Scientific Computing Using Matlab
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Lecture No. 32
Gershgorin Circle Theorem for Estimating Eigenvalues of a Matrix

So, hello viewers and welcome back to the course on Scientific Computing using Matlab, so today we will go for the lecture number 32 and we continue from the previous lecture that is Gershgorin theorem.

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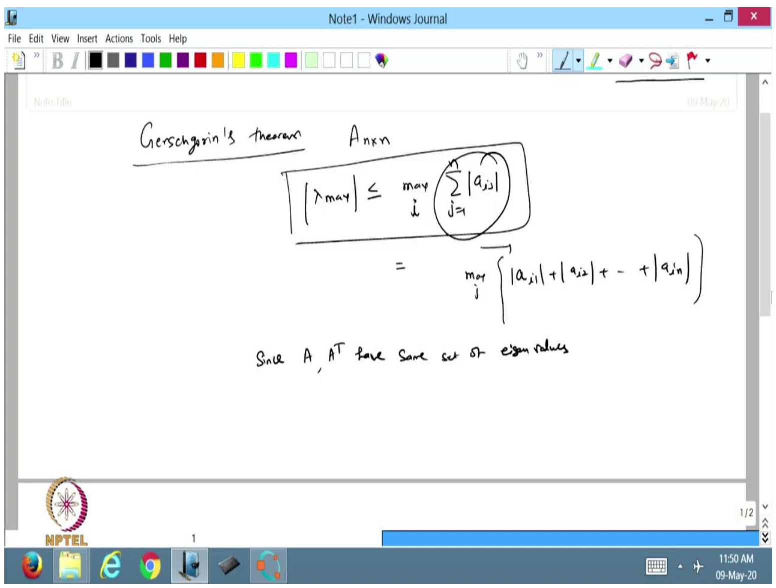

So, in the previous lecture we have discussed about the, the Gershgorin theorem and based on that one we found that if I have a matrix n cross n matrix, then the maximum Eigenvalue of this matrix, $|\lambda_{max}| \leq \max_{i} \sum_{j=1}^{n} |a_{ij}|$ So, I am taking the rows so in this case I am taking the sum of the magnitudes of the elements in the row and then I am taking the maximum over the all rows.

So that is, I think we should change this one to j because then this will be i because this will be equal to, first I will take this one, $\max_i\{|a_{i1}|+|a_{i2}|+\ldots+|a_{in}|\}$, so, it will move from all the rows and then I will take the maximum. So, this is. So, from here you can see from here that now the question is since matrix A and its transpose have the same set of Eigenvalues, so that we already know.

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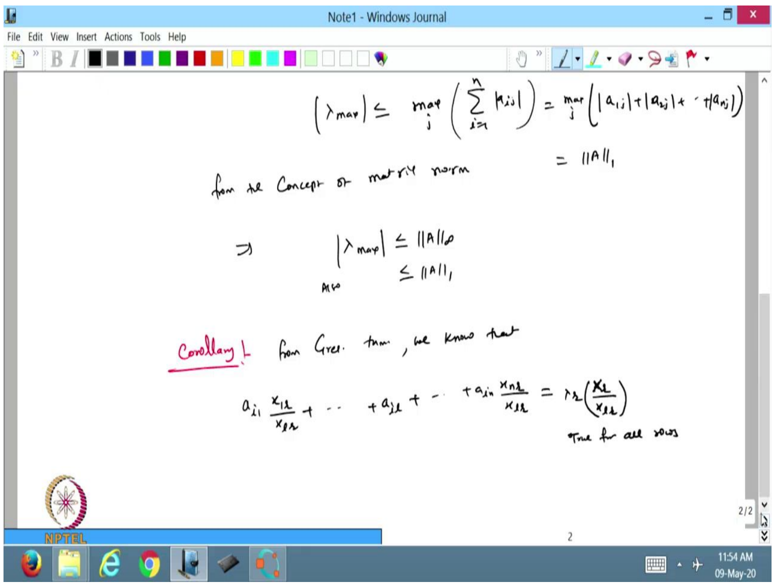

