Scientific Computing using Matlab
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Lecture 20
Gauss Elimination Method for Solving Linear System of Equation

Hello viewers, welcome back to the course on Scientific Computing using MATLAB. So today we will continue with our lecture 20. So, in the last lecture, we have just started with the Gauss-Elimination method for the direct method.

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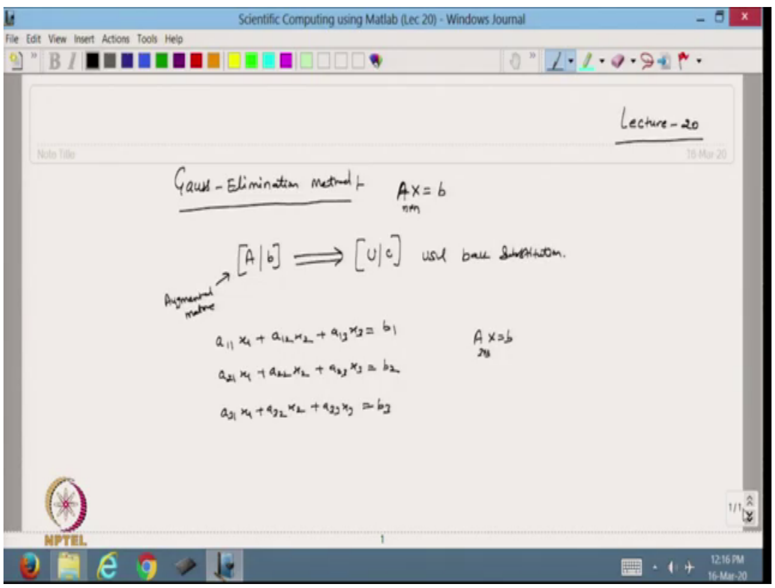

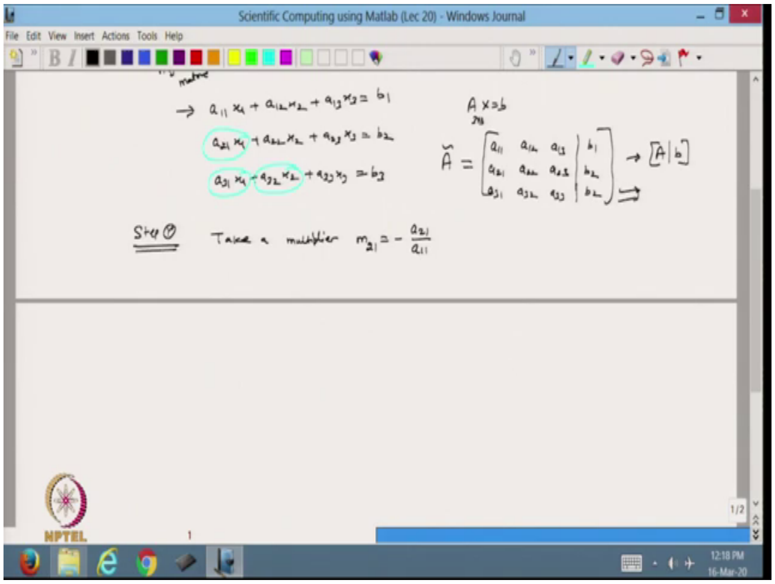

So, we are writing the Gauss Elimination method. So, in this case, what are we going to do? I have a system Ax = b that is n x n. And then I will write down my augmented matrix A. So this is my augmented matrix. And we will convert this matrix into an upper triangular matrix. And then using back substitution we will get the solution. We solve the system.

So let us start with the Gauss-Elimination method. So for example, let us take a 3 x 3 system. So I will just define a 3 by 3 system:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
$$

So this is my system, Ax = b and I am taking a 3 x 3 matrix. Now, this is my system of equations. Now what I want to do, I want to reduce this one into the upper triangular matrix. So I want to make this element 0. So this element, this element and this element, this one I want to make 0, because only then you will be able to get the upper triangular matrix. So let us start with this one.

