Scientific Computing Using MATLAB
Professor Vivek Aggarwal
Professor Mani Mehra
Department of Mathematics
Indian Institute of Technology, Delhi
Lecture 23

Welcome back viewers. So welcome to this course. Today we are going to start with lecture 23. So in the previous lecture we have discussed LU decomposition methods using the Gauss elimination and the Crout's method. So today we will go further. So in the previous lecture, we have discussed one example. In that case we were able to find the LU decomposition using Crout's method. So today I will take another example.

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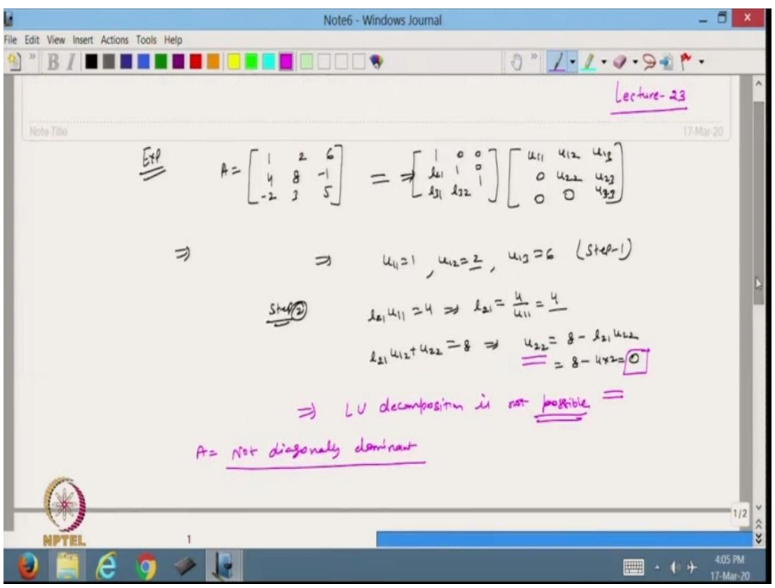

So let us take a matrix A and for this matrix is given as 1 2 6; 4 8 -1 and -2 3 5 and I want to reduce it to the LU decomposition. So in this case, I will take my L. So I will put this equal to 1 1 1; l21; l31; l32 into u11, u12, u13 and u22, u33 and u23. Now, I know that from here I can say that my u11=1, u12=2 and u13=6. So that is from step 1.

Now from step 2, if I take the second row, I will get l21*u11=4 and from here I will get the value of l21, so l21=4/u11=4. Now the next step will be to finding u22. So I will find out l21*u12+u22=8. So from here my u22=8-l21*u22=8-4*2=0. So that is the value 0.

So in this case what is happening? The value is coming 0 and if you remember that in the next steps to find the value of other elements some places we have to divide by u22 but u22 is coming 0. So in this case, we cannot divide by u22, so from here I can say that LU decomposition is not possible and if you look from the matrix A then you can see that it is not diagonally dominant. So in this case my matrix A is not diagonally dominant.

So it may happen that if I change this one and make this matrix as a diagonally dominant then if I do the LU decomposition, so in that case, it may possible that I will get the LU decomposition. So that is always there if your matrix is diagonally dominant, then this is LU decomposition is possible.

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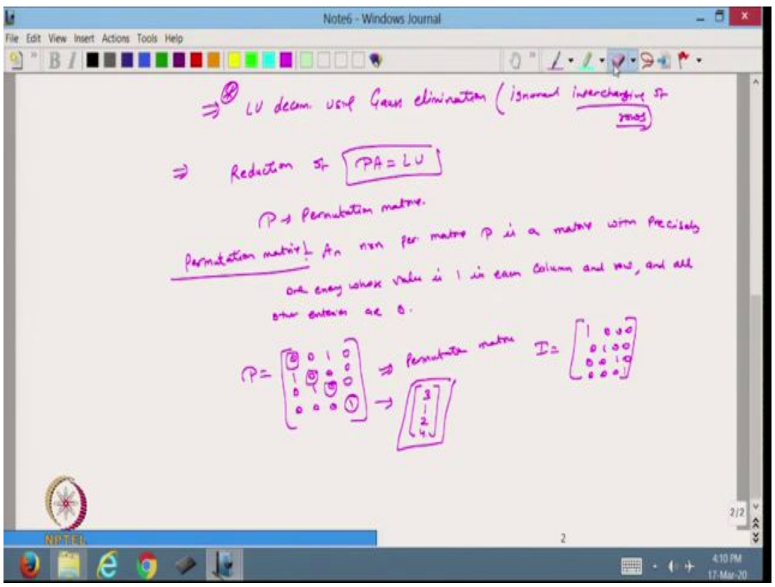