So, from here I can write that the λ_{max} is also less than the largest column sum in modulus, so I can also write the column sum. So, from here I can write that this max will be always less than summation, so this is, so I can write

$$|\lambda_{max}| \le \max_j\left(\sum_{i=1}^{n}|a_{ij}|\right) = \max_j(|a_{1j}|+\ldots+|a_{nj}|)$$

that is the column vector we are taking and then we finding the maximum of all these so that is the largest column and if we know from here from the norms, matrix norm so from the concept of matrix norm we know that the, this sum, the largest row sum, this sum is equal to 1. So, it is equal to infinity, so that is basically infinity now and this is the maximum column sum and this can be written as 1.

So from here, I can say that for any matrix I will get a maxis less than and also less than 1. So this is, we have done from the Gershgorin theorem. Now we want to do a corollary, so let us, this is the application of the Gershgorin theorem. So, now from here that we know that from Gershgorin theorem we know that we have written like this one,

$a_{i1}\frac{x_{1r}}{x_{lr}} + \ldots + a_{il} + \ldots + a_{in}\frac{x_{nr}}{x_{lr}} = \lambda_r \frac{x_r}{x_{lr}}$. So this is what we have, from the previous lecture we have seen that we can write this one as here. Now in this case, this is true for all rows.

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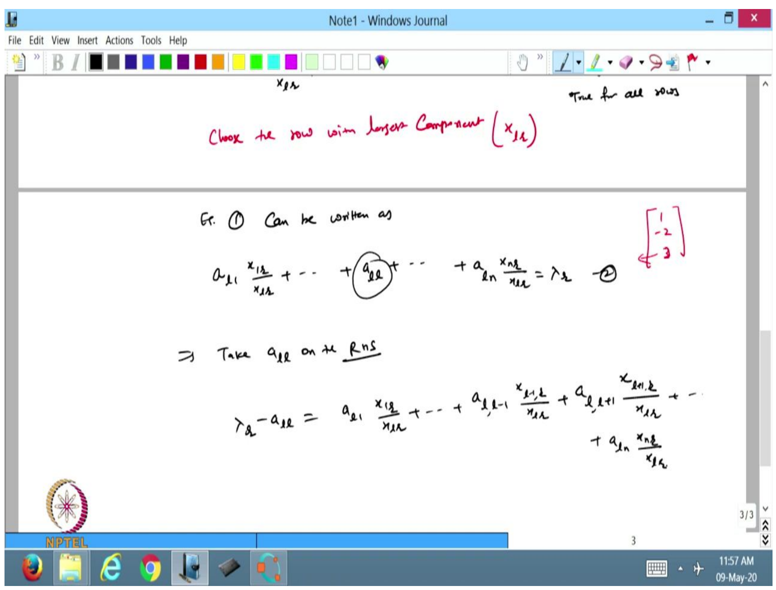

Now from here what do we do? We choose, choose the row with largest component, so that is xlr so we regard, we just take the row that corresponds to the xlr, like in the suppose for example, I have this vector 1, -2, 3 so in this case I am choosing the third row because third row has the largest element. So, I am choosing that row which has the largest element that is xlr.

So, from here I can write that, equation number I can write 1, so equation 1 can be written as, $a_{l1}\frac{x_{1r}}{x_{lr}} + \ldots + a_{ll} + \ldots + a_{ln}\frac{x_{nr}}{x_{lr}} = \lambda_r$. So from here I will get this equation, so that equation I just take the 2.

Now take the modulus value, so I just take the modulus of this, now from 2, we can write that modulus of r is less than equal to al1, al2 or maybe I can take from here, so before taking the modulus value I just take, so just taking before the modulus value I will take this component on the right hand side. So take a ll on the right hand, on the right hand side.

So I will take this, so I will get a, so I can write from here, I can write that this part I can write

$$\lambda_r - a_{ll} = a_{l1}\frac{x_{1r}}{x_{lr}} + \ldots + a_{l,l-1}\frac{x_{l-1,r}}{x_{lr}} + a_{l,l+1}\frac{x_{l+1,r}}{x_{lr}} + \ldots + a_{ln}\frac{x_{nr}}{x_{lr}}$$

, so this we will get.