So step 1. So, in the step1, we define a multiplier. So, I call it m_{21}. So, what is the m_{21}? What I will do is that I will divide this row by a_{11}. So, I will get the 1 only and to remove this value of a_{21}. So, what should I do? So, before that I just write the augmented matrix:

$$\tilde{A} = \begin{bmatrix} a_{11} & a_{23} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

So, now whatever I am going to do, I am going to do with this augmented matrix. Generally we represent it by tilde. Now take a multipliers a_{21}. So, what I will do, I will divide this first row by a_{11} and multiply by - a_{21} , and then I will add to eqn. (1), so this will make it 0. So I will take a multiplier:

$$m_{21} = -\frac{a_{21}}{a_{11}},$$

So this is a multiplier I am going to find out.

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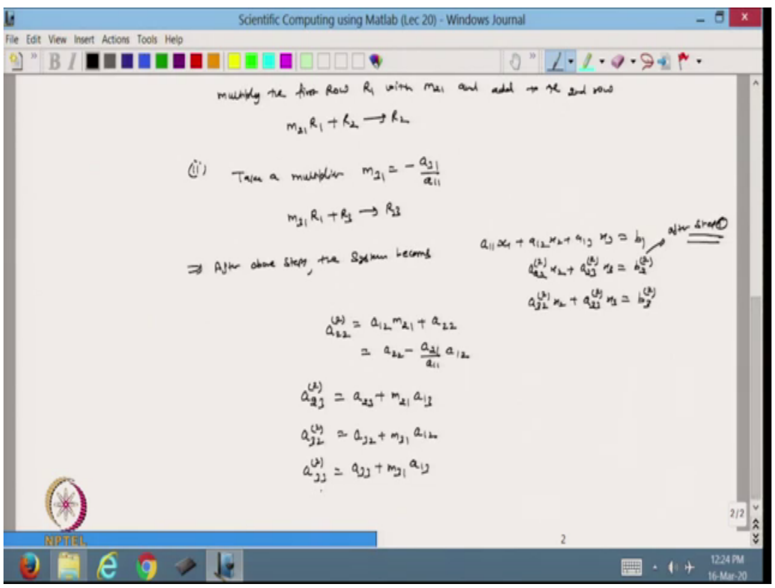

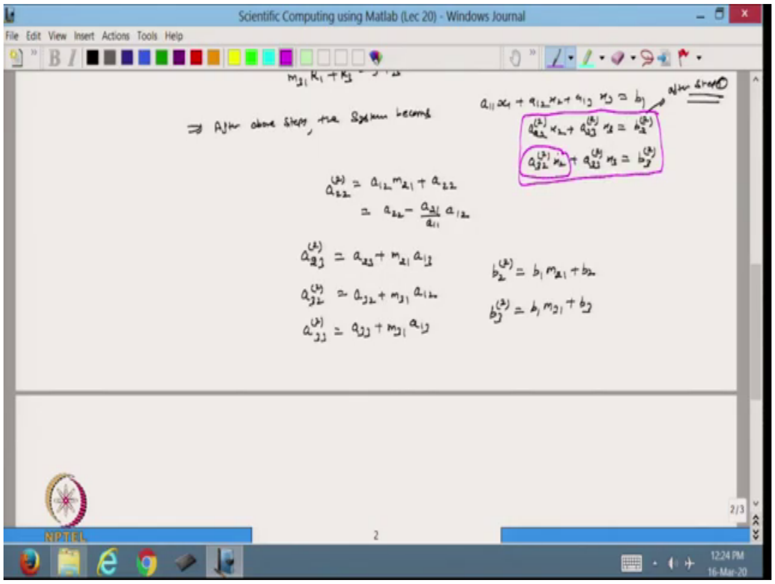

Now multiply the first row that is $m_{21}R_1 + R_2$ and add to the second row. Because I want to make this element 0 so that in my condition, so what I do, I will multiply the first row with m21 and add to the second row. So, I can write as a $m_{21}R_1 + R_2$. And the effect goes to the R_2. So, I am going to do the first one, the second one is the same.