So from here in the previous method where we have taken LU decomposition, so I have started with the method 1 LU decomposition using Gauss elimination. So in the LU decomposition using Gauss elimination we have ignored interchanging of rows. So in that case we have ignored the interchanging rows. So based on this one, so in the previous one LU decomposition is not possible using the Crout's method so that is it.

Now, we want to find out another method. So I am just telling you that in the beginning we have started with the LU decomposition using Gauss elimination and we have ignored interchanging of rows. So in this case, I am going to start with the new method and that is called the reduction of PA=LU decomposition form where P is a permutation matrix. So it is a permutation matrix so I can write the definition of permutation matrix.

So in matrix, an n cross n matrix permutation matrix P is a matrix with precisely one entry whose value is 1 in each column and in each column and row and all other values all other entries are 0. So that is called a permutation matrix. For example, if I define the P as, suppose I take a matrix, 0 0 1 0; 1 0 0 0; 0 1 0 0; 0 0 0 1. Suppose I take this matrix. 4 by 4 matrix I have taken. So that is a permutation matrix in each row and each column only one element.

So if I want to, this is a permutation matrix. So in the permutation matrix what we have done, we have started with the I that is the identity matrix, 0 0 1 0 and 0 0 0 1 and then we have interchanged the first row. So this permutation matrix if I can save, I can save it as 3 1 2 4 so it means I have interchanged the third row with the first row then this row with the second row and the second row with the fourth row.

So the fourth row is at the same place so that is my interchanging of the rows. So this is, it gives you only the interchanging row because we know that if I want to make the matrix a diagonal dominant then I have to interchange the rows. So that we also looked at the Gauss elimination method, to make the matrix diagonal dominant, we have done the partial pivoting.

And the partial pivoting was what was that, that was just to interchange the rows so that I can bring the highest element at this location and this location, this location and this location. Not in the permutation matrix, but in the given matrix A. So in this case what we are doing?

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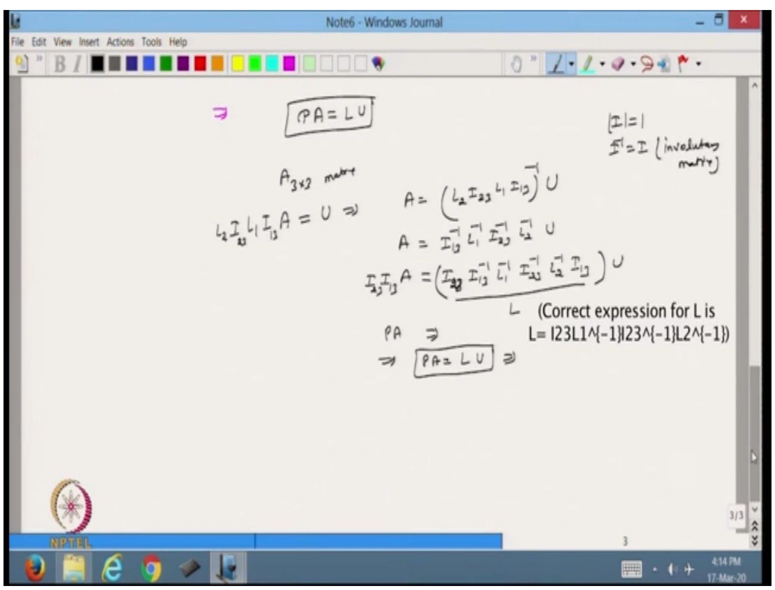

Now from here, I can say that I have a matrix A. Now I am applying permutation to this and then I am doing the LU decomposition. So that is called the LU decomposition method. So what I am doing now, suppose I have a matrix A. Suppose I take 3 by 3 matrices. So I will take A, then suppose I take an identity matrix I with the interchanging of first and third row.

