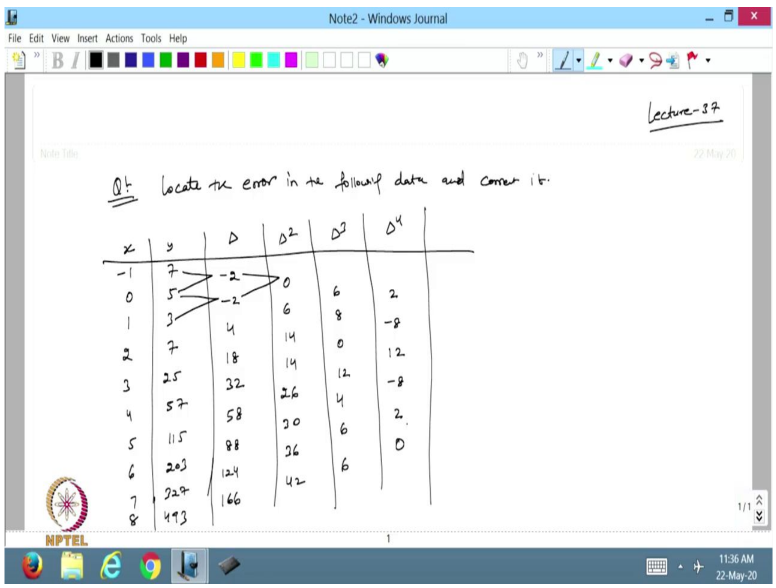

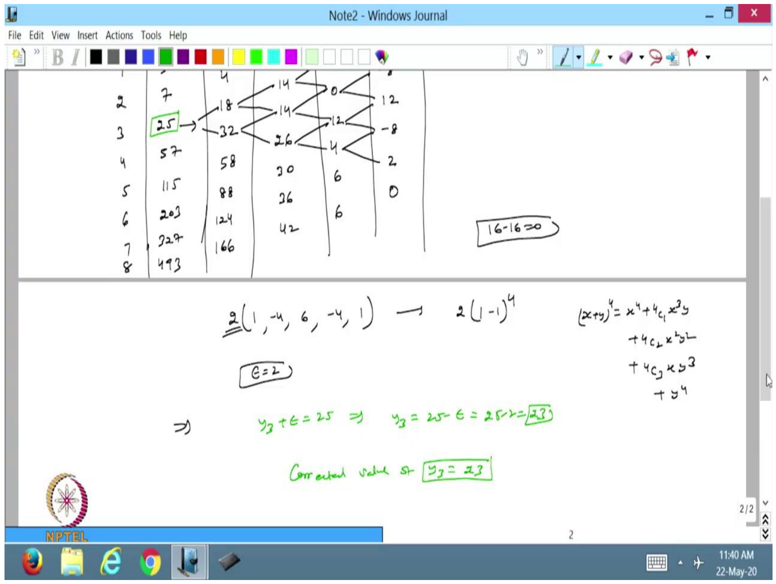

So, let us do one question based on the previous lecture. So, I wanted to locate the error in the following data and cracked it. So, that is a question. So, let us have the value this is the value is given to me x and y. So, x is -1, 0, 1, 2, 3, 4, 5, 6, 7, 8. So, these are a total ten values and the y value is given to me 7, 5, 3, 7, 25, 57, 115, 203, 327 and 493.

So, if you see from this table that it is giving me the expression that these values satisfying a function which is increasing function. So, that only we can say from this looking at the data. So, let us take the first difference. So, the first difference I am taking forward. So, this is 5-7 = -2 , 3-5 = -2, 7 -3 = 4, then 25-7 = 18, 57 -25 = 32 , then it will be 58 then it will be 88, then it will be 124 and this is 166.

So, this value is the first difference then I take the second difference. So, the second difference will be that it will be 0 then 4-(-2) = 6, this is 14 this is again 14 this is 26 and that is 30, that is 36 and this is 42. So, I am writing here this value in between now I am writing this value in between these values. So, that is the way we can write the table. Then I will take the third finite difference. So, this is 6-0 = 6, 14-6 = 8, then it is 0, then it will be 12, then it will be 4 then it will be 6 and then will be 6. Then I take the fourth difference.