Now, my next after doing this, I want to make this element also 0. So, how to make this one, the same one? Take a multiplier m_{31} because I want to make this element 0. So, I will call it m_{31}. I will call it $m_{31}R_1 + R_3$ and the effect goes to R_3. So, after these 2 steps, my system will reduce to.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 &= b_2^{(2)} \\
a_{32}^{(2)}x_2 + a_{33}^{(2)}x_3 &= b_3^{(2)}
\end{aligned}
$$

So that is the after step 1, or after iteration one or after step 1.

I am getting my system in this way. Now, if you see from here, now I have to deal with this 2 by2 system. So I have to remove this value to make it the upper triangular. So now I do step 2.

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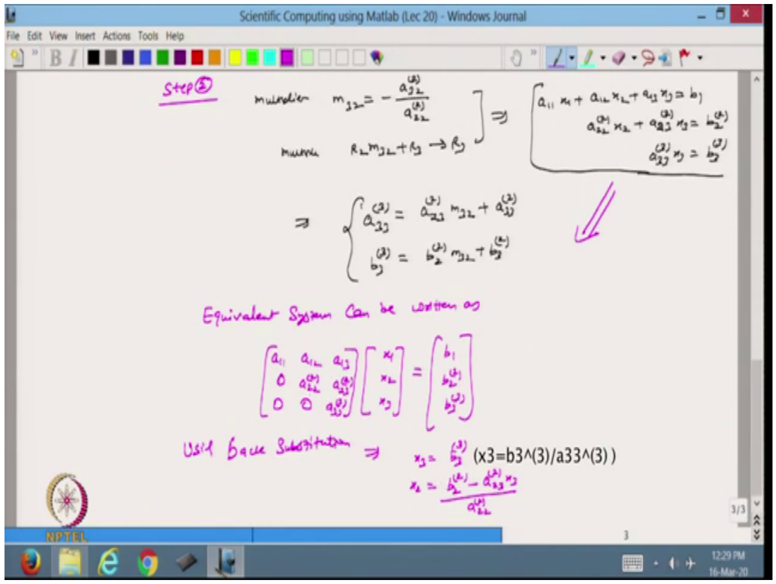

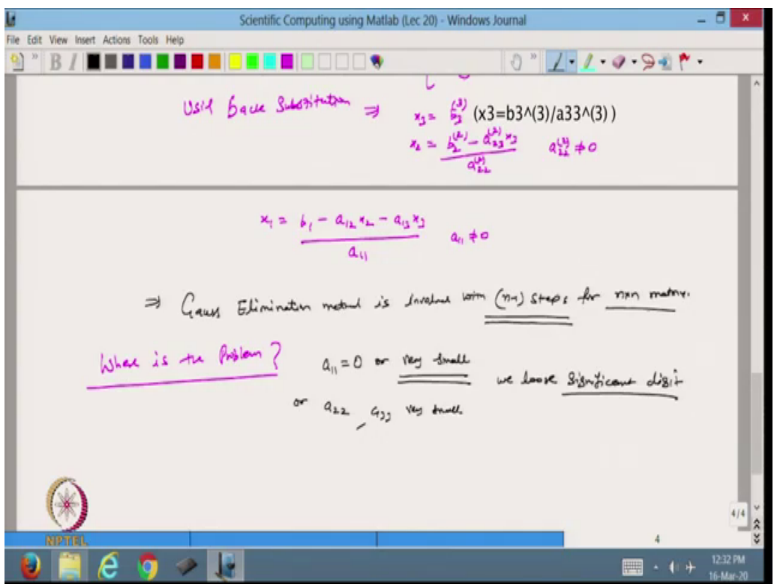

So in step 2, I am dealing with this system. So, this system I can call it a double star. So, I will apply the same way. So, in this case, what I will do is now, I will define my new multiplier. So I will define multiplier m_{32} because I want to make this element 0, so that will be - a_{32} /a_22 and whatever we are getting at this step, so, I will do the process here. So, I will write down multiply m_{32} R_2 + R_3 and the effect is going to R_3.

