Scientific Computing Using Matlab

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Lecture No. 28

Matlab Code for Gauss Seidel Method

Hello viewers, welcome back to the course on scientific computing using Matlab. So, in the previous lecture we have discussed the Matlab code for Gauss Seidel method, now we will continue from the convergence. So, today we will discuss the next lecture that is 28. So, let us do that.

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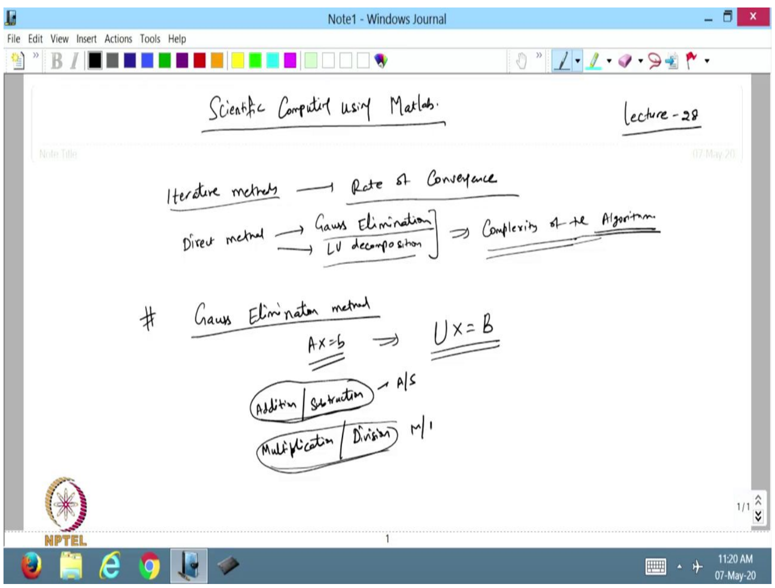

From the previous lectures that for iterative methods we are able to find the what is the rate of convergence? So, using the iterative method, I am able to find the rate of convergence. But, what about the direct methods? So, direct method means what about the Gauss elimination? Or what about the LU decomposition? So, can you talk about the rate of convergence of this one? So, in this case we can talk about the complexity of whatever we call it the algorithm.

Because in this case we are able to find the solution not in the form of iterative methods but we find the solution in just one complete process. So, that is not repeating again and again, so that is we can say that whatever the algorithm we have written for solving the system of equations using Gauss elimination or the LU decomposition we will find the complexity.

Complexity means that like for example, for a Gauss elimination method we know that we have a system Ax=b, and then we reduce into the upper triangular Matrix. So, Ux is reduced to a new Matrix. So, maybe I can call it as a B and then we reduce this one using the elementary operation and then while using the back substitution. We are able to find the solution of the given system now from here in transferring from the given system to the upper triangular system we do addition.

Subtraction we do multiplication and then division. So, in computers the memory used for doing addition subtraction is same and for multiplication division is same. So, this steps addition and subtraction iterative as 1 and the multiplication and division also treated as one. So, the abbreviation we use is in addition to subtraction. This is multiplication and division.

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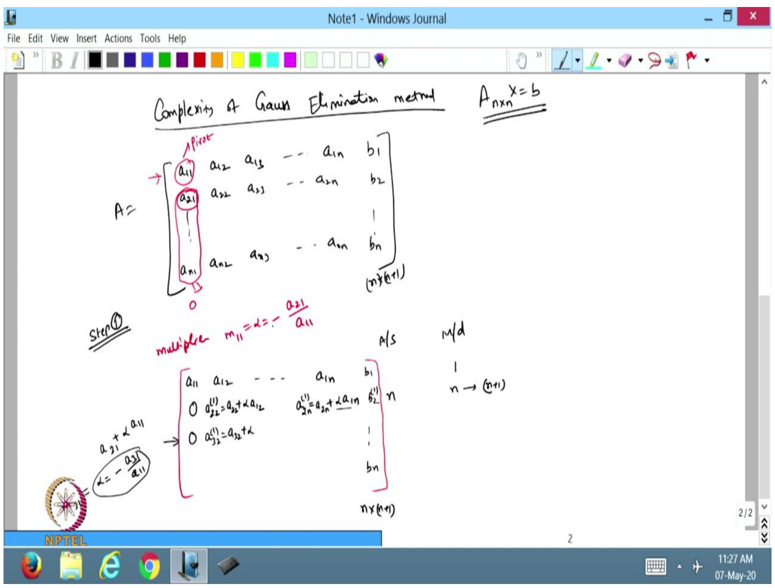

So let us try to find the complexity of Gauss elimination method because this word complexity we must have heard in the data structure the course related to the data structure in computer science wherever we are discussing any algorithm then we discuss about the complexity. So, now I want to find out what is the complexity of the Gauss elimination method. So, let us say I have n*n Matrix and this in my system.