So, it is 8-6 = 2, 0-8 = -8, 12 - 0 = 12, 4 -12 =- 8, 6-4 = 2 and this is 0. Now, from here we can see that the pattern looks like the binomial pattern. So, from here, I will just stop here and I will see what will happen in this case I am getting the errors. So, if you see from here, this is 2, 2, -8, -8 and 12 and if I add all this error together, it is 12 +2+ 2 = 16 and this is -8 and -8, so it is 16-16 = 0.

So, that gives me that error in this case because we have seen that the errors in any of the columns is 0, if I take the sum of all the errors and they follow the binomial fashion. So, in this here if I stop from here stop here and then I will see that my error is so I just take the two common then I can write this is 1, -4, 6, -4 and 1.

And from here if I see that I can write that this is the same as $(1-1)^4$. So, if you see from here, the coefficients will be so I can write from here $(x+y)^4$. So, this is $x^4 +^4 C_1 x^3 y +^4 C_2 x^2 y^2 +^4 C_3 x y^3 + y^4$.

Now, if I put this ${}^4C_1 = 4$, this is 6, this is 4, and if I put x = 1 and y = -1, I will get the same coefficients. So, this coefficient will be there. So, based on this one I can see that this is the error and following the binomial pattern. So, from here, I can say that my error $\epsilon = 2$. So, this is the error I have taken.

So, now from here, I can say that now I want to see that also that the error grows symmetrically so one above and one below. So, from here, you can see that if I choose this value 25, it will go this to this then from here it goes this to this value. So, this goes here and then this goes here like this one then it will go this way this way then it will go like so like this way.

So, that is my error. So, if you from here I can see that my error will definitely be in this value and from here I can say that my so it is value is -1 0, 1, 2, 3 so y_3. So, error is in the y_3. So, from here I can say that this is equal to the value is given to me $y_3 + \epsilon$. So, from here I can write that $y_3 + \epsilon = 25$ and from here my corrected $y_3 = 25 - \epsilon$ so it is 25-2 that is 23.

So, this is the corrected value of y_3 is 23. So, based on that the pattern of the errors and the symmetricness of the error we can find that the error is there in the value of 25 and we can correct it and this corrected value is 23. So, here we have used the finite difference table. Now, after doing this one. We will discuss a very important theorem.

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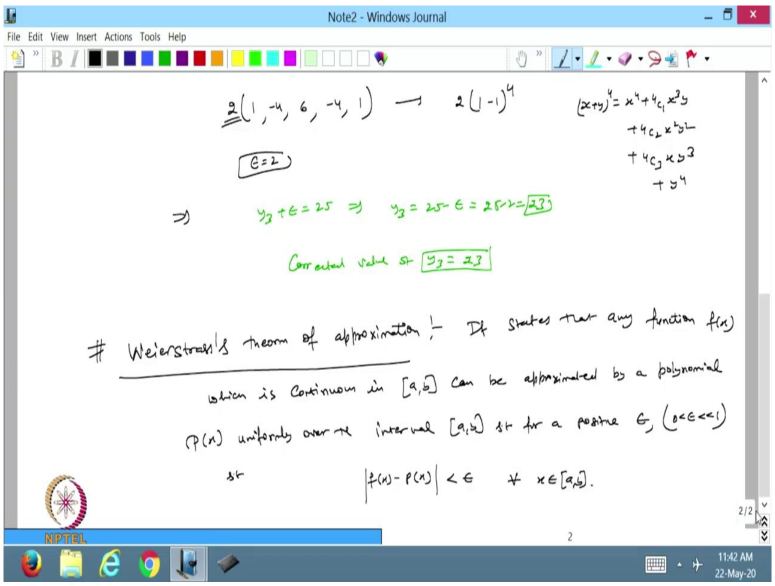

So, this is Weierstrass's theorem of approximation. So, it is states that any function f(x) which is continuous in the interval [a, b] so it is continuous in the interval [a, b] can be approximated by polynomial that is P(x) uniformly over the interval [a, b] such that for a $\epsilon > 0$ where ϵ epsilon is very small such that if I take the difference f(x)-P(x) for any x in the interval that I can make me less than ϵ for all x $x \in [a, b]$.

