Scientific Computing using MATLAB
Professor Vivek Aggarawal
Department of Mathematics
Indian Institute of Technology, Delhi
Lecture 22
LU Decomposition Method for Solving Linear System Equation

Hello viewers, welcome back to the course. So, today we are going forward and discussing lecture 22, so in the previous lecture we have discussed the Gauss elimination MATLAB code.

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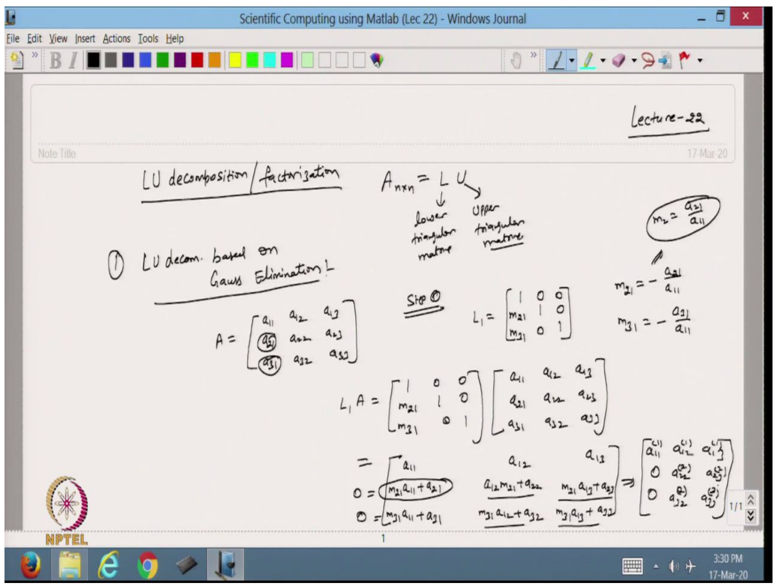

So, today we will discuss the next method and that is called LU decomposition or factorization. So, LU decomposition, so in this case what we are going to do? I have a matrix, so that is n*n matrix, now I am going to decompose this matrix or factorize with this matrix into the LU form. So, where L is my lower triangular matrix and this U is upper triangular matrix.

So there are different ways to do the LU decomposition, so I am going to start with the first one and that is based on Gauss elimination, so LU decomposition based on Gauss elimination, so in the Gauss elimination, we know that so let us I will take 3 by 3 matrix, so in the 3 by 3 matrix suppose I take this a11 a12 a13 a21 a22, so this is my matrix and then at the step 1 in the Gauss elimination I will make this element 0 and this element 0.

So, this will make with the help of this one, so this in this case what I will do that I take a matrix L I call it 1, because I am taking this for step one, lower triangular matrix with 1 1 as a diagonal element, 0 0 0 and here I am using my multipliers, so m21 m31 and this one I am taking 0, so this one this is the lower triangular matrix with a multiplier m21 and m31. So, let us see what will happen. This is my so suppose I pre-multiply L1 with A, so let us see what will happen.

So, this is 1 0 0; m21 1 0; m31 0 1 and you know that this is a multiplier m21 is what? So, m21 multiplier is minus a11 a21, m31 multiplier is minus a31 a11 in some books they do not take this negative sign, so in that case I can make this m2 as a 21 by a11 and then they will take the negative subtract this from the from this first row to make it 0, but I use the negative sign and I and I will add to the first row to make this one 0, so that both type of things can be taken.

So, this is my multiplier, so I will take the multiplier and then I multiply this by the matrix a21 a22 a31 32 33 and then if I multiply the first row, what will I get? I will get only a11, then I will multiply the second so I will get a12 and this is a13. So, suppose I get these values I will make a little bit bigger one, so a11 a12 and a13, now if I multiply this first column with this second row, so what I will get?