Now what I do is suppose this is my matrix A and that is a11, a12, a13,...,a1n, a21, a22, a23,...,a2n and an1, an2, an3,...,ann. Now what I do is that I will go step 1. So, in step 1 what I want to do, I will use this as a pivot. So, this is my pivot, my first pivot. So, I will use this pivot to find all elements below this 0. So, this one I want to make 0. So, what I do is that I will take the multiplier.

So, that we have already discussed. So, I will call it m11 or maybe I can call it α. So, that multiplier will be $m_{11} = \alpha = -a_{21}/a_{11}$ because I want to make this element 0 first this element 0. So, what I do I will divide this whole row by a11 and multiply by a21 with the negative sign and then so after this one.

So, after step 1. So, let us do what will happen in step 1. So, the first row will be the same as given to us. Now in this case, what I do is after step number 1 in the first step, I will make this 0. And then from here, I can write that this element a22 in the step 1 that will become $a_{22}^{(1)} = a_{22} + \alpha a_{12}$. Similarly, the next one will be there and the last it will be a2n, the new element will come that will be a1n.

That will be $a_{2n}^{(1)} = a_{2n} + \alpha a_{1n}$. So, I want to see how many numbers of addition and subtraction and how many multiplication and division. So, in this case if you see from here, here I am dividing multiplying this one by this factor. So, in this case, I have done one multiplication now I will ignore the steps involved to make this element 0. Now from here, you can see that in this case what I am doing.

I am one multiplying and one dividing or one multiplying and one addition similarly next is one multiplying and one division multiplication and addition here also one multiplication and one addition. So, from here you can see that now I have to from here, so I will change this Matrix. Now we will put the right hand side element in the last column because whatever the operation we are applying on The Matrix. That is also applying on the right hand side.

So, I will again make this element b1 b2...bn and then this will be matrix now this also will become b1 and $b_2^{(1)} = b_2 + \alpha b_1$, so that will be this one. So, now from here you can see that the dimension of this Matrix is n*(n+1), so that is n*(n+1). Now, I am not doing anything with this one, so the remaining columns I am doing so I can see the number of additions I am putting is n.

And the number of multiplication is also n. So, to reduce the first element in the second row 0, I am doing n additions and n multiplication and one multiplication I have done for this one also because I was multiplying by this factor and then adding this one. So, here I can say that the total number of multiplication will be increased by n+1. So, this will be increased by 1, so this is the n+1 multiplication I am doing.

So, the same thing I will do again to make this element 0. So, in this case also, I am multiplying this with this factor alpha. So, to make this element zero what I am doing I am doing this element as $a_{31} + \alpha a_{11}$ where $\alpha = -a_{31}/a_{11}$. So, this is my Factor now. So, now in this case also, I will be doing the same thing and then I am writing here a32 so that will be

$a_{32}^{(1)} = a_{32} + m_{31} a_{12}$.

So, that is alpha I am again choosing. So, this is the α you can choose. So, let us take this as m only because this alpha will be the same in that case. So, I will call it m, so this α is m11 and all this one now this α I can delete and I can make it m31 and then I will put a12. So, 31 12 and so on and in the end I will get a3n in the step 1. So, it will be $a_{3n}^{(1)} = a_{3n} + m_{31} a_{1n}$ and this will be again the b. So, this b so this will be b31.