So, in this case everything depends upon that how small the epsilon I am taking if I choose very very small epsilon then the polynomial will be of higher degree and then this difference can be made less than ϵ based on the value of given ϵ. So, that is the Weierstrass's theorem of approximation. Now, we will use this theorem later on.

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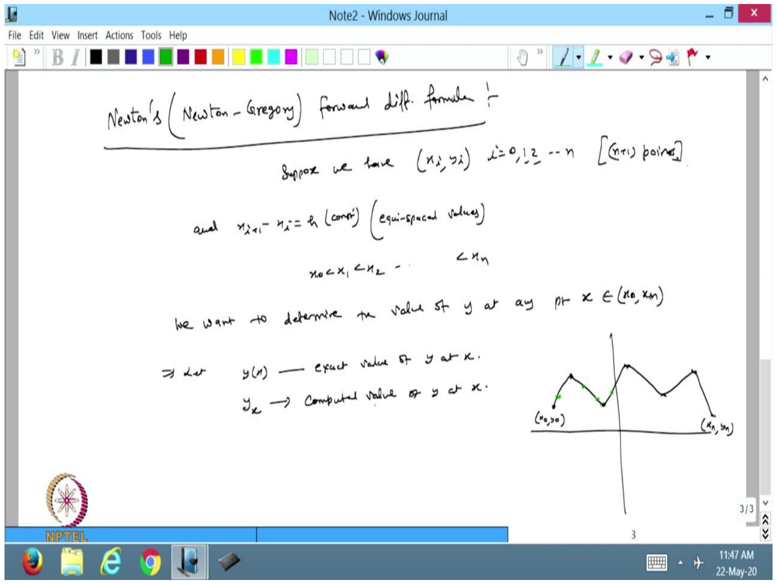

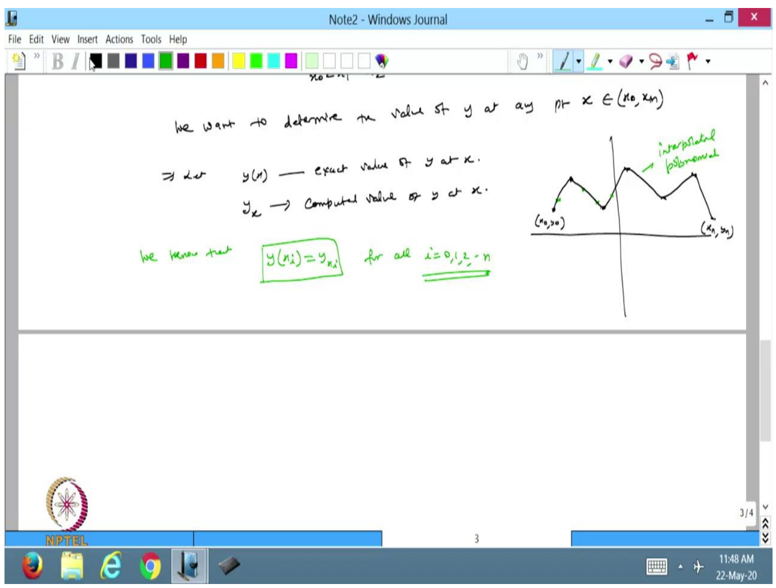

So, now I will write down the interpolating polynomial. So, let us do it Newton's or we call it Newton Gregory forward difference formula. So, suppose we have data and data is given to me in the form of $(x_i, y_i), i = 0, 1, \dots, n.$ So, total n+1 points are given and so suppose we have this data and $x_{i+1} - x_i = h$, where h is constant for each i. It means that equi-spaced data equi-spaced values.

So I can write this x_i as $x_0 < x_1 < x_2 < \cdots < x_n$. Now, we want to determine the value of y at any point $x \in (x_0, x_n)$. So, this is an open interval I am taking because if I take a closed interval the value at x_0 and x_n is already known to us.

So, I am taking the open interval. Now, I want to determine the value of y at any value x in the open interval. So, do what we represent now, so let y(x) gives me that exact value of y at x and y_x, so that is the computed value of y at x. It means that I have some data like. So, some data is given to me. this value is given to me and based on this one I will write a polynomial interpolating polynomial. This is supposed to be my interpolating polynomial.