I will get m21 a11 plus a21 then I am doing this one for the second row so it will be a12 m21 +a22, then m21 a13+a23. So, next will be m31 a11 this one +a31 then m31 a12 and then +a32 m31 a13+a33. So, from here you can see that what is this, I am multiplying the first row with the multiplier, whatever the multiplier and adding to this, so this part will be 0 and this will be 0. And then what am I doing here? I am multiplying this one with a multiplier and adding to the second one.

So, this will be the new value it will come, this is the new value, new value and new value, so from here you can see that after first step I am getting whatever the matrix I am getting is a11 a12 a13 0 0 and then a22 a23 a32 and a33, I can write as 1 1 here, so this is same as this one. So, I am getting the new values after step 1, so this is the new value I am getting. So, that is my matrix after the step completion of step 1.

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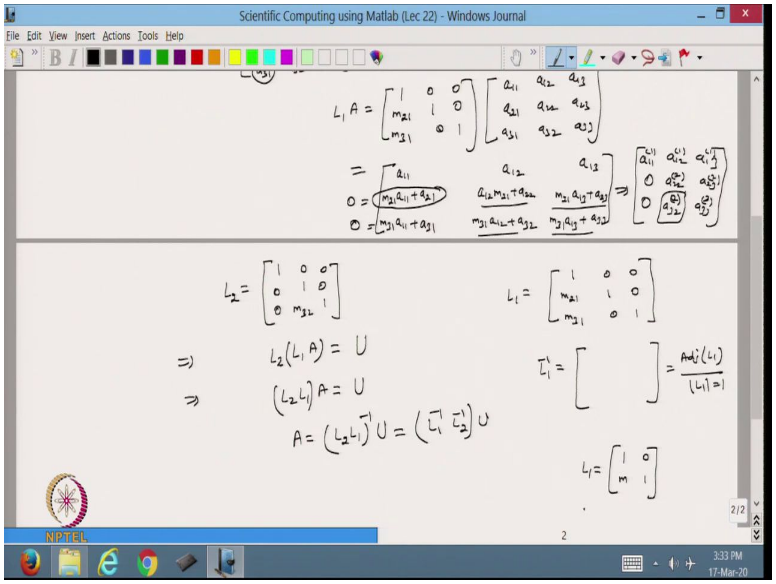

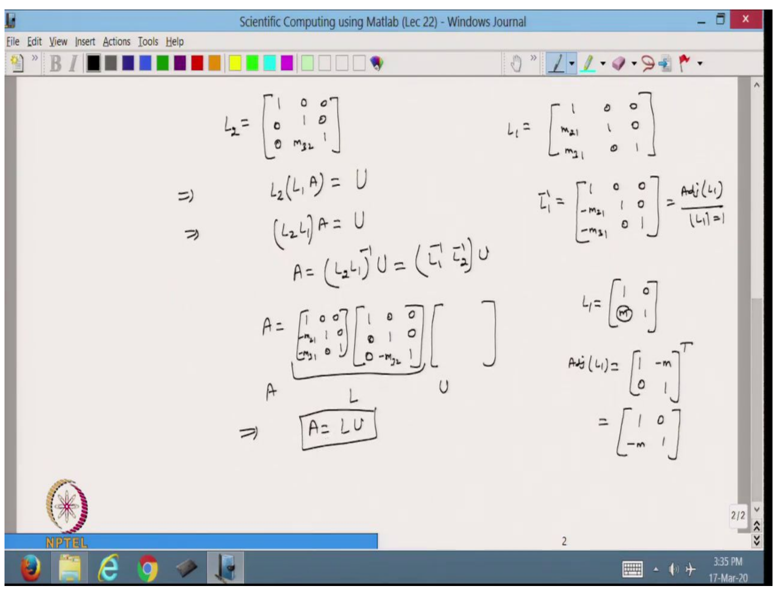

So, based on this one I can say that, because now I am able to make this 0, now I have to make this 0, so based on this one, I can say that I will take my L2 now, so L2, what will be the L2? It is 1 1 1; 0 0 0 and now in this case it will be 0 0 because I want only a multiplier here, so I will write it m32, so m32 multiplier I will take to make this element 0.