So that I know that if I pre-multiply the same will happen the A. Then I am implying L1, then suppose I am doing L so I can take supposed 2 3. Then I am taking L2. So this is my L2 and then after that I am getting my U. So from here I can say that my A will be what? $A = (L_2 I_{23} L_1 I_{13})^{-1} U$ so that is my A and I know that this I determinant is 1 I inverse is also I so this type of matrices are called unitary matrices.

So from here, if you see from here, I can write this matrix as $A = I_{13}^{-1} L_1^{-1} I_{23}^{-1} L_2^{-1} U$ and that is my A. So from here I can write, $I_{23} I_{13} A = \left(I_{23}^{-1} L_1^{-1} I_{23}^{-1} L_2^{-1}\right) U$ So this is why I am getting U. Now from here this is my I, so that will come my PA and if you open this one just do the calculation then it will give you PA=LU.

So that is my L and this is my U. So this will give me the lower triangular matrix and this will give you the upper triangular and my PA will be this one. So that is the matrix with the partial pivoting with the permutation of the rows. So this is what we have to do when we come across the ways as we have done the example for the Crout's methods.

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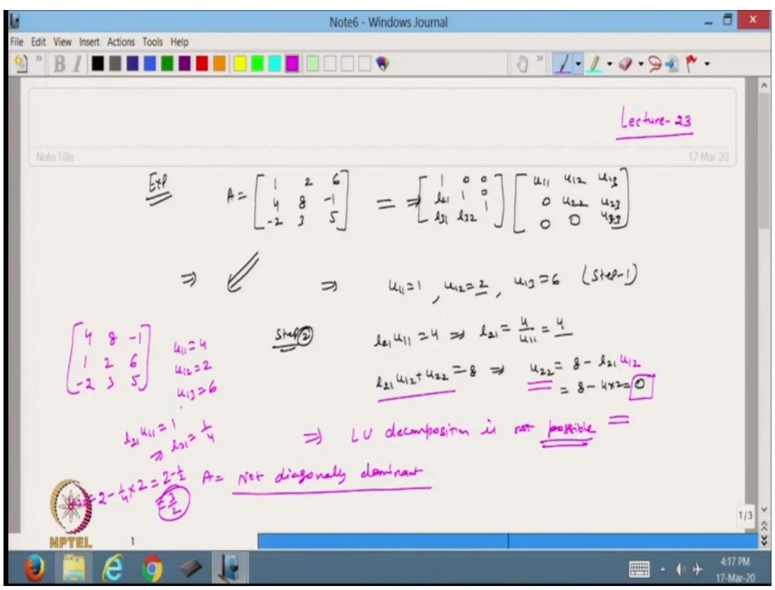

In the example of the Crout's methods if I change this matrix in this form. Suppose I write 4 8 -1 and then 1 2 6 here and then -2 3 5. So in this case my u11=4, U12=2, U13=6 and then my l21*U11=1. So from here I can say my l21=1/4 and then if I want to find my l21, so my L so this one I want to do, so my u22 will be 8 not 8 it will be 2. So u22=2-(¼)*2=3/2, so now it is becoming non-zero. So if I do this one, then it may be possible that we will get the matrix, it may not be possible that you will get the Crout's method for this one.

So now after doing this one, all this will make the MATLAB code later on for Crout's method for PA LU decomposition.

