Scientific Computing Using MATLAB
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Lecture - 25
Continued

Hello, viewers. Welcome back to the course on Scientific Computing Using MATLAB. So, today, we are going to discuss the lecture 25. So in the previous lecture, we have discussed the iterative method that is the Gauss-Jacobi method.

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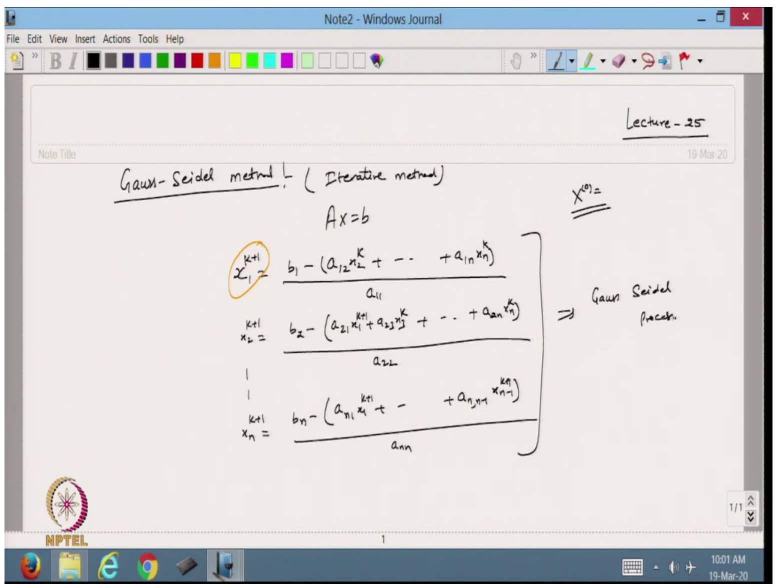

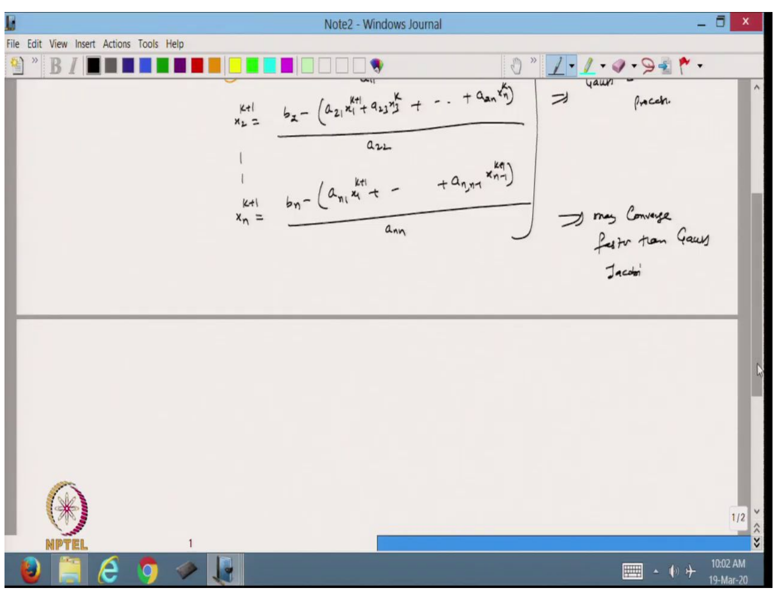

So now we will continue with another method and that is called the Gauss-Seidel. So this is also an iterative method. Now, it is a small change in the Jacobi method. So if we remember, then from the Jacobi method or we got, we have a system of equation Ax=b and we got this iteration process, $x_1 = (b_1 - (a_{12}x_2 + \ldots + a_{1n}x_n))/a_{11}, \quad a_{11} \neq 0$; then x2 is $x_2 = (b_2 - (a_{21}x_1 + a_{23}x_3 + \ldots + a_{2n}x_n))/a_2 2, \quad a_{22} \neq 0$

So it is $x_n = (b_n - (a_{n1}x_1 + \ldots + a_{n,n-1}x_{n-1}))/a_{nn}, \quad a_{nn} \neq 0$, and we know that the matrix, my matrix a is diagonally dominant and these coefficients, they are the diagonal elements are not zero in this case.