And based on this one I will get the new matrix, so from here I can say that first I am putting L1 A, so whatever the matrix I am getting then I am pre-multiplying by A2 and once I do this one, I will get my U that is my upper triangular matrix. So, based on this one, I can say that $L_2(L_1 A) = U$ Now, I know that the product of the lower triangular matrix is again the lower triangular matrix and in this case, the diagonal elements are 1 1, so both the cases, so this matrix is nonsingular so I can take the inverse.

So, based on this one I can write $A = U^{-1}$, this can be written as $L_1^{-1} L_2^{-1}$ and I know that the inverse of a lower triangular matrix is also lower triangular for this also, so in this case if you see from here so my L1 is this 1 0 0 and this is m21 1 0; m31 0 1 and I want to find its inverse, L inverse, so L inverse will be what? Because the determinant is 1 only so that will be equal to the adjoint of L1 divided by determinant.

So, the determinant that is already 1, so it will be the same as the adjoint of this one L1. And what is the adjoint of L1? So, if I take this one, suppose I take a 2 by 2 matrix, so 1 1 0 m, so this is my L1, so the I know that the adjoint of L1 will be I will put the cofactor of this, so what is the cofactor of this? Is 1 1, what is the cofactor of 0? So, it is m, so plus minus so it is -m and this is 0 and 1.

And then I will take transpose, so this will be equal to 1 1 -m 0. So, just you if you see this one the L inverse is just the same one except the sign is changing at this value. So, if you see this one, it will be same as 1 0 0; -m21 1 0; minus m31 0 1, so just the inverse will be same just the sign will change here, so from based on this one, I can say that this is my A and my L1 will be 1 0 0; -m21 1 0; -m31 0 1, so that is my L1 inverse.

And similarly I can find the L2 inverse, so that will be 1 0 0; 0 1 0; 0 -m32 1 and then the upper triangular matrix. So, this is a lower triangular matrix I will call it L and this is my upper triangular matrix and that is A, so based on this one I can say that A is equal to LU. So, I am able to reduce my matrix A into the lower triangular, into the upper triangular matrix. So, once I am able to find this value then how can I solve this one?

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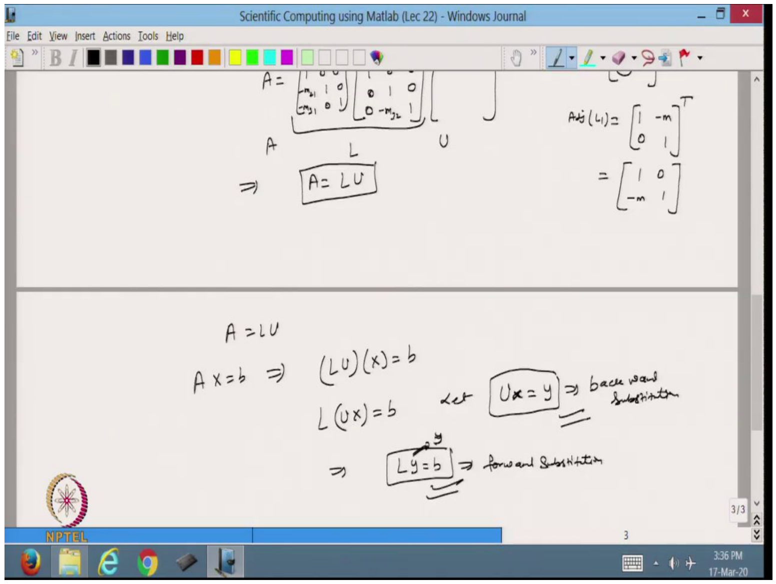