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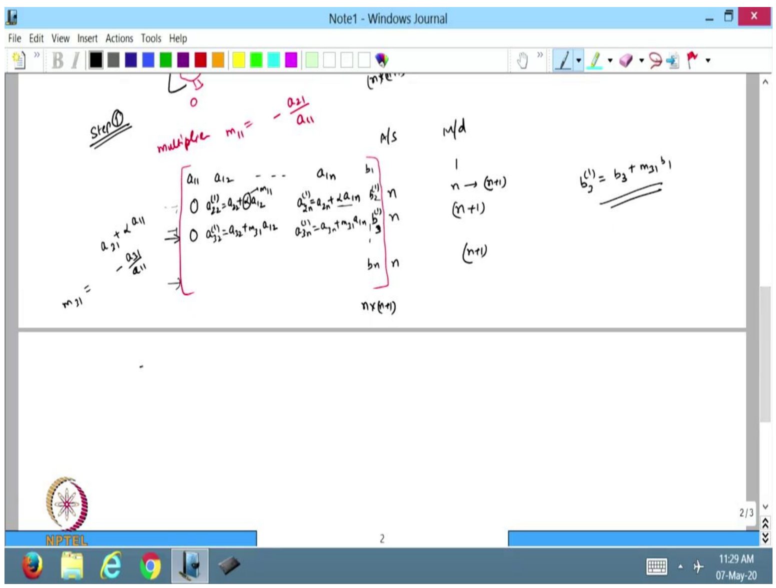

And b31 I can write as $b_3^{(1)} = b_3 + m_{31} b_1$, so that will be my b31. So, now in this case also, you can see that I am doing n number of addition and n number of multiplication. So, one process is multiplying by this one so I can directly write n+1 so the same thing is going to happen. So this is second row third row, so in the last row I am going again n and n+1.

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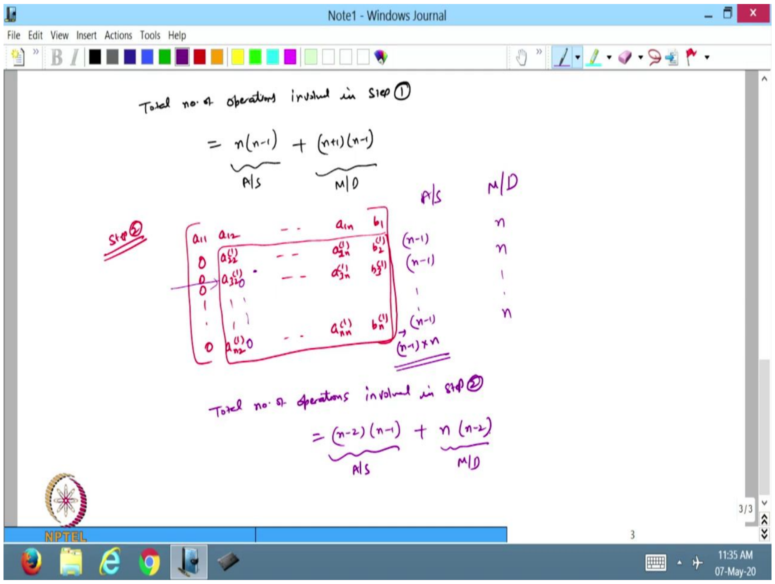

Now, from here you can see that the total number of steps involved is the total number of operations because these are the operation addition and multiplication total number of operations involved in step 1. So, that is my step 1, so it will be so now this is n n n and you can see that I am this n rows. So, the first row I am leaving, So it will be n plus n minus 1 time.

So, I can write that this is n*(n-1) so that is my addition or subtraction plus. The n+1 into n-1 times, (n+1)*(n-1), so that is the multiplication or division. So, these are the total number of steps involved in Step 1. Now let us do the next one the step 2. So, in step two what is happening now, I am leaving this one. So, my Matrix will be a11 a12...a1n and b1 this is 0, 0, 0 all are 0 here and I am getting a22 a2n b2 and then a32 a3n b31 and so on an2 so this will be an2 ann and this is bn.

So, I am able to do this. Now I will concentrate on this Matrix. So, the size of this matrix if you reduce this one, so you can see that in this case the first row is gone now. So, the reduced matrix this matrix has the size n-1 rows into n. So, one it is reduced by 1. So, this is (n-1)*n now the same thing will happen again.

