Scientific Computing Using Matlab

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Lecture 18

Linear System of Equations

Hello viewers, welcome back to the course on Scientific Computing Using Matlab. So today we are going to start with the lecture 18 and we are going to start with the unit next unit.

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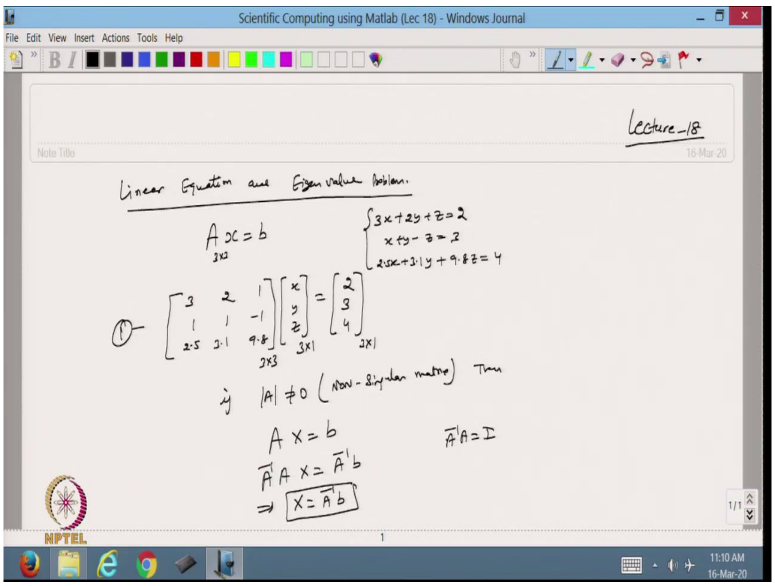

So this is about linear equations and Eigenvalue problems. So in this case we are going to deal with the first with the linear equation. So suppose we have a system like A x = b because you must have seen this type of system anywhere. Most of the place because whenever you are dealing with the like I have this equation

$$3x + 2y + z = 2$$
$$x + y - z = 3$$
$$2.5x + 3.1y + 9.8z = 4$$

So I do not know whether the solution exists or not but suppose I take like this one. So this type of equation you can solve because I can write this system in the form of A x =b

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & -1 \\ 2.5 & 3.1 & 9.8 \end{bmatrix} ; \; b = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} ; \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

So this is the linear system of equations which has in this case deals with the 3 by 3 matrix. And that is 3 x 1 and this is 3 x 1. so this is a column vector and that is a matrix. So now we know that if I have to solve this system and then if the determinant of A is non zero so it is a nonsingular matrix, then we can solve this system A x = b.

So now if the A is a nonsingular matrix I can re-multiply this with A^{-1} and I know that A^{-1} A= I that is the identity matrix. So from here I can write x = A^{-1} b. So this one we can do we can find the inverse.

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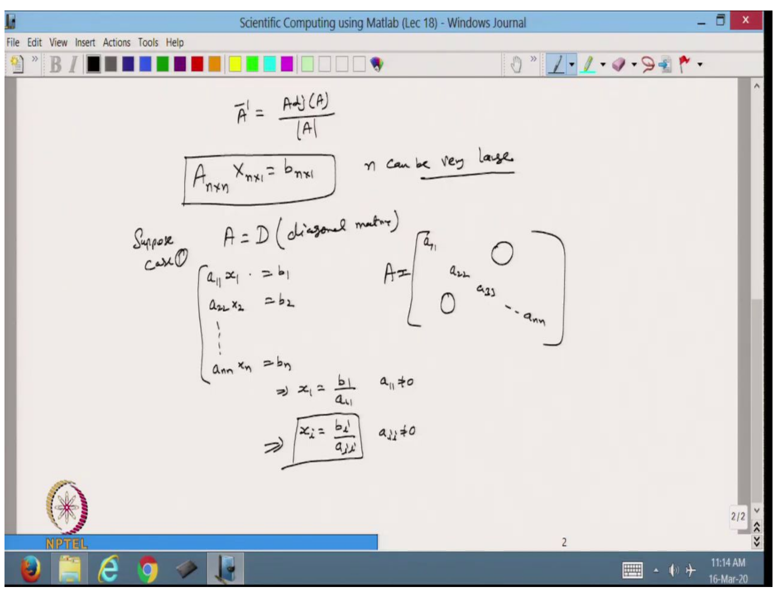

And the inverse we can find with the help of adjoint, so we also know how to find the A inverse

$$A^{-1} = \frac{adj(A)}{|A|}$$

. So this is the way we generally solve the 3 by 3 system and what we have done in the school level.