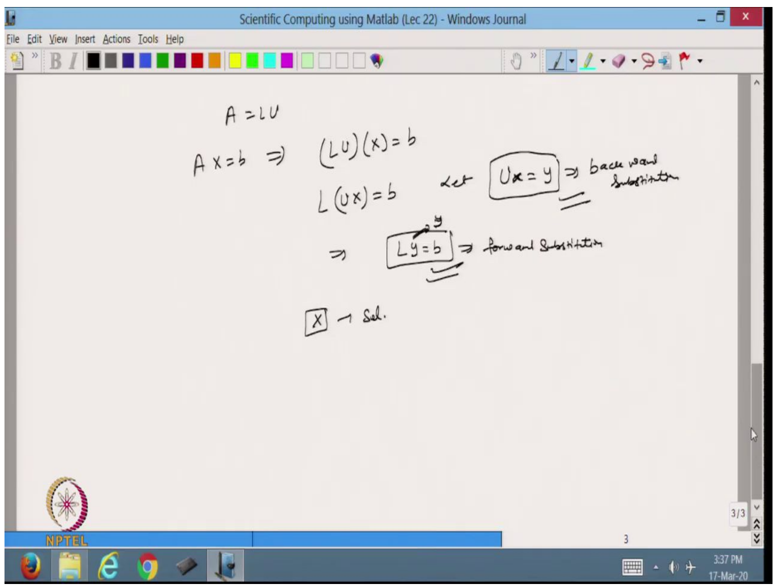

So, once I get this, I can say that my A is equal to LU, now my system was AX=b, so based on this one, I will find the value (LU) (x) =b. Now, from here I can write my Ux=b. So, let Ux=y, so from here, I can say that Ly= b. So, I know the value of L, I know the value b, I will first find the value of y. So, this value of y will be coming from forward substitution, because it is a lower triangular matrix.

So, once I get the value y I will put the value of y and then I will use the backward substitution. So, now based on the forward substitution and the backward substitution I am able to solve my system and then I will get the solution, so X is my solution. So, this is what we have done with the help of Gauss elimination method, that is the LU decomposition.

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So, the next one is the second one is Crout's, so in this case, what are we going to do? I am going to reduce matrix A into the LU decomposition form, so where I am taking my L as 1 1 1, 0 0 0 suppose I take 3 by 3 matrix so L I am taking 21 l31 l32 into U I am taking as u11 u22 u33 let I am take the 3 by 3 matrix u12 u13 and u23 and all other zero. So, in this case, I have 3 elements unknown here and 6 unknown here's, so from here I have 9 unknowns and A is a 3 by 3 matrix, so in this case also 9 values.

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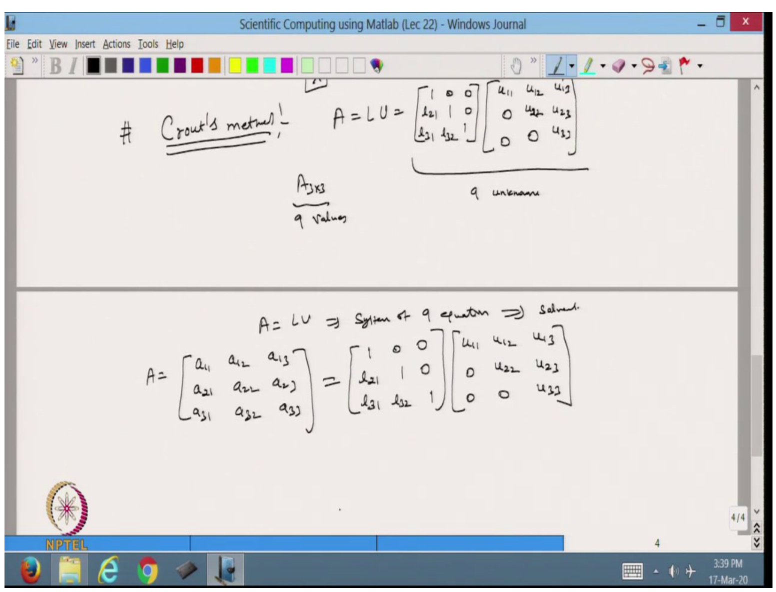