So, from here the, I can see from here that the number of operations involved so addition subtraction, multiplication and division now from here you can see that earlier, it was n n number of steps now it is I am looking at this so it has n columns. So, it will be n-1 n-1 and so on n-1 so now n-1 addition, because now what I am doing is that I am making first this element 0 and all this element 0 and the next element will be addition and then multiplication addition and multiplication and so on.

So, in this case, earlier it was n plus 1 column. So, 1 is reducing so I am getting n minus n. So, in this case, I already have n columns and I am leaving this one so I will get n-1 columns left and then in the multiplication division the same thing as the number of columns was n+1 so I am taking n+1. So, now from here I can see that this will be n, n and n. Now, from here I can see that the total number of operations involved in step 2.

So, that will now see the number of rows are the same. The only thing is that the number of rows is n-1 so it will be n-2 so it will be n-1 n-1 n-1 n minus two times because I am starting from here. This is already the number of rows by n-1 so I am leaving the first one. So, it is (n-2)* (n-1) so that is (n-2)*(n-1).

So, this is I am now taking n so n*(n-2). So, this is total n and this is n-2. So this is n n-2 and this is my n-2. So, that is the total number of so, this is my addition subtraction and this is multiplication and division. Now from here you can see that earlier it was n n-1, now it is n-1 n-2 n+1 n-1 it is n n-2. So, if you see from here then

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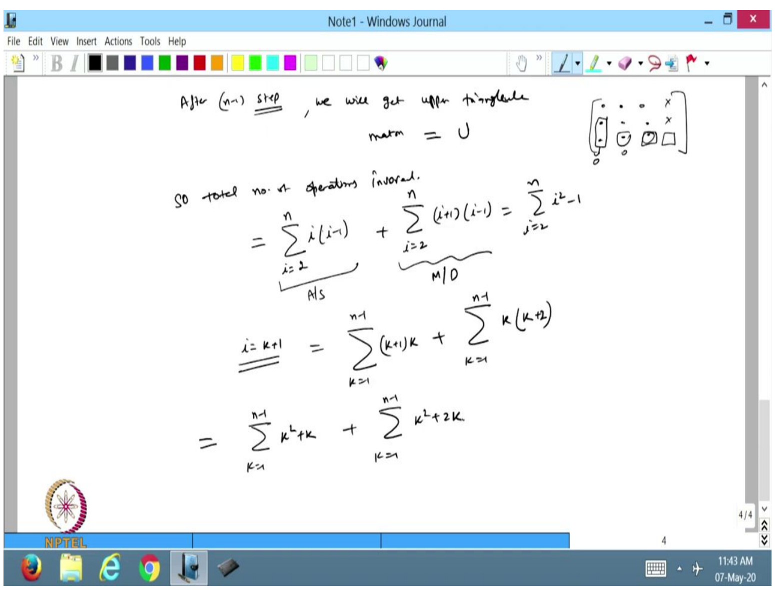

Then after after, so it is starting from step number 1. So, after n-1 steps because when I have a 3 by 3 matrix. So, in the first element, I have a 3 by 3 matrix. So, 3 by 3 matrices in the first step. I will reduce this one 0 and the next step I will reduce this one 0, and then it will be upper triangular. So, if I have n number of rows then after n minus 1 step, we will get an upper triangular matrix matrix that is U and then we are able to solve this one.

So in the n-1 step now, I want to find the, so, total number of operations involved. So, I can add all this one together. So, now I can n n-1 n-2 to n-1 and so on. So, if you see from here, then I can write that $\sum_{i=2}^{n} i(i-1)$. So, this is the number of addition or subtraction.

Because in the end you will get only this element left. So, i is moving from 2 to n because it is n it is n n-1 when I put 2, it is equal to 2 in 2 minus 1 2. So, there are only two additions I have to make for finding the last one. One addition for this and one addition I will do for the right hand side vector. So, this is only two additions we have to do in that case. For the 3 by 3 matrix. So, similarly I can go from here. So, in the end I have to do two additions and then two multiplications.

So, this one I can do as, now you can see from here. It is n n-2 earlier it was n+1 -n. So, whatever the element I am taking a minus 2 is coming here. So, it is n n -2 so from here we can see that I can write it as $\sum_{i=2}^{n}(i+1)(i-1)$. So, this one I can write. So, when i=n it is (n+1)*(n-1) then n-1 it will be n* (n-2) so on it will keep going.