So what we do now, I can make this as an iterative process. So, from here you can see that I will start with this one and I will write this as $x_1^{k+1} = (b_1 - (a_{12}x_2^k + \ldots + a_{1n}x_n^k))/a_{11}$ because I have the initial approximation x0, so this is a given approximation to me. I will put the value there and I will get this value here.

Now what I do is that I will use this value. So whatever the value, updated value I am getting, I am using this value here and I will write this as $x_2^{k+1} = (b_2 - (a_{21}x_1^{k+1} + a_{23}x_3^k \ldots + a_{2n}x_n^k))/a_{22}$. So in this case, what I am doing, I am using the updated value of x1 to find the value of x2.

In the Jacobi method, we were using the same k everywhere to find this whole vector as a, to find the updated value of the whole vector, but here we are not doing that. We are taking whatever the updated value is available to me. I will use that value to find the updated value of the other variable.

So, in this case, we will keep going like this one, and in the last one, I will use $x_n^{k+1} = (b_n - (a_{n1}x_1^{k+1} + \ldots + a_{n,n-1}x_{n-1}^{k+1}))/a_{nn}$. So whatever the value is available to me, I will use this one, and then this process is called the Gauss-Seidel process.

So this is not much difference between the Gauss-Jacobi and the Gauss-Seidel, only thing is that so this is a more updated value. And from here, we can also see that this method may converge faster than Gauss-Jacobi.

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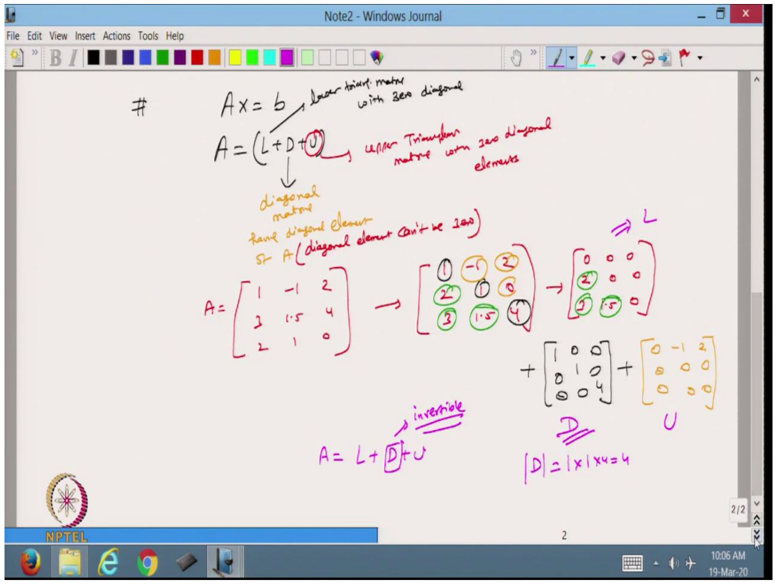

So now, we will find out that in this case, so I know that the sufficient condition is there for the convergence of the Gauss-Jacobi and Gauss-Seidel method. Now will find out the condition of convergence of that method. Now, what is the other?

So I have a process Ax is equal to b. So this is the process given to me. What I can do is convert my matrix A into this form, $A = L + D + U$. So this is, I can call it a lower triangular matrix with zero diagonal, so all of the elements of the diagonal are 0. This one is a diagonal matrix, so a diagonal matrix with the diagonal elements of A, having diagonal elements of A as diagonal elements. And this one is an upper triangular matrix, upper triangular matrix with zero diagonal elements.

For example, I have a matrix, suppose I take the matrix A is equal to 1 -1 2, 3 1.5 4, 2 1 0, so suppose I take this matrix. So I do not want the diagonal elements to be zero. So this one I can write, diagonal elements cannot be 0 because if this is the case, then I will change this one to this form, 1 -1 2, 2 1 0, 3 1.5 4.