So, this is my interpolating polynomial and now I want to find the value of y at any x. So, suppose this is my point (x_0, y_0) and this is my (x_n, y_n). So, my value is given to me at this point, this point, this point. Now, I want to find the value of the function y at any x in between somewhere. So, it might be x can be here it can be here it can be here here any value I can want to find and y(x) is the value exact value the function at this point and y_x is the computed value of this one.

So, this is my interpolating polynomial. Now, from here, we also know we know that $y(x_i)$ and the given value there is same as y_{x_i}. So, this value is the same. Because I know that the nodal points my function is also passing from the given data point and my interpolating polynomial also passing through these data points. So, this is true for all $i's$ that is $0, 1, \ldots, n.$. So, this is true that the computed value is the same as the exact value. So, this is one of the things we already know from the interpolating polynomial.

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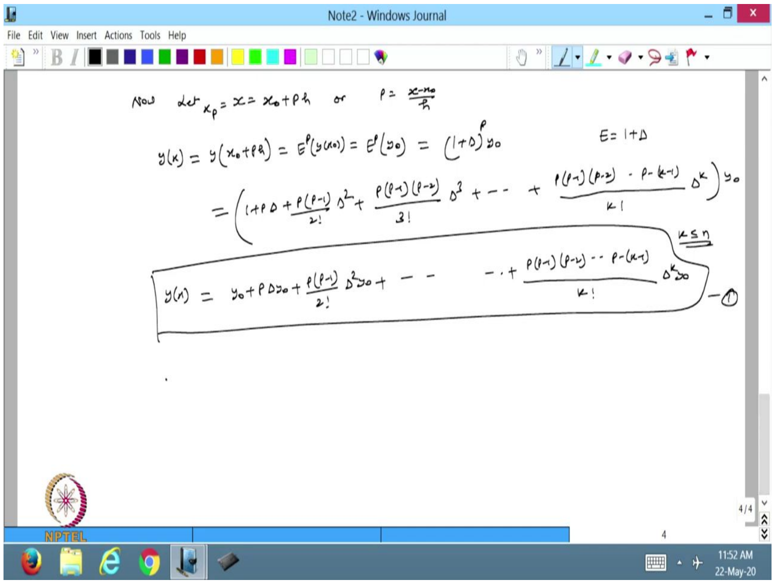

Now, let I want to write $x = x_0 + ph$, I introduce a variable p, h already known to me and x_0 is the initial point of the data and I call it x_p xp or from here I can say that my $p = \dfrac{x - x_0}{h}$. So, this p I am introducing. So, now I want to find the value of the function at x.

So, let us do this one. So, I want to find y is equal to x. So, $y(x) = y(x_0 + ph)$ and this one I can write as a $E^p(y(x_0))$. So, $y(x_0)$ is y_0 y naught so I can write a $E^p(y_0)$. Now, from E^p I know that this can be written as $(1 + \Delta)^p y_0$ because we know that E can be written as $1 + \Delta$.

So, I am putting this value. Now, from here, I am expanding this one using the binomial expansion. So, from here, I can write this as

$$1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots$$

. Now, I know that if I have the n+1 number of points, then the n^{th} finite difference and n^{th} difference operator will be a constant value and the next value will be 0.

So, it will go up to the $\frac{p(p-1)(p-2)\cdots(p-(k-1))}{k!}\Delta^k$. So, this is my polynomial is going up to the k^{th}kth finite difference and this is equal to y_0. So, this k is always less than equal to n. So, definitely this polynomial will be terminating polynomial and you will terminate so let it terminate after the k^{th} finite difference and that k is I know that they will always be less than equal to n.

So, from here I can write now I can take the y_0 apply this operator on the y_0 so it will be

$$y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots + \frac{p(p-1)\cdots(p-(k-1))}{k!}\Delta^k y_0.$$

So, this is my polynomial. I am getting the polynomial in p. Now, the problem is that on the left side I have the y at x and the right hand side I have the all the terms in terms of p so that we can convert into back into the x form. Now from here I can write so this is my 1.