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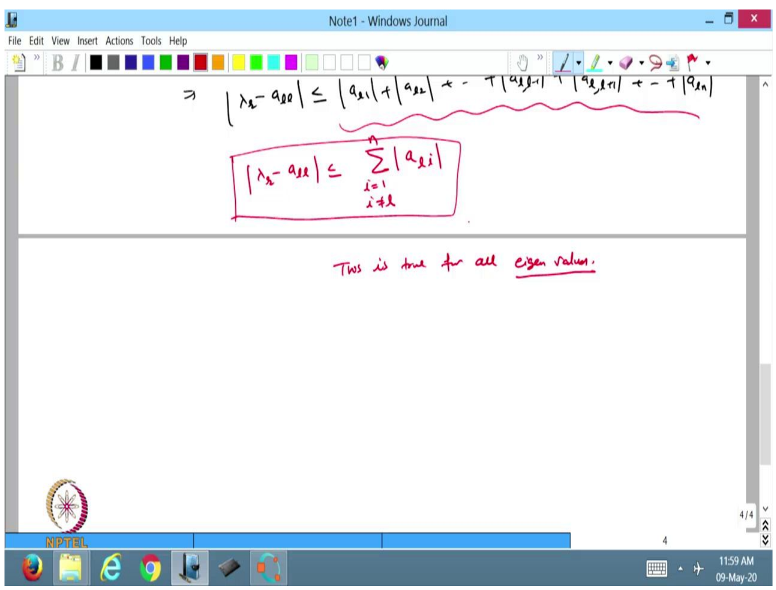

Now after doing this one and then I take the modulus value and from here I will take this modulus value then this modulus value will be written as, $|\lambda_r - a_{ll}| \leq |a_{l1}| + |a_{l2}| + \ldots + |a_{l,l-1}| + |a_{l,l+1}| + \ldots + |a_{ln}|$, so from here I will get this value. So, now you can see from here that this is again the sum of the modulus of the ith row except that a ll, so that is equal to the sum, so this can be written as, now from here I can write that the $|\lambda_r - a_{ll}| \le \sum_{i=1, i \ne l}^{n} |a_{li}|$.

So, except this value so this can be written from here this value and this is true for all Eigenvalues, because I have chosen that only that row which has the largest value, so similarly we can choose for the other Eigenvalues and then we will find out this. So, this is true for all Eigenvalues.

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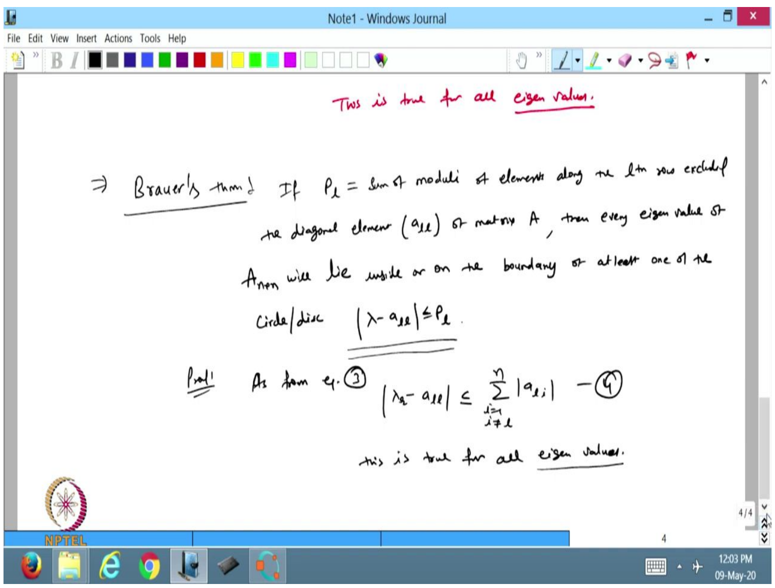

So, now from here we will get one another theorem that is called Brauer's Theorem, so Brauer theorem says that if Pl that is equal to the sum of moduli of elements and along the lth row excluding the diagonal element that is all of matrix A, then every Eigenvalue of A, that matrix A that is n cross n matrix will lie inside or on the boundary of at least one of the circle or disc I can say $|\lambda - a_{ll}| \le |p_l|$.