Now, there is no diagonal element that is 0. So I will convert this one into the form like this one now. I will write L, so it is 0 0 0, 0 0 0, 2 3 1.5. So I will write like this one. So in this case, I have taken this element, this element, and this element, so that is the same as this one. And then I will write plus my diagonal matrix. So the diagonal matrix, I write 1 1 4. So this is the same as this one here and all others are 0, plus then I take the other matrix, upper triangular.

So I left with only this, this, and this, so I will take this as a minus 1 2 0 and this is all other elements as 0. So from here, I can cite that this is my L, this is my D, and this is my U. So I can say that my matrix $A = L + D + U$, and D matrix is invertible; invertible means it does not have, because I know that the diagonal element, the determinant of the diagonal matrix is 1 into 1. So that will be 1*1*4, so 4 is the value. So if any of the diagonal elements is 0 then it will be 0 and it is not invertible. So we want our D to be invertible, so that is why we have to make the changes. So any matrix can be written as a form of L+D+U.

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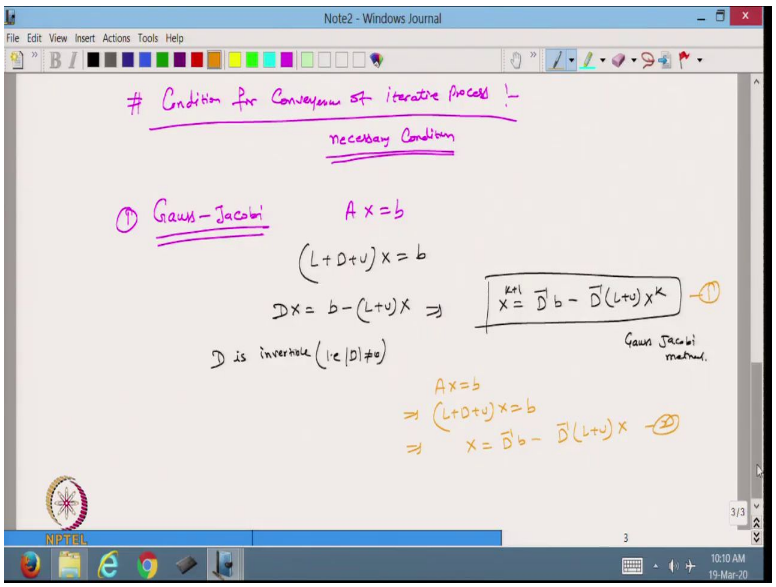

Then, let us see that condition for convergence of the iterative process. We know that the diagonal dominance is there, but diagonal dominance is the sufficient condition. Now we are talking about the necessary condition. So this is the necessary condition we are going to find, a necessary condition that will happen if the matrix is not the diagonally dominant.

So if the matrix is not diagonally dominant, then it may converge or may not converge. So let us start with the first one, the Gauss-Jacobi. So in the Gauss-Jacobi, you know that I have Ax=b, so that is my (L+ D+U)x=b. So this is given to me.

Now, from here I can write, so from the Gauss-Jacobi method, you know that I will take all the elements of L on the right-hand side and all the elements of U on the right-hand side. So from here, I can make this system as I can write this system as $Dx = b - (L + U)x$, so this can be written here.

And then, D is invertible, that is non-zero. So from here, I can write

$x^{k+1} = D^{-1}b - D^{-1}(L+U)x^k$, and this is my iterative process. So that is the short form of the Gauss-Jacobi method, so this is my Gauss-Jacobi method.

Now, I also know that I can write my system as, so this one, I can write as Ax=b. So, from here I can write (L+D+U)x=b. So from here, I can write my x can be written as $x = D^{-1}b - D^{-1}(L+U)x$, where x is the exact solution. So I can take it as a 1 and I can take it as a 2.