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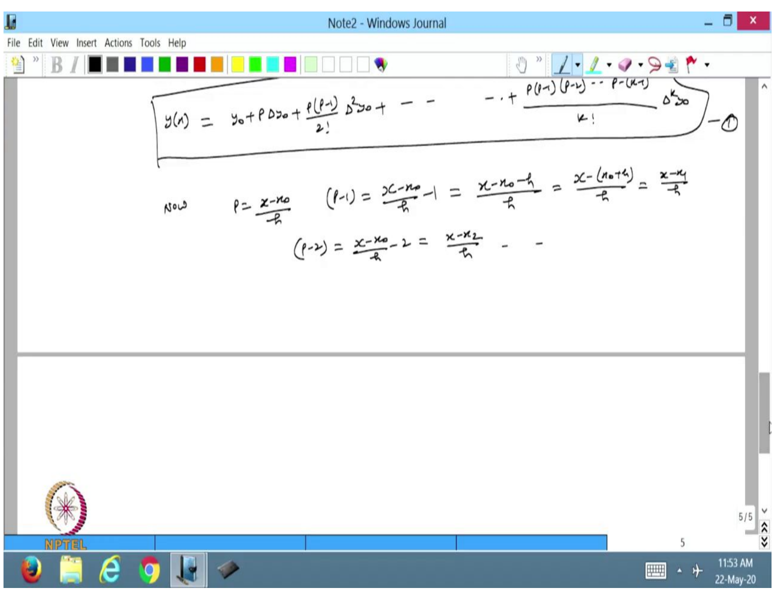

Now, p can be written as $p = \frac{x - x_0}{h}$. So, p-1 can be written as $\frac{x - x_0}{h} - 1$. So, this is

$\frac{x - x_0 - h}{h}$ and that can be written as $\frac{x - (x_0 + h)}{h}$ and this is $\frac{x - x_1}{h}$. Similarly, I can define p-2 so you can see that it will be $\frac{x - x_0}{h} - 2$ and this will be $\frac{x - x_2}{h}$ x so on. So, this way we can write all the terms in the term of p and from here my equation number 1.

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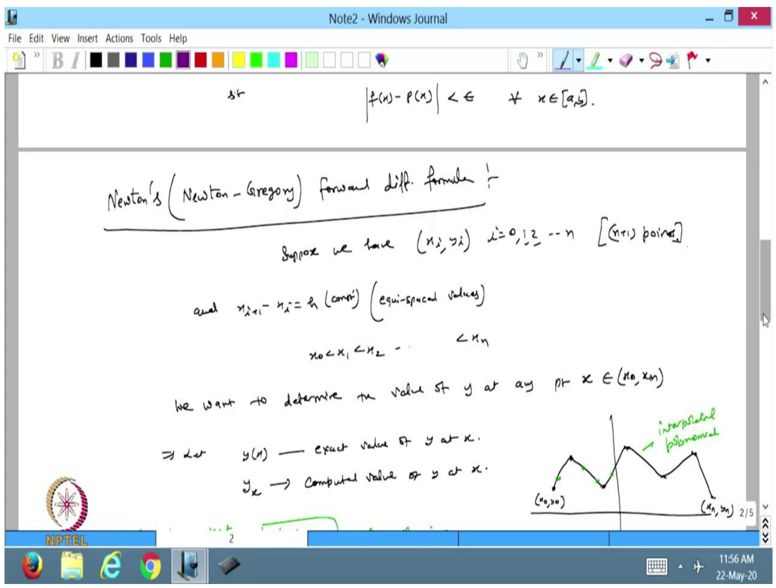