So, after doing this. I will get my system because here only I have to deal with only one element then I will get the system.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{22}^{(2)}x_2 + a_{23}^{(2)}x_3 = b_2^{(2)}$$
$$a_{33}^{(3)}x_3 = b_3^{(3)}$$

So, equivalent system can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} \\ 0 & 0 & a_{33}^{(3)} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{(2)} \\ b_3^{(3)} \end{bmatrix}$$

Now, using back substitution we are getting :

$$x_3 = \frac{b_3^{(3)}}{a_{33}^{(3)}}, \quad x_2 = \frac{\left(b_2^{(2)} - a_{23}^{(2)}x_3\right)}{a_{22}^{(2)}}, \quad x_1 = \frac{(b_1 - a_{12}x_2 - a_{13}x_3)}{a_{11}}$$

So, by the back substitution, now we are able to solve my system.

So, this is called the Gauss-Elimination method, but only if you see that the so, we have done this 1 for 3 by 3 matrix. So, the same way we can go for n by n matrix and So, from here I can write down how the Gauss-Elimination method is involved with n -1 steps for n x n matrix. Because in the previous one we have a 3 by 3 matrix so within 2 steps we are able to solve this one.

So, if I n x n matrix using the n -1 steps we are able to convert the matrix to the upper triangular and then we can solve the system. So, this Gauss-Elimination involves n-1 steps. Now, the problem comes here. So, where is the problem? So, the problem is that, what will happen if $a_{11}=0$ or very small? So, if it is 0 then we cannot proceed or if it is very-very small then we already know that we lose a significant digit.

In the previous lecture we have seen that if we are dividing by a very small number then we lose significant digits. So, what will happen if I have a_{11} or a_{22} or a_{33} are very small. So, in that case may Gauss-Elimination method will not work and it will not give you the exact solution.

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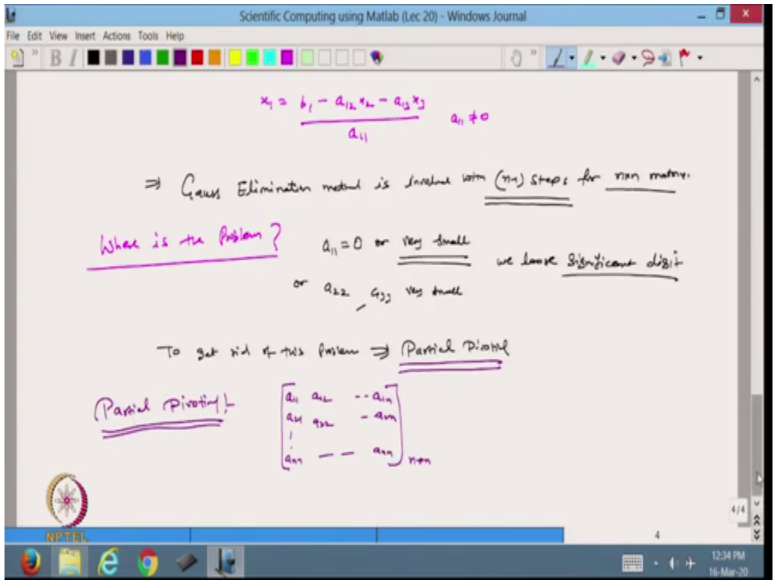