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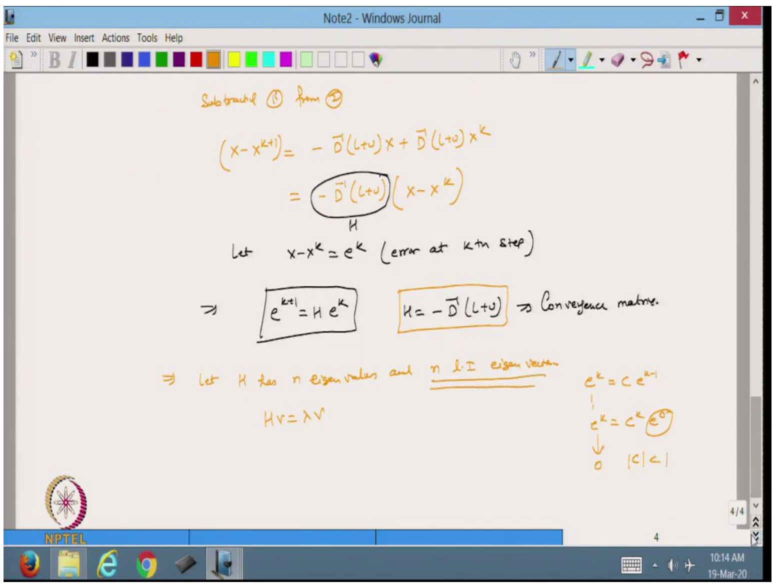

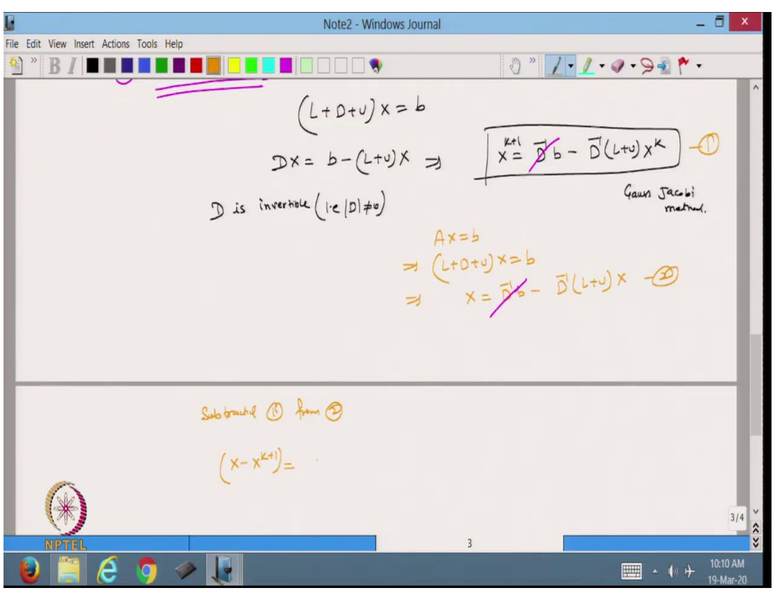

Now subtracting 1 from 2. So I can write 1 from 2. So from here, I will get

$$(x - x^{k+1}) = -D^{-1}(L+U)x + D^{-1}(L+U)x^k$$
$$= -D^{-1}(L+U)(x - x^k)$$

So this is what? It is the exact solution minus the approximation at the kth step. This is the exact solution and minus the approximation of the k plus 1th step. So, let $x - x^k = e^k$, so that is my ek. So that is, I call it error at kth step. So from here, I can write from this that I can write $e^{k+1} = He^k$, where my $H = -D^{-1}(L+U)$ and this is called a convergence matrix.

So everything depends upon what is this matrix and if you remember from the previous lectures when we were dealing with non-linear equation like Newton methods or secant methods, so in that case, if you see that this is just a number, then you know that x(k+1), some number is there c xk. So if, not that the error. So $e^k = ce^{k-1}$, like this one and then we can change this one into $e^k = c^k e^0$, if you remember by the continuous substitution.