From here I can write so this is the total number of steps involved in multiplication or division. Now from here I can write this as now you can see that it is i from 2 to n then what I do is that I just transform that i =k+1 now if i=2 then the k=1. So, if I change this one then i get

$$\sum_{k=1}^{n-1}(k+1)k + \sum_{k=1}^{n-1} k(k+2).$$

Because i-1=k. So, i-1=k and i+1 will be k+1 k+2, so this is k+2. So, k+2 will come. Now after doing this one, from here I can get that $\sum_{k=1}^{n-1} k^2 + k + \sum_{k=1}^{n-1} k^2 + 2k$.

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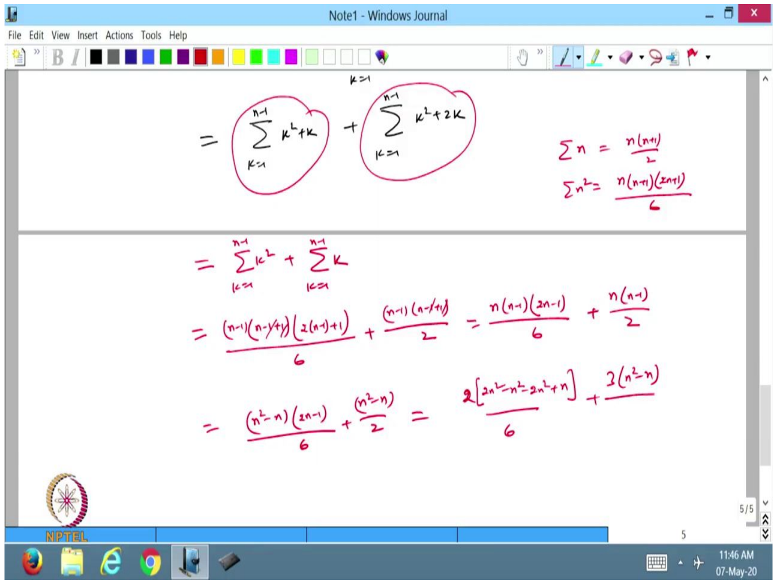

So, now from here you can see that now I know the summation. So, Summation from so this is already known to me that $\sum n = n(n+1)/2$ so that we already know the summation and $\sum n^2 = n(n+1)(2n+1)/6$. So, it is up to n. So, from here I can write, so, first time doing this one. So, it is

$$\sum_{k=1}^{n-1} k^2 + \sum_{k=1}^{n-1} k$$

$$= \frac{(n-1)(n-1+1)(2(n-1)+1)}{6} + \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)(2n-1)}{6} + \frac{n(n-1)}{2}$$

. So, from here, I can write this as I can write from here.

And now after doing this one you will see that I will write this as so this can be written as

$$\frac{(n^2-n)(2n-1)}{6} + \frac{n^2-n}{2} = \frac{2(2n^3-n^2-2n^2+n)}{6} + \frac{3(n^2-n)}{6}$$

$$= \frac{4n^3-6n^2+2n+3n^2-3n}{6}$$

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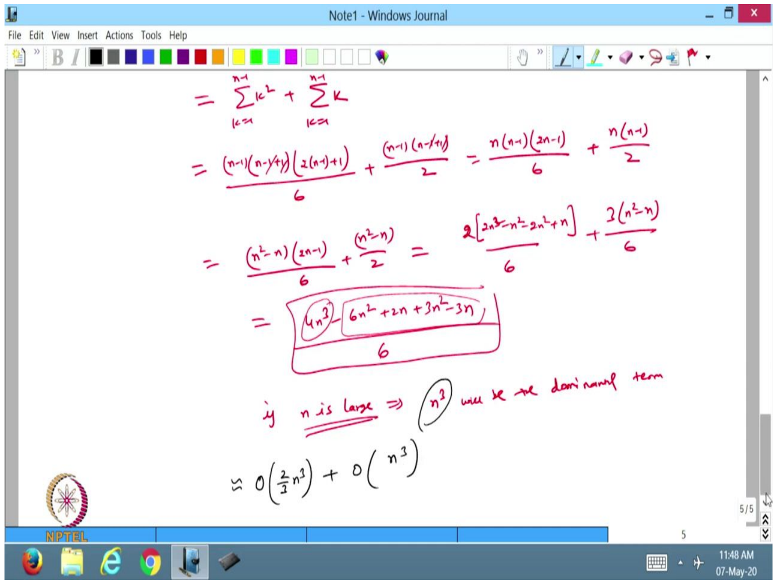

and now from here the same thing I can do from this factor also the almost the same thing will come except the two are there. Otherwise the same form will come. Now if I get this type of value now, you can see that if n is large implies that n^3 will be the dominating term.