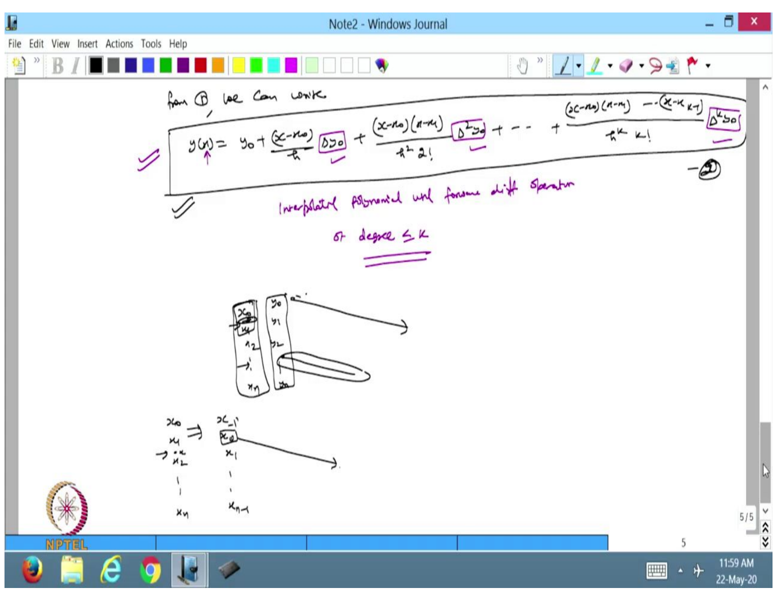

So, from here we can write

$$y(x) = y_0 + \frac{(x-x_0)}{h}\Delta y_0 + \frac{(x-x_0)(x-x_1)}{h^2 2!}\Delta^2 y_0 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})}{h^k k!}\Delta^k y_0$$

This is my equation number 2. So, that is the interpolating polynomial using the forward difference operator Δ. So, that is the value that is the polynomial we are getting. Now, if at onceI am able to find this polynomial then you just put the value of any x here and I can get the interpolating value of that using this equation number 2. So, that is I can write at an interpolating polynomial using forward difference operator.

So, now question is how we can use this one. Now, if you see from here, I need to find what is this one? What is the second difference? And what is the k^{th} difference? So, to implement this equation number 2, we have to first have to make the finite difference table and from the finite difference table we have to calculate the value these operators this difference is and then substitute that values putting the value of all these data points $x_0, x_1, \cdots, x_n$.

Then from after substituting all this value we will get the polynomial of degree. So, this from here I can say this is the interpolating polynomial using forward difference of degree so from here you can say that the degree will be I am going up to x_{k-1}. So, degree will be k so maximum it will be k. So, it is degree will be I can write less than equal to k.

If I stop here, it is degree will be only one if I stop here, it degree will be only maximum up to 2. So, I am going up to kth its degree will be always less than equal to k. Because it may happen that the coefficient becomes 0. So, in that case degree will be less than k. Otherwise it is the kth degree polynomial. So this is called the interpolating polynomial. Now, the question is that when I can apply this Newton's forward difference. So, this is a Newton's forward difference formula. So, when I can apply?

So, I can apply this values whenever I need to solve I know I have a values suppose I have value. So, this is the data given to me $x_0, x_1, \cdots, x_n$. So, I can apply this value whenever I want to approximate the value at some x and x is lying near here. Because in that case if it is lying here, then I can use all this values whatever the values. So, this is the x_0. Similarly, I can have value $y_0, y_1, \cdots, y_n$.

So, if I want to approximate the value of x somewhere in the middle or somewhere in between the x_0x naught and x_1, then using the finite difference operator I can use all the values. So, because from the previous lecture we know that this value will move like this one up to the kth finite difference.

So, in that case, I can use all these values and my approximation will be better. But suppose my x I want to find and x is lying somewhere here. So, in that case what will happen I can use only this value and all the values below this one. So, below this one will be very less values. So, in that case in my polynomial, whatever the interpolating polynomial we are talking about the degree of that polynomial will be very less as compared to the degree of the polynomial when the value of x is lying here.

So, in that case the approximation will be very not as good as we are getting in the case of the value of x lying here. So, everything depends on where the value of x is lying. So, in this case our value of x is lying here somewhere on the top of the table, but sometimes it may happen that I want to find I have the values like this one $x_0, x_1, \cdots, x_n$ and my x is lying not here but it will be lying here somewhere. So that is my x.