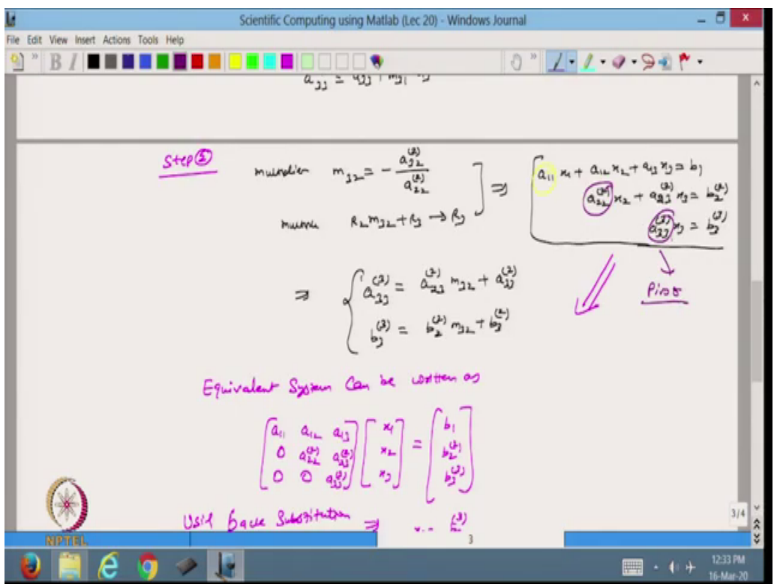

So, to get rid of this problem, what do we do? We take a pivoting, partial pivoting. So, in the previous slide, these elements we call it, this element maybe this element and this elements, they are called pivot. Pivot means it is a very important element. So, now, we have to deal with how we can remove this difficulty, when we have these elements at the diagonals are very small in number.

So, that can be done with the help of partial pivoting. So, what is the partial pivoting? So, in the partial pivoting what we do is suppose we have a system. So, let you have a system I have a n x n system.

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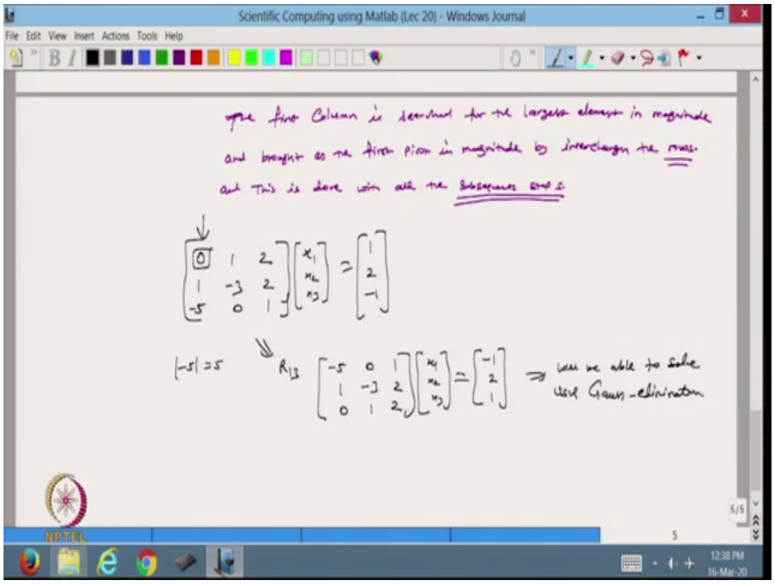

So we will do step 1. So in step 1, what do we do? So then the step 1 of the elimination, the first column is searched for the largest element in magnitude. Okay so we have to deal with the magnitude. It may be negative also and brought as the first pivot in magnitude by interchanging the first and by interchanging the rows. Rows means whatever the row has the largest element with the first row.

So, we can interchange that one and the same procedure and this is done with all the subsequent steps. So, this is the procedure basically. So, the first column is searched for the largest element in magnitude and brought to the first pivot in the magnitude by interchanging the row and this is done with all the subsequent steps. So in this case what will happen?