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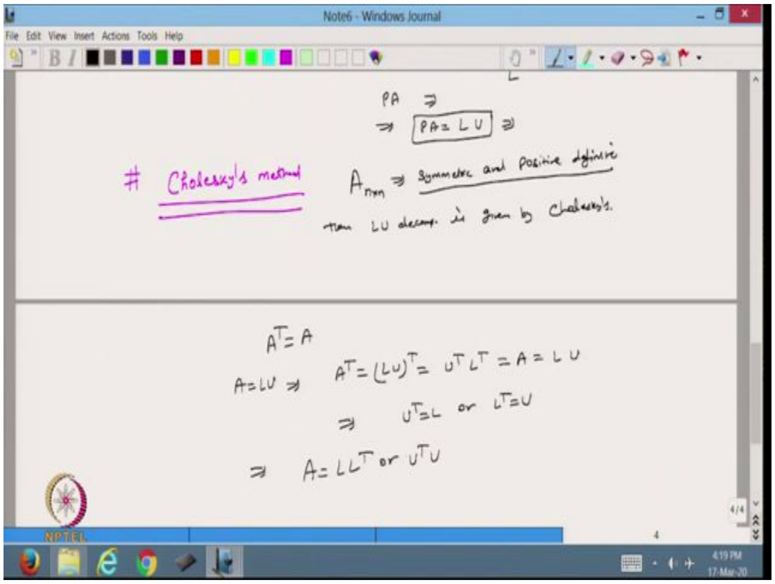

Now we take another very important method and that is called the Cholesky's method. So what is going to happen in the Cholesky method? So if I take a matrix A n*n and this matrix is symmetric and positive definite. So two conditions are there; symmetric and positive definite then the LU decomposition is given by Cholesky. So in this case if I take the matrix A symmetric it means $A = A^T$ transpose and then what will happen. I have my LU decomposition.

So I will take $A^T = (LU)^T = U^T L^T = A = LU$. So from here I can say that either $U^T = L$ or $L^T = U$. So from here, I can say that my matrix $A = LL^T$ or UU^T.

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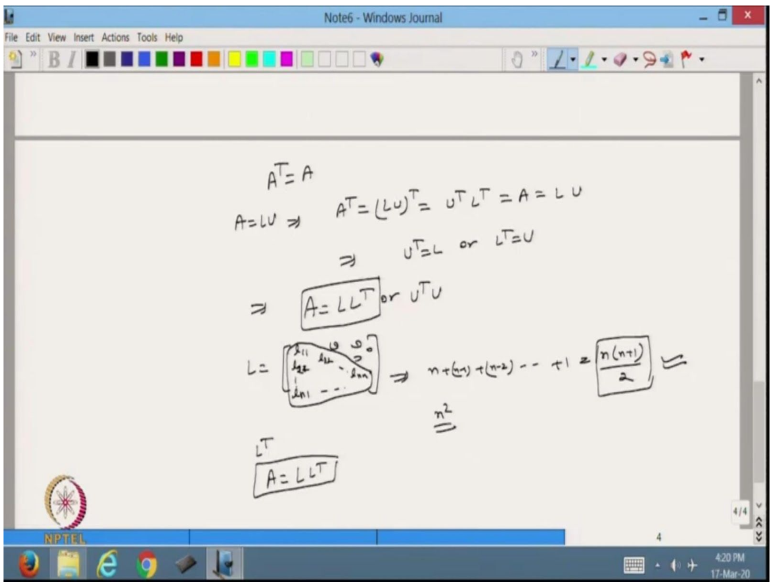

So in this case what is going to happen? Now suppose L is, suppose I take this method this is the notation. So let my this is l11, l22, lnn so l12 L sorry 21 ln1 this one so now my L transformation will be the same element, but it will become the upper triangular matrix. So from here I can say that I need to find only these elements and how many elements there are so we have $n + (n-1) + (n-2) + \ldots + 1 = \frac{n(n+1)}{2}$. So now I have to find $\frac{n(n+1)}{2}$ from this because in the matrix we have n^2 elements. So instead of n^2 elements, we need to find only these elements. So we have to find out a lot of elements and based on this L then I can find my L transpose and then I reduce my $A = LL^T$ notation. So this method is only for the symmetric matrix and the positive definite matrix.