Then we know that if this error is, some error is there and I want that this tends to the, this error tends to 0, then we know that the ck, now this can be possible when the c is less than 1. Because when the c is less than 1, the more power of c will be much smaller and then this will go to 0. The same concept, we can apply here but here, we are dealing with the matrix. So how can we deal with such types of things?

Now, I know that this is my matrix. So let H have n eigenvalues and n linearly-independent eigenvectors. So I know that I am saying that $Hx = \lambda x$, so not x, I should not take this x. So, suppose that I am taking $Hv = \lambda v$. So it will, it is n*n matrix, so it will have a n number of eigenvalues. So I am considering that this eigenvalue is there and eigenvalue is they, we are getting the L, linearly-independent eigenvectors so that we are considering here.

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Note2 - Windows Journal

$$(x - x^{k+1}) = -D^{-1}(L+U)x + D^{-1}(L+U)x^k$$

$$= \underbrace{-D^{-1}(L+U)}_{H}(x - x^k)$$

Let $x - x^k = e^k$ (error at kth step)

$\Rightarrow$ $\boxed{e^{k+1} = H e^k}$ — (2) $\boxed{H = -D^{-1}(L+U)}$ $\Rightarrow$ Convergence matrix.

$\Rightarrow$ Let H has n eigen values and n l.I eigen vectors

$Hv = \lambda v$

$e^k = c\, e^{k-1}$

$e^k = c^k e^0$

$\downarrow$

0 $\quad |c| < 1$

$\Rightarrow$ Eq. (2) can be written as

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Note2 - Windows Journal

$$e^{k+1} = He^k = H(He^{k-1}) = H(H)(He^{k-2}) - -$$

$$\boxed{e^{k+1} = H^{k+1} e^0} \quad \text{— (3)}$$

Let $e^0 = \begin{bmatrix} x_1 - x_1^{(0)} \\ x_2 - x_2^0 \\ \vdots \\ x_n - x_n^0 \end{bmatrix} = \sum_{i=1}^{n} c_i v_i$ $\quad c_i$ scalars (real)

$\boxed{Hv_i = \lambda_i v_i}$

$$\Rightarrow H e^0 = H \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} c_i H v_i = \sum_{i=1}^{n} c_i H v_i = \sum_{i=1}^{n} c_i \lambda_i v_i$$

$$\Rightarrow H^2 e^0 = H(He^0) = \sum_{i=1}^{n} c_i \lambda_i (Hv_i) = \sum_{i=1}^{n} c_i \lambda_i^2 v_i$$

$$\Rightarrow \boxed{H^{k+1} e^0 = \sum_{i=1}^{n} c_i \lambda_i^{k+1} v_i}$$

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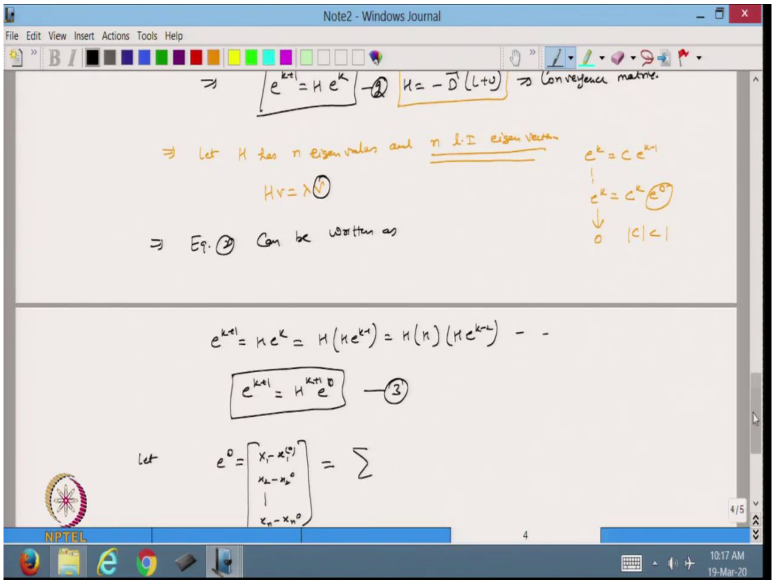

Now, from here I can say that, so using this one value, I can write. Now what I do is that from equation number, maybe which equation is there, one is there. So I can call it the 2 equations. So equation 2 can be written as $e^{k+1} = He^k = H(He^{k-1}) = H(H)(He^{k-2})\ldots$.