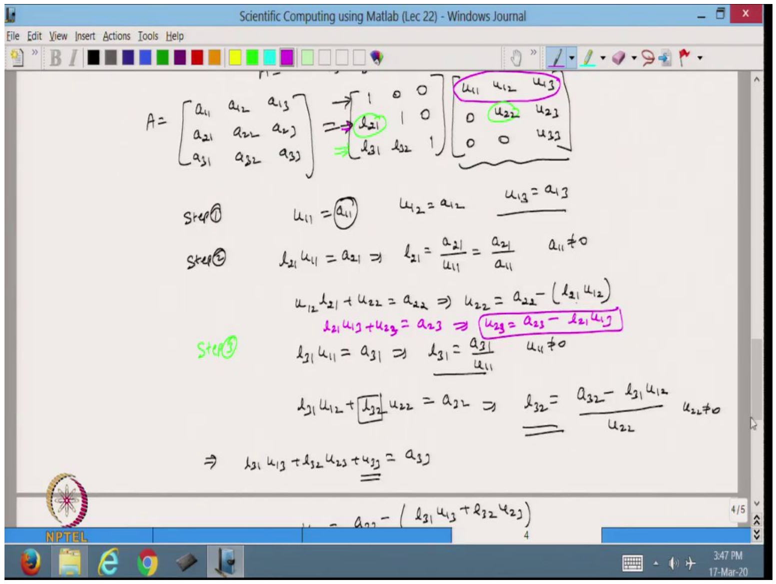

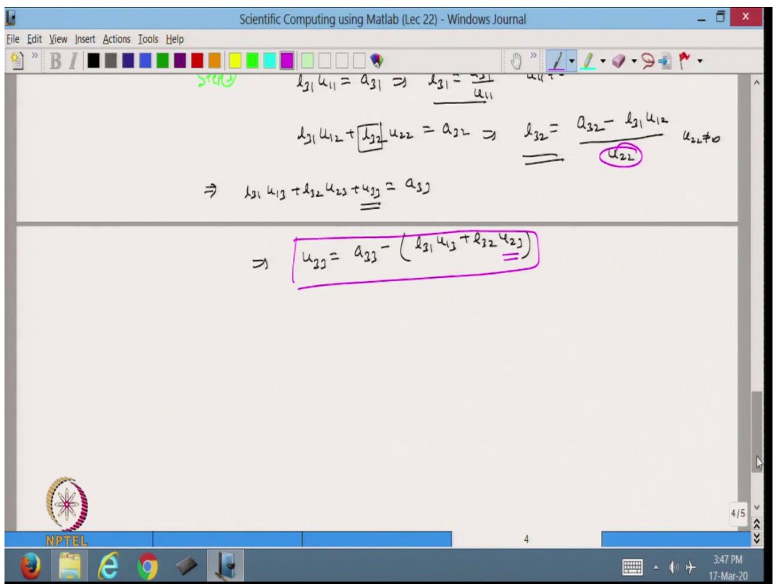

So, now if I compare this two matrices, so A=LU, so then it should be same as component wise each element of the matrix should be same, so based on this one I will get 9 equations, so I will get system of 9 equations and then I can solve this one to find the value, so which can be solved. So, how do you solve this one? So, let us see this one. So, I just take 3 by 3 matrices and then it will be more clear, so I will take my A as a11 a12 a13, so this is equal to 1 0 0, l21 1 0, l31 l32 1 into u11 u12 u33.

Now, the first step is to find the value of this first row, u11 u12 U13. So, step 1, so if I multiply this matrix with the first row, I will deal only with the first row of all the matrices. So, what will I get? I will get u11=a11, u12=a12, u13=a13. So, from here I am able to find the value of u1 u2 u3, so, this is my step 1. Now, step 2, so step 2 means I am going to use my second row, because the first row of this is known to me now, so this is I am able to solve, now I will use this to find out l21, l22 and l23. So, let us see how.