So now for this, this is whatever we have done from here so I just call it the equation number 1, 2 and this is 3. So, proof our statement thus is this one, so as from equation 3 we can say that $|\lambda_r - a_{ll}| \le \sum_{i=1, i\neq l}^{n} |a_{li}|$. So, this is from the equation, so I call it 4.

Now, this is true for all Eigenvalues because I have taken a one Eigenvalue that is the rth Eigen value and that rth Eigenvalues satisfy this relation, so I will choose another Eigenvalue I will get the another relation, I choose another Eigenvalue then I get the another relation, so this is true for all Eigenvalues from here. So, using this one we can find the bounds for all the Eigenvalues of a given matrix and using this matrix we can give the estimation of α that where the α is lying.

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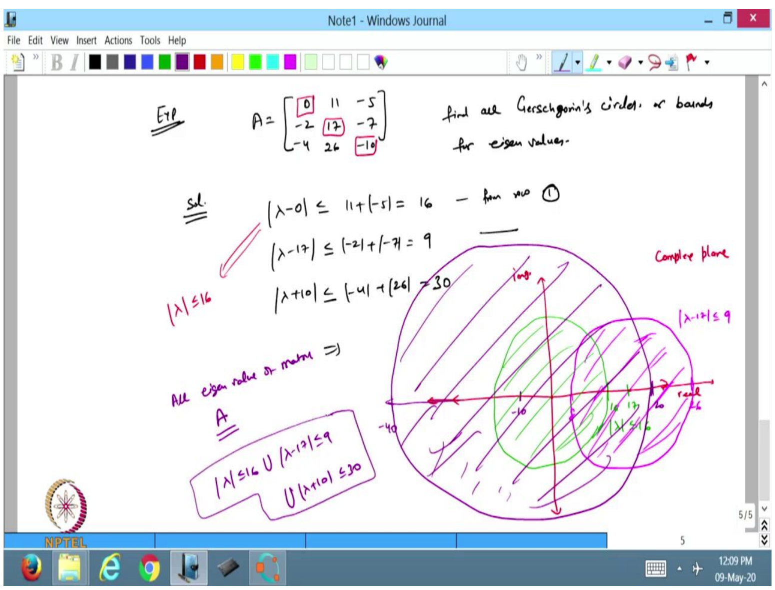

So let us give one example of what the meaning of this one is. So, for example, let us take the same matrix that we have taken, this is my matrix A so, 0, 11, -5, -2, 17, -7, -4, 26, -10. So, the question is find, find all Gershgorin circles or bounds for Eigenvalues. So, from here if you see so, I choose the diagonal element so this is my diagonal elements this, this and this.

So, from here I can write that for the first circle I can write $|\lambda - 0| \le 11 + |-5| = 16$. So, that is from row 1, now from the second row I will get $|\lambda - 17| \leq |-2| + |-7| = 9$, so this is from the second row and from the third row I will get $|\lambda - 10| \leq |-4| + 26 = 30$ and that is 30. Now, from here I can see that, from here I will get the bounds, so this will give me, it gives the $|\lambda| \leq 16$.

Now what I do is that I try to plot in the complex plane, so I am plotting this one in the complex plane. So, this is the real part and this is the imaginary part, now I take the Eigenvalue, first is that the λ is less than equal to 16, so I take a circle with the radius 16. so it is a very big circle we have to draw so let u take suppose, this is my, this one, so that circle I take and with the center is 0 in this case, so this is my, so λ is less than equal to 16 so that becomes the disk, so this is the part, given disk we have to take.

Now, from the second one is that $|\lambda - 17|$ is less than 9. So, this is going up to 16, so this is 16, now I have to choose 17 so suppose this is 17 so I will take this as a center now and I will take a circle of radius 9 so, let us take a circle of radius 9 so, 17 and it will go up to 9, so it is 26. So, suppose I take this as a, this place I choose as a 26 and 17 minus 9 is 8, so suppose this is my 8. So, from here I will find a circle, write a circle, drawing a circle with the center this point.