So in the end, I can write this as a $e^{k+1} = H^k e^1$ and e1 is the initial approximation or we can write this one as $e^{k+1} = H^{k+1} e^0$. So, whenever we are dealing with e^{k+1}, then this value will come. So I can write this as an equation. So I will call it 3.

Now, we consider that let my e0, what is my e^0? $e^0 = [x_1 - x_1^0, x_2 - x_2^0, \ldots, x_n - x_n^0]$, so that is a vector basically. So I call this vector that this vector can be written as a linear combination of these eigenvectors, so whatever the eigenvector I am taking; so these eigenvectors.

So let $e^0 = [x_1 - x_1^0, x_2 - x_2^0, \ldots, x_n - x_n^0] = \sum_{i=1}^{n} c_i v_i$. So I am considering this one that this vector can be written as a linear combination of the eigenvectors of the matrix H. So from here, I can write, now $e^{k+1} = H^{k+1} \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} c_i H^{k+1} v_i = \sum_{i=1}^{n} c_i$. And $H^{k+1} v_i$ is what? So instead of this, I just write

$$He^0 = H\sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} c_i H v_i = \sum_{i=1}^{n} c_i \lambda_i v_i$$ because H of vi is equal to lambda i vi, the ith eigenvalue. So from here, I can get my H e naught is equal to this one.

Now from here, I can again apply $H^2 e^0$, so that will be $$H^2 e^0 = H(He^0) = \sum_{i=1}^{n} c_i \lambda_i H v_i = \sum_{i=1}^{n} c_i \lambda_i^2 v_i$$. So, if I keep going like this then from here, I can write that $$H^{k+1} e^0 = \sum_{i=1}^{n} c_i \lambda_i^{k+1} v_i = e^{k+1}$$. So this can be written like this.

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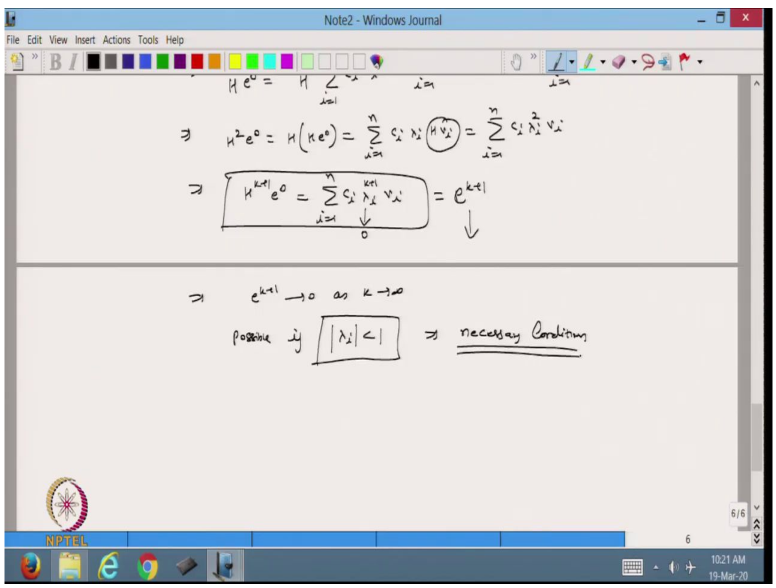

Now from here, and that is equal to e^{k+1}. Now if I want that, my method that $e^{k+1} \to 0$ as $k \to \infty$, so this is possible if $|\lambda_i| < 1$. Because if the magnitude is less than 1, then this quantity will go to 0 if $k \to \infty$. So if the $k \to \infty$, this value tends to 0 because ci is constant so that it cannot be changed; and vi is also the constant vector that cannot be changed.