So, I will multiply this first column with second row I will get l21*u11=a21, so based on this one I can say that my $l_{21} = \frac{a_{21}}{u11} = \frac{a_{21}}{a_{11}}, \quad a_{11} \neq 0$. So, now it is giving the value of l21, now based on this one, now I will do the multiplication of the second column with the second row, so I will get a12*l21+u22=a22.

So, based on this one I can say because this is known to me, this is all three knows from where my u22 will be a22-l21 u12, so that is giving me the value of 22, so I should keep the elements in this order, so that I can write 21 12, so it means 22. So, that gives me the value of u22. So, now from here you can see that I am able to find the value of this and this element. So, now I will use the third one. So, this is my step number two, now I will get the step 3, so in the step 3 I will use the third row with all this one, so I will get from here, so my first will be $l_{31} u_{11} = a_{31}$ and from here I can say that my $l_{31} = \frac{a_{31}}{u11}, \quad u_{11} \neq 0$.

Then I will get l31 u12+ l32 u22=a32. So, once based on this one, now I want to find u22, this is already known to me, u22 is already known to me, where is the u22? This one, so now I want to find the value of this one, so from here I can write $l_{32} = \frac{a_{32} - l_{31} u_{12}}{u_{22}}$. Because u22 is already known from step number two, so based on this one I am able to find provided $u_{22} \neq 0.$

So, this is there and the last step, so from the last step, I will get l31 u13 +l32 u23 +u33=a33. Now, from here I will find the value of u33 will be a33-l31 u13 + l32 u23, so this all the values are already known to me l31 is there, so l31 I am just found, u13 is already known to me, l32 this one, u23, so I just finished the two steps I will do the third step also, so this one it will give you l21 u13 + u23=a23. So, from here I will get the value of u23. So, u23=a23-l21 u13, so that is the way we can find the value of u.

So, based on this one I can put the value of u123 here and then I will get the value of u33. So, this is the way I can find all the values provided because in some places I have put the condition that this should not be equal to 0, this should not be equal to 0, so that is the condition. So, based on this one, now we can write the algorithm for this, so what is the algorithm for crout's method?

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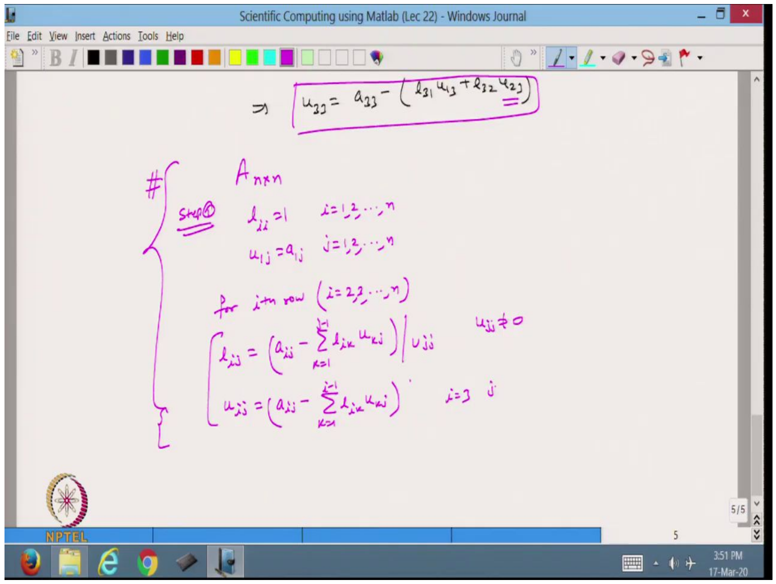