So let us have some discussion about the positive matrix and the symmetric matrix. Now why do we need this symmetric matrix?

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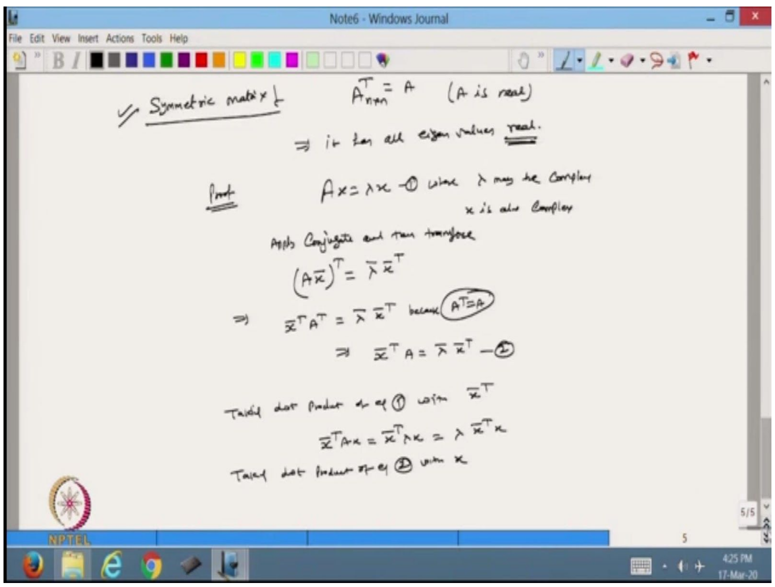

So we know the symmetric matrix. So suppose A is n*in a matrix then it is called the symmetric matrix if A transpose is equal to A. So when I talk about the symmetric matrix I assume that A is a real matrix. So with the benefit of the symmetric matrices is that it has all Eigenvalues real because when we have the real Eigenvalues, then we can say that whether it has a positive sign or negative sign so the benefit with the symmetric matrix is that it has a all the Eigenvalues that is real and this we can also prove.

I know that for the Eigenvalues I can write my $Ax = \lambda x$ where λ may be complex because I do not know. I have to prove that it is real or complex. So if it is a complex A is real and this is a complex so x is also complex. Now what I do is that I take conjugate so this is equation number 1. Apply conjugate and then transpose so $(A\bar{x})^T = \bar{\lambda}(\bar{x})^T$

Now because λ is a scalar so transpose is nothing. So I will get the same value. So from here, I can write this equation as $(\bar{x})^T A^T = \bar{\lambda}(\bar{x})^T$. Now from here, I know that $A^T = A$ so that gives me $(\bar{x})^T A = \bar{\lambda}(\bar{x})^T$ so that equation I write it as is 2.

Now what I do is that I am taking the dot product. Taking dot product of equation 1 with X bar transpose. So what will I get? I will get $(\bar{x})^T Ax = (\bar{x})^T \lambda x = \lambda(\bar{x})^T x$. Now the same thing I am going to do with taking the dot product of equation 2 with x. So I will take x, for equation number 2.

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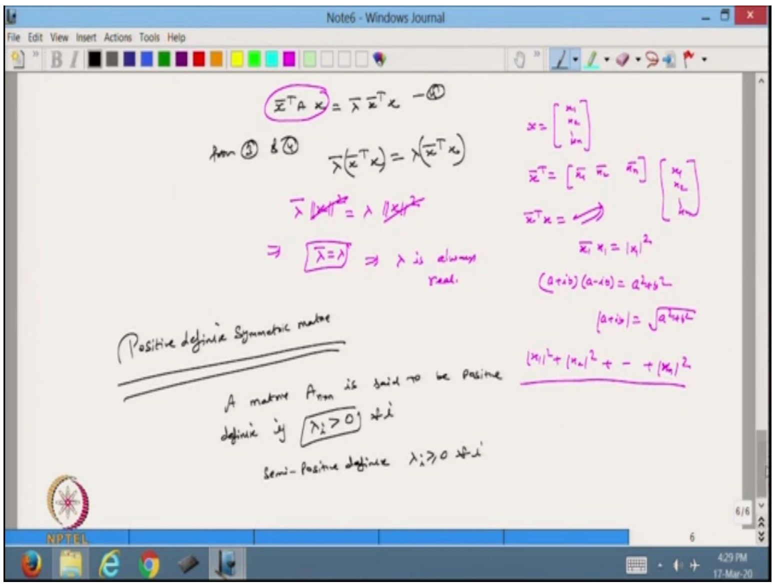