So if my lambda i is going to 0 then I can say that my error is going to 0. So from here, this is a possibility when I take the magnitude of lambda i that should be less than 1 because then, the more power of lambda will become much smaller and then it will tend to 0. So that is called the, so this is the necessary condition.

So necessary condition means that if the iterative process is convergent, the error is going to 0, then my λ_i should be less than 1. So this I have considered. So that is the necessary condition. So this is what we have calculated for the Gauss-Jacobi method.

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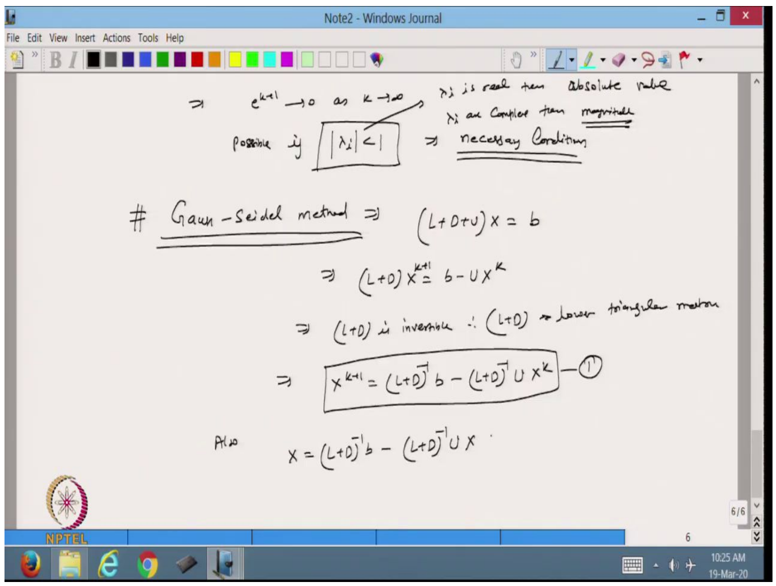

The same way I will see what will happen in the case of Gauss-Seidel. So this magnitude I have taken, if λ is real, then absolute value and if λ is complex, then magnitude, it should be less than 1 or so Gauss-Seidel method.

So in this case, I can write my same (L+D+U)x=b. And from here, I can write from here, so this is my, suppose this one. So from here, in this now I know that it can be written as $(L+D)X^{K+1} = b - Ux^k$, because in this case, we are keeping this, so now I can write this as this. Because in the Gauss-Seidel method, if you see then I am updating the value of x, this one.

So in the first case, it is okay, in the second case, the left part will be the updated value. So the left part will be the updated value, it means that I have to choose the lower triangle matrix in the updated way. So from here, I can write like this and from here, L+D inverse I am taking is invertible, because L+D will be a lower triangular matrix, and D, I already told you that in the diagonal element, no element is zero, so that will be the determinant will be non-zero.

So from here, I can write my $X^{K+1} = (L + D)^{-1} b - (L + D)^{-1} U x^k$. So this is there, so that is I can pick it as equation number 1. So that in my short form of the Gauss-Seidel method. Also, I can write from here, $x = (L + D)^{-1} b - (L + D)^{-1} U x$, where x is the exact solution. So the same way, subtracting.