Like suppose I have a system like this one,

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & -3 & 2 \\ -5 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

if I want to apply the Gauss-Elimination then this element a_{11}= 0. So, what I do, I will find out the first step is that I will find out the largest value in magnitude. So may be in magnitude the largest value is this one because if I take the modulus value of minus 5, that is a 5. So what I do in this case, I will do the R_{13}.

So, after doing the R_{13}, what I will get,

$$\begin{bmatrix} -5 & 0 & 1 \\ 1 & -3 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Now, I can apply my Gauss-Elimination method and based on this Gauss-Elimination method, because it is already 0, so then it is useful for us.

So, in this case, I will find the upper triangular matrix and we will be able to solve using Gauss-Elimination. So this is called the partial pivoting. So generally we go, I will always go for partial pivoting. But there is another term also complete pivoting.

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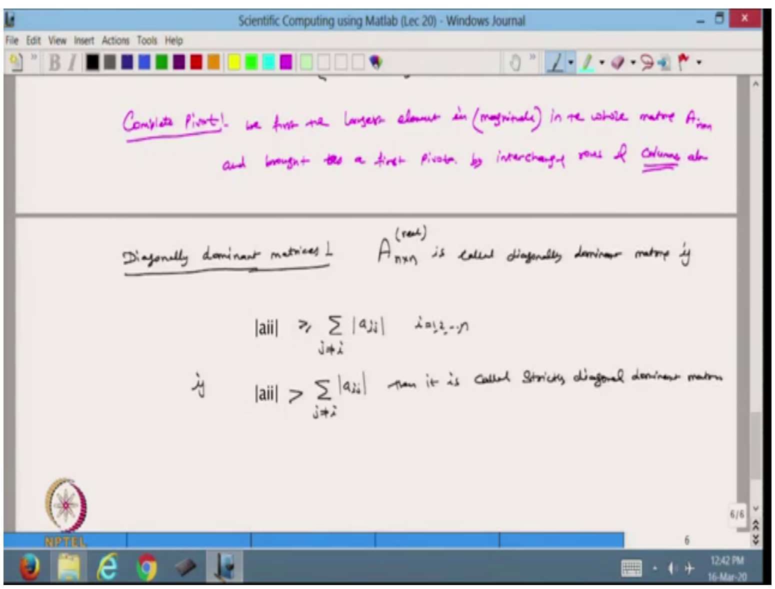

So we hardly go for this complete pivoting. So in that case, we find the largest element in magnitude in the whole matrix. So, my matrix is n x n. So, we will find the largest element in magnitude in the whole matrix and bring it as a first pivot. So, in this case what is happening? So, first we went by interchanging rows and columns also. So, in this case the variable may be shifting. So, that is called the complete pivoting.

So, the problem with that is that in this case we can interchange the row so, row wise okay but when we interchange the column, then the position of the variable will change. So, then it will make life much more complicated. So, we generally do not go for complete pivoting, we always go for the partial pivoting.

Now, there are some other types of matrices that are very useful when we are dealing with the Gauss-Elimination method and we call it diagonally dominant matrices. So, this is called the diagonally dominant matrices. For example, I have a matrix A. So, a matrix is so I am talking about the real matrix, okay is called diagonally dominant matrix if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad i = 1, 2, 3, \ldots, n.$$

It means that the diagonal elements, the magnitude of the diagonal element is always greater than equal to the summation of the magnitude of the other elements. So if this is true, then we call it the diagonal dominant matrix if,

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \quad i = 1, 2, 3, \ldots, n.$$

Then it is called a strictly diagonally dominant matrix.