So, in that case what we will do we shift this one and x_0 I call it x_{-1}, so x_1 I call it x_0 and this will be x_{n-1}. So, in that case, we just shift these indexes so that this becomes x_0 and then after this I can use the finite difference table and then I can use all these values corresponding to this line and putting this here in this equation and get the interpolating polynomial.
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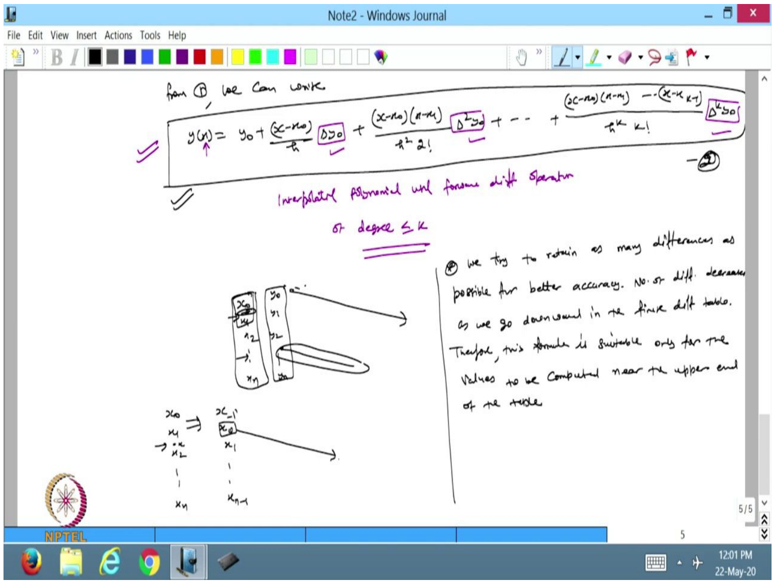

So, from here I can say that to implement this formula number 2, I can write that we try to retain as many differences as possible for better accuracy. The number of differences decreases as we go downward in the finite difference table that we have just discussed. So therefore, this formula is suitable only for the values to be computed near the upper end of the table. So, that is there.

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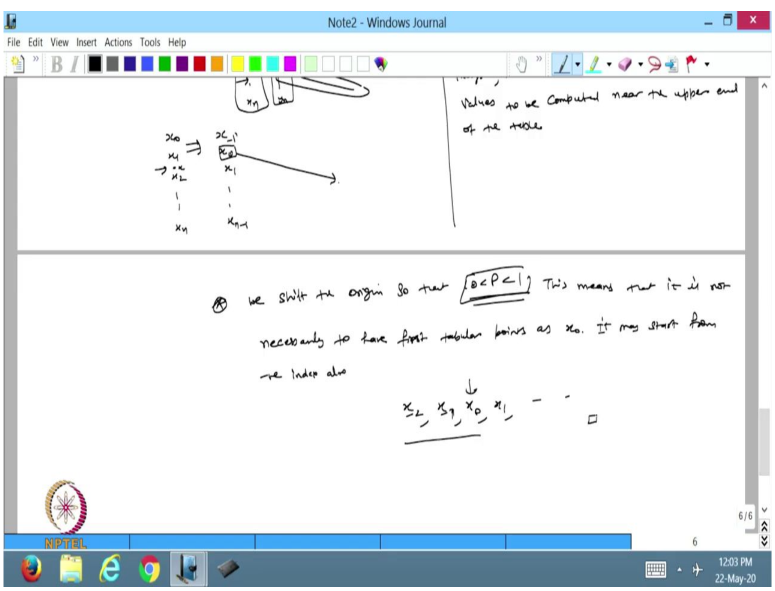

And the second one is I can write the next important point is that we can shift we shift the origin. So, that the value of p is lying between 0 and 1. So this means that I just for example, I have taken here that my x is lying here. So, I am shifting this x_1 to x_0 and x_{-1}. So, in that case my value the p will be always lying between 0 and 1. So, we shift this this means that it is not necessarily to have the first tabular point as x_0 it may start from negative indexing also like it starts from x_{-2}, x_{-1} then x_0 then x_1 like this one.

So, in this case, I have shifted two times to make this x_0 earlier but I was shifted x_0 here and then the previous values I call it x_{-1} and x_{-2}. So, this way we can shift just to choose that the value the p is lying between 0 and 1. So, I should stop here. So, today we have discussed how the error propagates in the finite difference table.

And then we have discussed the Newton forward difference formula to approximate the value at any x and that x lies in the upper of the and the upper part of the finite difference. So, we will continue with this in the next lecture. So, thanks for watching. Thanks very much.