So, this is my circle with lambda minus 17 less than equal to 9. So, now it is less than equal to so it becomes a disk, so this is my another disk and from the third one I choose, so I choose the lambda plus 10 so it means the λ minus 10, so it is 16 it is also going up to 16 so suppose it is somewhere minus 10 and then I have to choose a radius of 30. So, 30 means I have to go up to 40 and -10+30 is 20, so suppose this is my 20, so I have to choose a very big circle like this one and this is my other disc.

So, from here it says that, from here that all, all Eigenvalues of matrix A will lie in the union of this circle or all the Eigenvalues in the matrix A will lie in any of the circle or I can take the union. So, from here we can take that, that I can find all the Eigenvalues of the matrix A into the $|\lambda| \leq 16 \cup |\lambda - 17| \leq 9 \cup |\lambda + 10| \leq 30$ So, from here I will get, this is the, the bounds for all the Eigenvalues of the matrix, given matrix.

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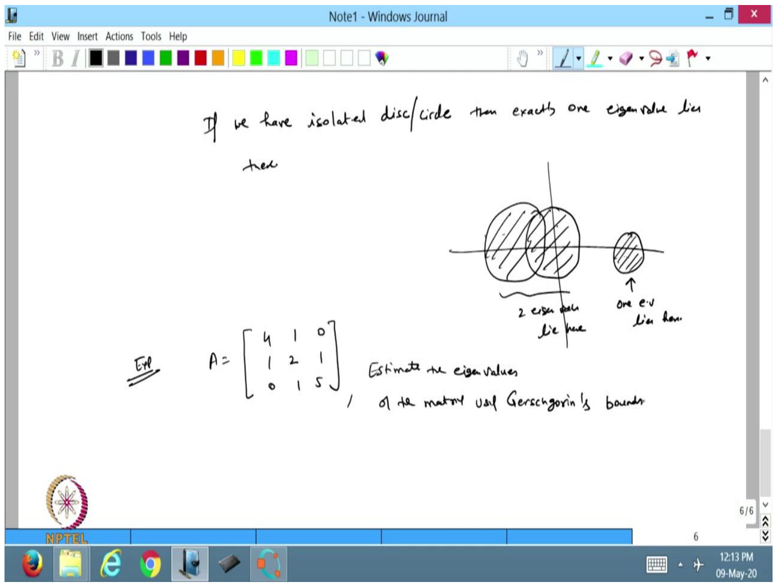

Now, from here I can draw one conclusion is that, so from here I can draw one conclusion that if if we have isolated disc or circle, then then exactly one Eigenvalue lies there because we have seen from here that in this case this was the first disc that is also intersecting with the second disc and I have taken the third disc that is also intersecting with all these two.

So, this is completely inside this one and this is also intersecting, so in this case there is no isolated disc so that is why we say that all the Eigenvalues will lie in the union of this one, but what will happen if I have this type of disc? Suppose I get one disc like this one, another disc I will get like this one and another disc I am getting like this one, so in this case and suppose I have a 3 by 3 matrix.

So, in the 3 by 3 matrix I will get three discs, so one disc will be there so in that case the one Eigenvalue will lie here definitely and the two Eigenvalues will lie in the union of this and this. So, that is the meaning of isolate value, so one Eigenvalue lies here and two Eigenvalues lie here. So, that is the bounds we can find.

So, let us take one more example, I have a matrix A so 4, 1, 0, 1, 2, 1 and 0, 1, 5. So, this I have chosen, then the question is, estimate the Eigenvalues of the matrix, using Gershgorin bound.