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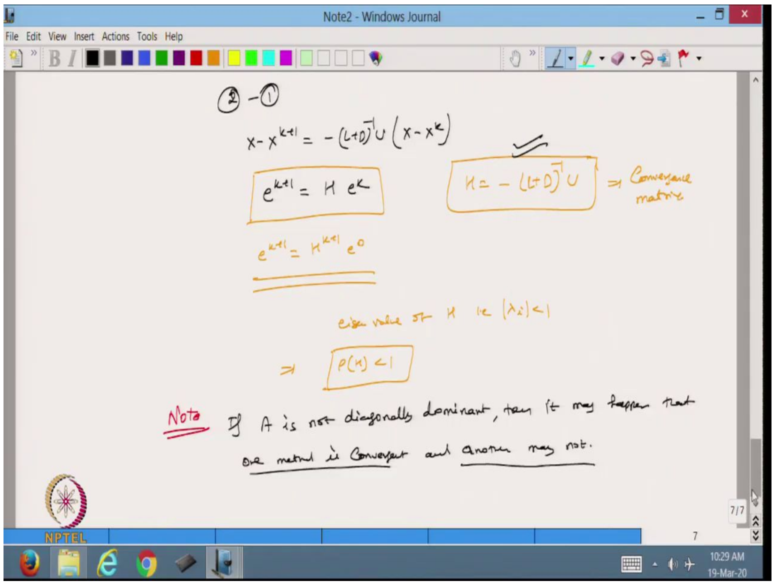

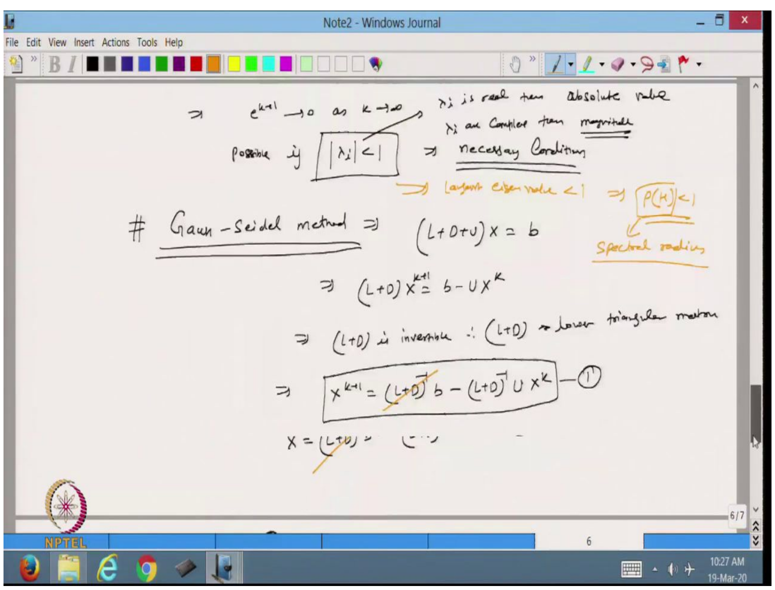

So 2 minus 1, so it will give you $x - x^{k+1} = -(L + D)^{-1} U(x - x^k)$. So from here I can write that this is my $e^{k+1} = He^k$. So from here, I can say that my $H = -(L + D)^{-1} U$. So this is my convergence matrix.

So in this case, my convergence matrix is different from the Gauss-Jacobi method, and from here, the same condition will happen that my $e^{k+1} = H^{k+1} e^0$. So in this case also, the same way the convergence that the eigenvalues of H, that is $|\lambda_i| < 1$.

So this is possible, I can say from here that the largest eigenvalue should be less than 1, it can be also written as a, and this is called spectral radius. Spectral means the set of all eigenvalues and the largest eigenvalue the spectral radius. So this one can be written as there, so with the same condition that the necessary condition for the convergence is that it should be less than 1, the largest eigenvalue should be less than 1. So that is also the necessary condition in the terms of Gauss-Seidel method.

So now we are able to find the Gauss-Jacobi method and the Gauss-Seidel method and this H and H may be different. The note is there, one note that if matrix A is not diagonally dominant, then it may happen that one method is convergent and another may not because in both the cases, this matrix is different. So the matrices are different, the eigenvalues are different, so it may happen that one method will converge and another method may not. If the matrix is diagonally dominant, then definitely, it is going to converge.

So this is, we stop here and this is all about the Gauss-Seidel method and their convergence. So maybe in the next class, we will go further. So thanks for watching. Thanks very much.