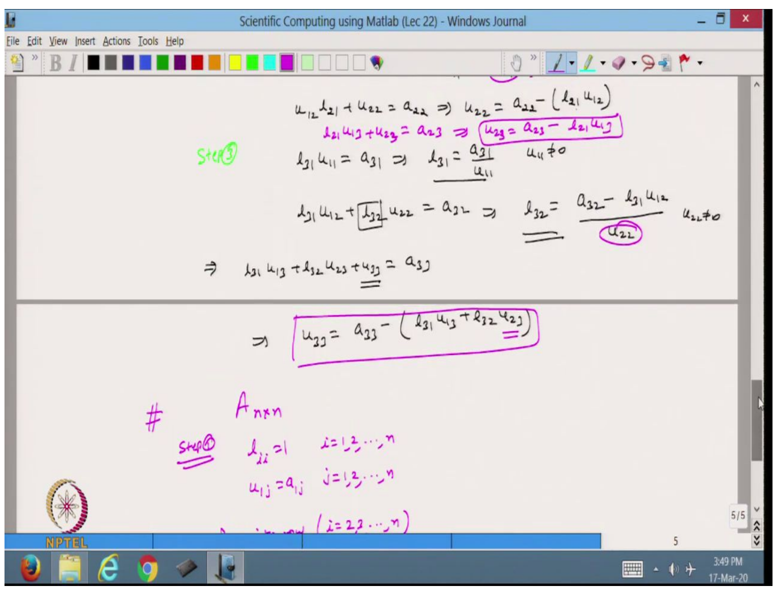

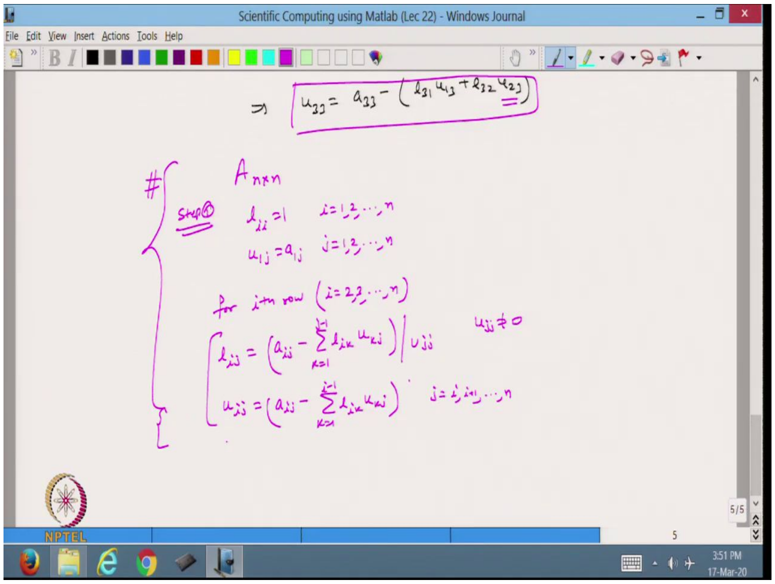

So, this is the algorithm that A is my n*n matrix then step 1, my $l_{ii} = 1, \quad i = 1, 2 \ldots, n$ and my $u_{ij} = a_{1j}$, where $j = 1, \ldots, n$,, so this is the first step, then for ith row, so whatever the row I am taking so i is basically it is 2, 3...n, so any ith row I am talking about my

$$l_{ij} = \frac{\left(a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}\right)}{u_{jj}}, \quad u_{jj} \neq 0$$

and then I will can find the value of uij, so uij is no division by any number, so in this case, it is

$$u_{ij} = \left(a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}\right), \quad j = i, i+1, \ldots, n$$

So suppose I take i=3 in this case, so k will be 1 to 2, so it is l31 l31 u13 because I am taking j is also 3 +l32 u3, so that is the so this is the algorithm to find out the solution using the Crout's methods. So, this is true for all j for which j=i,i+1,...,n. So, that is the algorithm for using Crout's methods.