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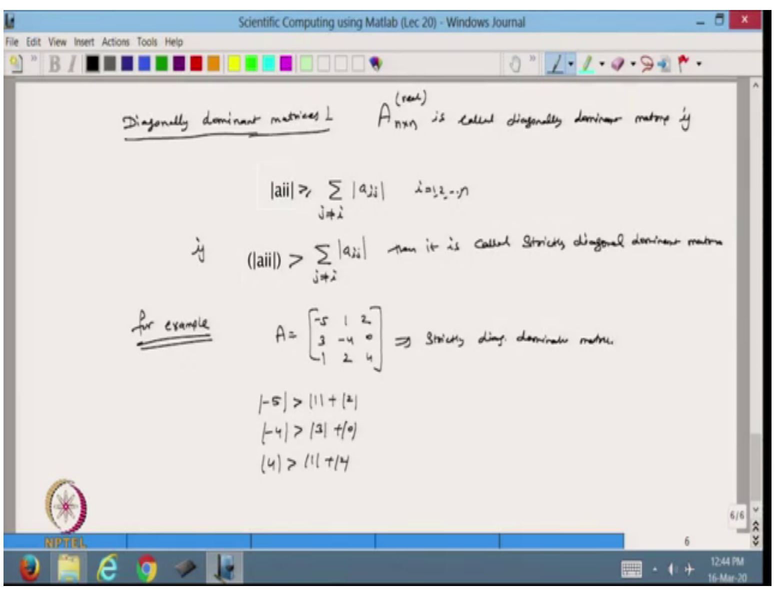

So for example I take a matrix A:

$$\begin{bmatrix} -5 & 1 & 2 \\ 3 & -4 & 0 \\ 1 & 2 & 4 \end{bmatrix}$$

So in this case what is happening now:

$$|-5| > |1| + |2|$$
$$|-4| > |3| + |0|$$
$$|4| > |1| + |2|$$

So this is true, then I say that this is a strictly diagonally dominant matrix.

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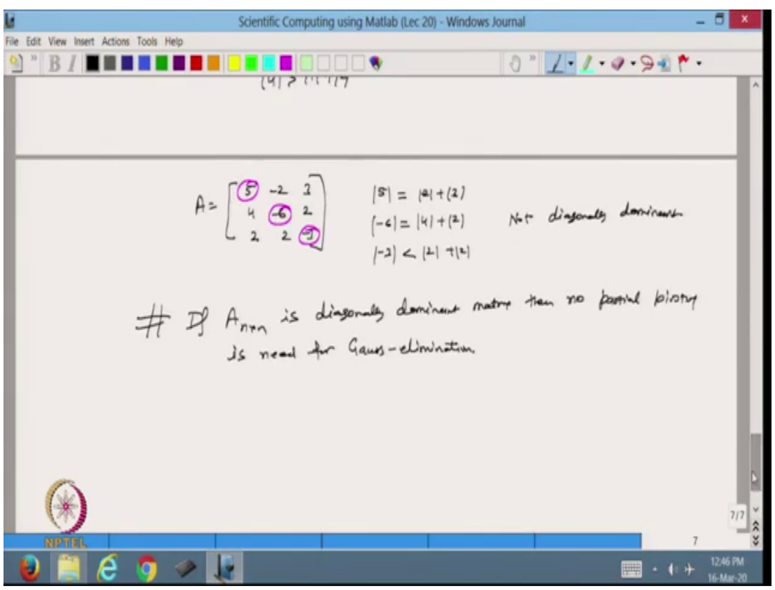

Or maybe I can take another example:

$$A = \begin{bmatrix} 5 & -2 & 3 \\ 4 & -6 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

with

$$|5| = |-2| + |3|$$
$$|-6| = |4| + |2|$$
$$|-3| < |2| + |2|$$

So this here it is not satisfying this condition, so not diagonally dominant.

So from here, I can make a statement that if a n x n is a diagonally dominant matrix, then no partial pivoting is needed for Gauss-Elimination. So, in that case no need of doing the partial pivoting for applying the Gauss-Elimination method. So, this is all about in this lecture.

So, I hope you have understood how we can apply the Gauss-Elimination method and what is the partial pivoting or the diagonally dominant matrix. So, in the next lecture we will continue with this one. So, thanks for viewing. Thanks very much.