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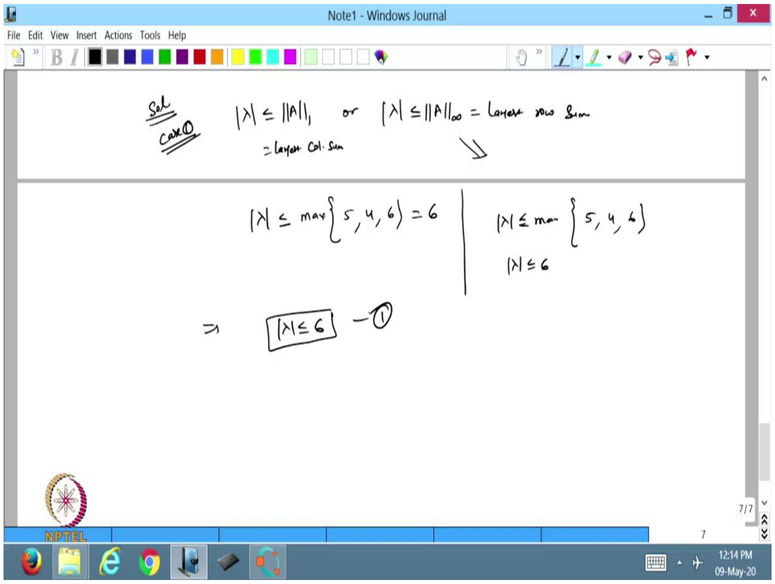

So now, we have this matrix, so let us start doing this one. So, now in this case I know that from the Gershgorin theorem we know that, that I will apply this formula that $|\lambda| \leq \|A\|_1$ or I will get the $|\lambda| \leq \|A\|_\infty$.

So, one norm is the maximum column and infinity norm is the maximum row. So, first is this one, so case one, so in this I will find out all the bounds whatever is possible there, so let us take this one, so this is the maximum largest column sum and this is equal to the largest row sum. So, from here I can say that lambda, so I choose the largest column sum, so this is the, I want to find the maximum of first column sum, so that is 4 +1, 5, 2+1, 3 +1, 4 and this 6, so that is equal to 6.

Now, from here I will get, now from here I will get λ is maximum overall row sum, so $|\lambda| \leq max\{5, 4, 6\}$. Now from here I will also get $|\lambda| \leq 6$, from here I will get my magnitude of λ is less than equal to 6. So, this is the first bound I will get, case 1.

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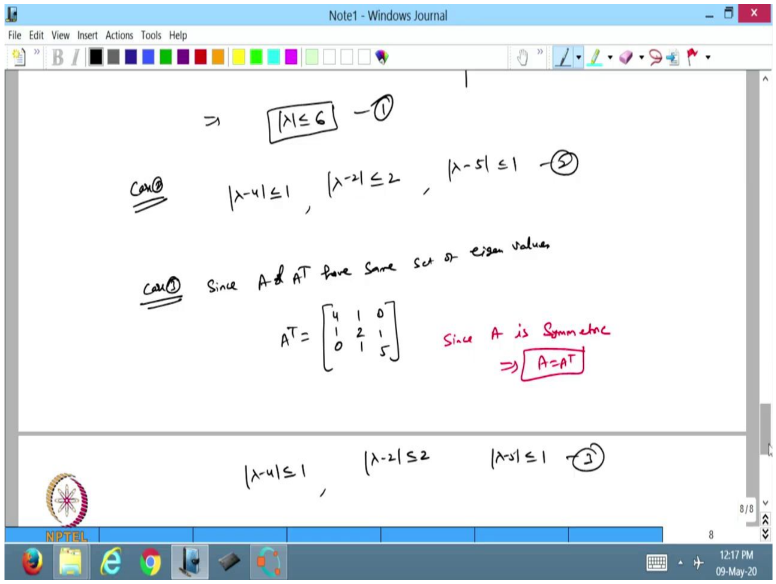

Now case 2, I will apply the Gershgorin, so in that case $|\lambda - 4| \leq 1$, $|\lambda - 2| \leq 2$ and $|\lambda - 5| \leq 1$, so that is the Gershgorin disc we are getting for this matrix. Now from here, if you see this one so this is case number 2 and I will take case 3 since, A and A transpose have the same set of Eigenvalues, so what do we do in this case? Instead of finding the rows I can take the column, so in that case I will get my matrix, so that is the matrix I will get. So, I get the matrix A transpose, so this A transpose will be 4, 1, 0, 1, 2, 1 0, 1, 5.