So from here, I can write $(\bar{x})^T Ax = \bar{\lambda}(\bar{x})^T x$. So that is a dot product I have taken, 3 and 4. So from equation number 3 and 4 because this is just a vector so I can take the transpose. So from 3 and 4 you will see that this and this are the same. So from here I can write that

$$\bar{\lambda}(\bar{x})^T x = \lambda(\bar{x})^T x$$

So this is just because if suppose X is a vector, x1, x2, xn and it is a complex valued so x bar transpose will be (X1 transpose) Xn conjugate and then multiplied by x. So suppose I am doing this one, so X transpose X will become this into x1, x2, xn and I know that $\bar{x}_1$ with x, the complex number multiplied by its conjugate will give you the modulus value because I know that $(a + ib)(a - ib) = a^2 + b^2$ and $|a + ib| = \sqrt{a^2 + b^2}$.

So this is the modulus square. I can say that. So based on this one, if you see from here, then this quantity is basically I can write from here $\bar{\lambda}\|x\|^2 = \lambda\|x\|^2$. So this one I can write because from here I will get $|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2$. So this I take as a norm. So this value will be there.

So from here I can eliminate this and from here I will get these values. So from here, I can say that lambda is always real. So for the symmetric matrix, the Eigenvalues are always real. After that I will define the term positive definite. Positive definite and that matrix is already symmetric, positive definite symmetric matrices because the matrix A is already submitting and I am discussing the positive definite.

So in this case from here, I know that $Ax = \lambda x$ and positive definite means that means that a matrix A that is n*n is said to be positive definite if all Eigenvalues, all Eigenvalues are greater than 0 and it is called semi-positive definite this is for all I. Lambda I is greater than equal to 0 for all I. It means that my all Eigen value should be greater than equal to 0. So every positive definite it is strictly greater than 0 otherwise greater than equal to 0.

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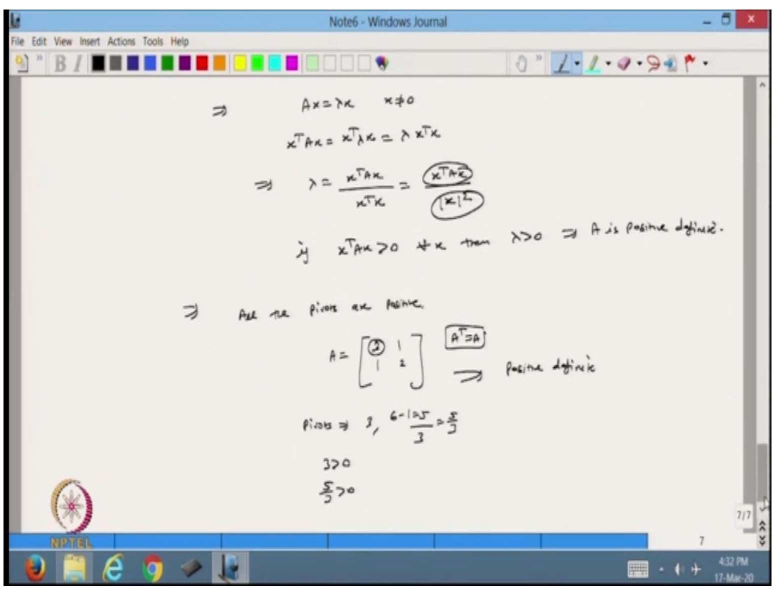