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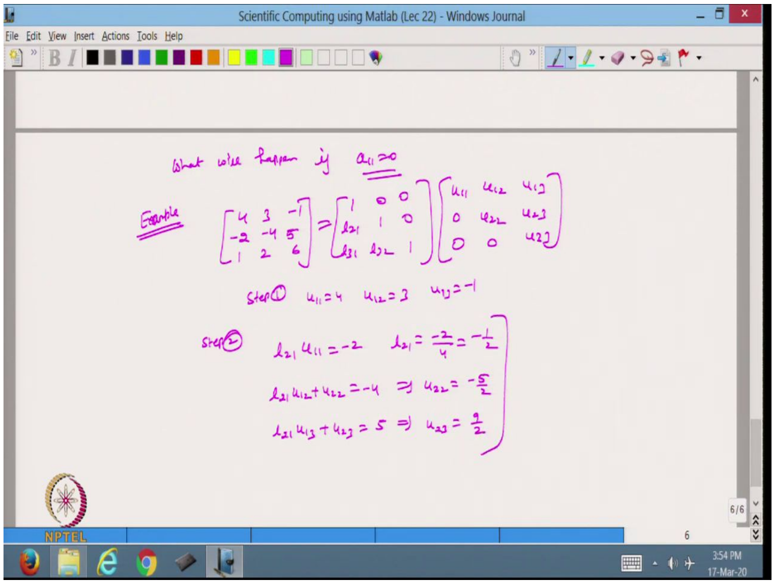

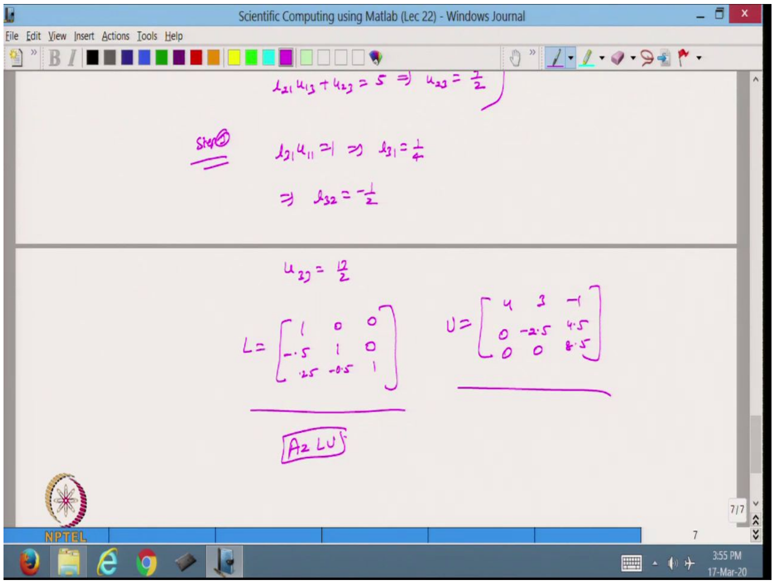

Now, the problem is in the Crout's method as we have seen, what will happen if my a11=0? So, let us take one example, I take a matrix 4 3 -1, -2 -4 5, 1 2 6, so this is my matrix and I will apply the Crout's method for this one, so l21 1 0, l31 l32 1 and this is my u11 u12 u13, so in this case my you u11=4, u12=3, u13=-1, so that is a step 1, so after that step 2.

So, step 2 gives me that value of l21 u11=-2, so l21=-2/4, so it is -1/2. Then I will find the value l21 u12+u22=-4, so that gives me the value of u22=-5/2. Now, the last one is l21 u13+u23=5. So, that gives me the value of u23=9/2, so this is what I have done.

Now, in step 3 I will find the value so l31 u11=1, so that gives me my value of l31=1/4. And if you do the calculation more, then you will see that l32=-1/2 and u33=17/2. So, based on this one we are able to find the L, so L is coming 1 0 0, -0.5 1 0, 0.25 -0.5 1.

So, this is my L and U is coming 4 3 -1, 0 -2.5 4.5, 0 0 85, so that in my LU decomposition using the Crout's method. So, we will stop here today and in the next lecture, we will show some results or some cases when the Crout's method is not applicable and then we will go further. So, thanks for watching. Thanks very much.