Now, I can apply the Gershgorin theorem here, so from here I can write $|\lambda - 4| \leq 1$, $|\lambda - 2| \leq 2$ and $|\lambda - 5| \leq 1$. So, in this case if you see I am getting the same value because 4 this 1, 2, this is a symmetric matrix so no problem.

So, I can say that since, A is symmetric which implies that A is equal to A transpose, so I get the same bounds. So, from here I can say that, from this I can call it 1 and then 2 and then 3 so, 2 and 3 are the same.

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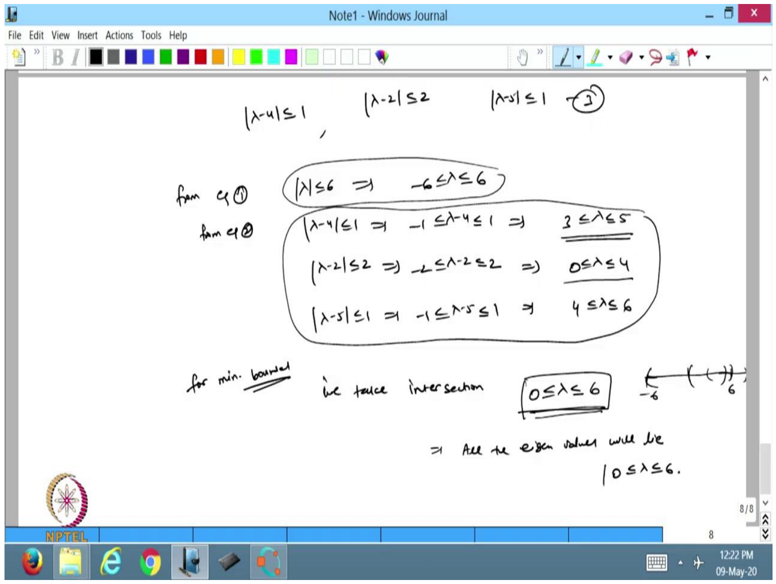

So, from 1 I will get $|\lambda| \leq 6$, that gives me that the lambda is lying from -6 to 6, so that is the one of the bound, from the second bound I will get from equation number 2, so $|\lambda - 4| \leq 1$ that implies that $\lambda -$ 4 is lying from -1 to 1 and that gives me λ is less than equal to I am adding 4 here, so 4 it will go 4-1 it will be 3 and this is 5.

So, that is the corresponding range of λ, $|\lambda - 2| \leq 2$ so that gives me $-2 \leq \lambda \leq 2$ and if I add, so it will be λ from 0 up to 4, so that is the another bounds and the third bound is $|\lambda - 5| \leq 1$ that implies that $-5 \leq \lambda \leq 1$ so that implies that the lambda will lie from, adding 5 here so it will be 4 and it will be 6.

So, based on this one, so this bounds we have calculated independently, this bounds were calculated independently so if I want to find that the minimum bound, so for or I can say the stick bound I will take the intersection, so intersection gives me that my λ is moving from -6 to 6 in this case but it is moving from 3 to 5, 0 to 4 and 4 to 6. So, if I take the intersection of all these I will get my λ will lie -6, 3, 0, 4 so let us take this one, this is suppose I take 6 it is go -6, here I am going 3 so 5 and the λ is moving from 0 to 4, and this 4 to 6.

So, if I get from here, widely I can say that my intersection is here 4, so if I take 4 and then from here I get 6, so I get this intersection point from 0 to 4, 4 to 6 and 3 to 6, what if suppose I want to take the left bound so left one minus 6, 3, 0, 4 so if I take the intersection in this case and then from right hand side, no from here I will get this 0 and from right hand side I will get 6.

So, that is the maximum all the bound we will get, so you will see from here that from here I can say that all the Eigenvalues will lie within this range 0 to 6. So that we, all about the Eigenvalues. So, we stop here. So, today we have discussed the application of the Gershgorin theorem and then we have discussed another theorem that is the Brauer's that is also the application of the Gershgorin theorem and then we have discussed two examples based on this one.

So, in the next lecture we will continue from this one, so I hope you enjoyed this lecture, thanks for watching, thanks very much.