Because if n suppose in this case also if I put n =5, so this will be the dominating term. It may happen that this will be cancelled out and this will dominate but one thing is true that if n is large enough, then definitely these terms and cube will be the dominating terms of so that is true in that case this n^3 will be the dominating term. So, from here I can see that I can write that the total number of operations in this case is approximately of order $\mathcal{O}(\frac{2}{3}n^3) + \mathcal{O}(n^3)$.

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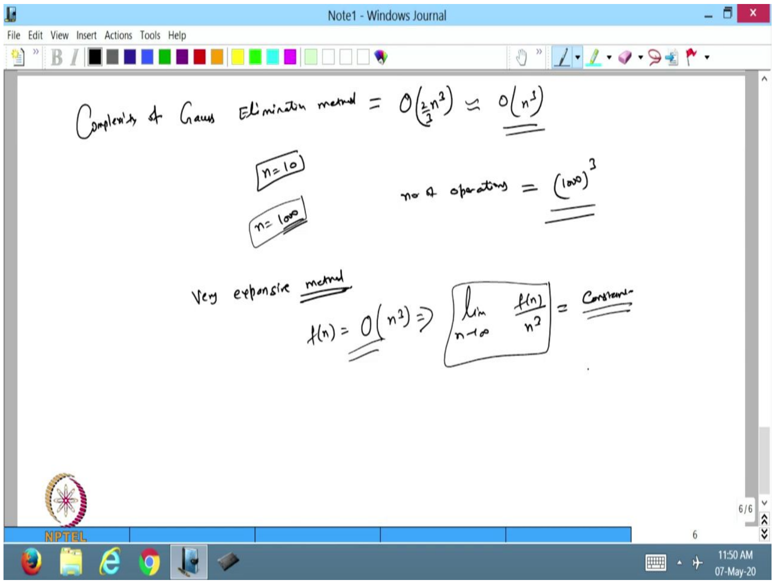

Now from here, I can see that from here I can say that the Gauss elimination method from here I can say that the Gauss Elimination method the complexity of Gauss elimination is order of n^3 or $\frac{2}{3}n^3$. The same as n^3 does not matter. I can write as an order of n^3. So, that means that suppose I have maybe n=10. So, if I take n is equal to 10 by 10 matrix, then the number of operations we have to do is 10^3.

So, if n is large enough, so 10 10^3 means 1000 operations we have to do and if n is maybe 1,000 then you can see that the number of operations will be 1000^3. So, you can see how many operations we have to do 10^9 it will go. So, in that case we have to do so many of the operations to solve this type of system. So, in this case, I can say that this method is very expensive.

Because the number of operations and this factor I just want to define, what is the meaning of O, this is a big O. So, if I write $\mathcal{O}(n^3)$ it means if I put $lim_{n\to\infty}$ so I can say that some function $f(n) = \mathcal{O}(n^3)$ that implies that the $lim_{n\to\infty}\frac{f(n)}{n^3} = constant$. So, that is the meaning of the $\mathcal{O}(n^3)$.

So, that is the meaning of big O and it is used in the complexity. So, that is the complexity used by the Gauss elimination method. So, now we will stop here. So, today we have discussed the complexity of the Gauss elimination method and we found that this complexity is n^3.

And that is very expensive because when we increase the size of a matrix, then the number of operations will go n^3 and it will increase very fast. So, that is why we say that the Gauss elimination method is quite expensive. So, today I hope that you have learned how we can find the complexity of the direct methods and also we have discussed that rate of convergence for the iterative method. So, in the next lecture we will continue from this one. Thanks for viewing this lecture. Thanks very much.