So from here, I know how to find the Eigenvalue. I can find the Eigenvalue as $Ax = \lambda x, \quad x \neq 0$ and λ is my Eigenvalue. I can multiply this with a A transpose and I can write this one as so from here, I can write $x^T Ax = x^T \lambda x = \lambda x^T x$ and from here I can write my $\lambda = \frac{x^T Ax}{x^T x} = \frac{x^T Ax}{|x|^2}$. So from here this quantity is always positive. Suppose this is positive so if $x^T Ax > 0 \quad \forall x$ then $\lambda > 0$ and then we will say that the A is positive definite. So in that case, I will say that A is positive definite. Another way is to find out how we can find out that A is positive definite so then we can say that all the pivots are positive.

So all the pivots will be positive. So what are the pivots? Like I have a matrix A and suppose I take the matrix maybe 3 2 1 1. So this is symmetric matrix $A^T = A$ Now in this case, what are the pivots? So the first pivot is 3 and the next pivot is so the first pivot 3 and the next is so next is I will call it I can reduce this with a Gauss elimination or the other method is finding the determinant. So it is a 6 minus 1 that is 5 divided by 3. So that is 5 by 3.

So in this case my pivot 3 is positive, 5/3 is positive so from here I can say that this is positive definite. So this matrix is a positive definite matrix.

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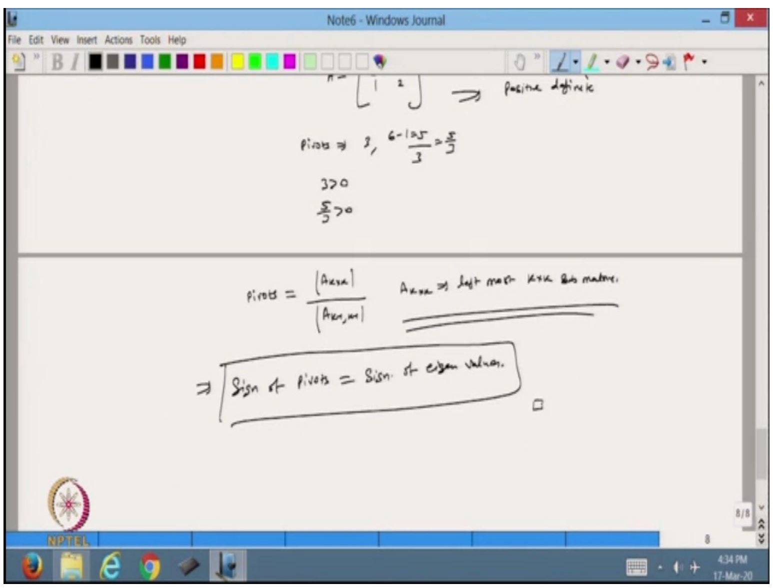

And from here, I can give one information: what is the pivot? How can you find the pivots? So pivots can be found with the determinant because we consider the left most square matrices. So in that case we will get

$$pivots = \frac{|A_{k\times k}|}{|A_{k-1,k-1}|}$$

where A k means where A is the leftmost K*K sub-matrix and this is the previous one.

So like this is a 2*2 matrix and this is the previous one. So I have done taking the determinant of this divided by this. So that is a pivot. So based on the elements of the pivots because it is very easy to reduce this matrix into the Gauss elimination type and then we can find the pivot. So based on the sign of the pivots I can say that whether my matrix is positive definite or not. So from here I can say that the sign of pivots is the same as the sign of Eigenvalues.

So based on the sign of the pivots I can find out whether the matrix is positive definite or the negative definite or the mixed type. So that is all about the positive definite matrix. So we will stop today.

So in the in the today class so we have discussed about that how we can apply the permuted matrix LU decomposition method and then we have discussed the another method that is Cholesky method for symmetric matrix and the positive definite matrix and the next class we will discuss about the other method also. So thanks for watching. Thanks very much.