Now things have changed because now what will happen we may have the system A_{nxn} . Suppose it can be a very large system and this is my vector n cross one and this is equals to b_{nx1} . So that is my system and n can be very large. So in this case now it is very difficult to find the inverse of the matrix or even it is very difficult to find whether the matrix is singular or nonsingular. Suppose somehow we find that it is a nonsingular then it is also very difficult to find the inverse for a very large system, even with the help of the computer.

So in this case we are unable to solve the system then we have to use different methods to find out the solution of the system. Now suppose I take a matrix A = D, a diagonal matrix. So my matrix A is a diagonal matrix, and let us write the system, so in this case my matrix will be

$$a_{11}x_1 = b_1$$
$$a_{22}x_2 = b_2$$
$$a_{33}x_3 = b_3$$
$$.$$
$$a_{nn}x_n = b_n$$

So my matrix is a diagonal matrix and based on the diagonal I will get the system of equations so from here you can see that, I can solve this equation exactly

$$x_1 = \frac{b_1}{a_{11}}, \; x_2 = \frac{b_2}{a_{22}}, \; \ldots, \; x_n = \frac{b_n}{a_{nn}}.$$

So there is a solution we are able to find.

Now in this case it does not matter because my matrix is a diagonal matrix so it can be very large system but for the very large system if the matrix is the diagonal matrix we can find the exact solution with a very small time with help of this one, so this expression give you the exact value of the solution. So we are able to do this one, so I can write this as a case 1 because my matrix is a diagonal matrix, a very simple matrix.

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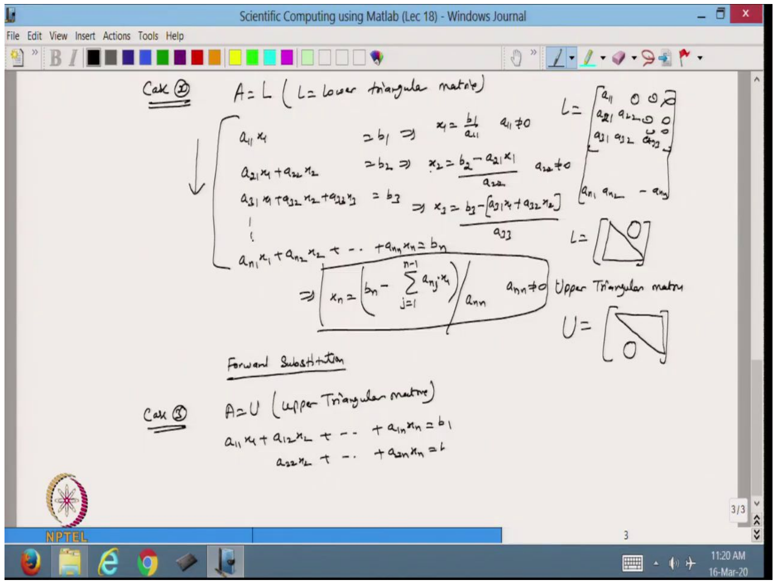

So let us take another example, case 2. So in the case 2, suppose I have my matrix A and that is equal to the lower triangular matrix, so L = lower triangular matrix. So the lower triangular matrix

$$L = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & & a_{22} & 0 \\ \vdots & & \ddots & \vdots \\ a_{n1} & & \dots & a_{nn} \end{bmatrix}$$

So this matrix is basically if you see then this is the values the elements in the matrix and all the elements above the main diagonal are 0, so this is the lower triangular matrix. Similarly, we can find the upper triangular, upper triangular matrix so that it will be U.

$$U = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & & a_{22} & a_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & & \dots 0 & a_{nn} \end{bmatrix}$$

So let us say that A is a lower triangular matrix, so if it is a lower triangular matrix then by equation, we get

$$x_1 = \frac{b_1}{a_{11}}\ ,\ x_2 = \frac{(b_2 - a_{21}x_1)}{a_{22}},\ \ldots\ ,\ x_n = \frac{\left(b_n - \sum_{j=1}^{n-1} a_{nj}x_j\right)}{a_{nn}}$$

So we are starting with the x_1 so that is called the forward substitution and the system is able to be solved using the forward substitution. So in this case also you can see that we are able to find the value of the solution exactly even for the large number of systems. So the same case I can do the third case when my A is equal to U that is the upper triangular. So in the upper triangular matrix what we will get I get my system of equations:

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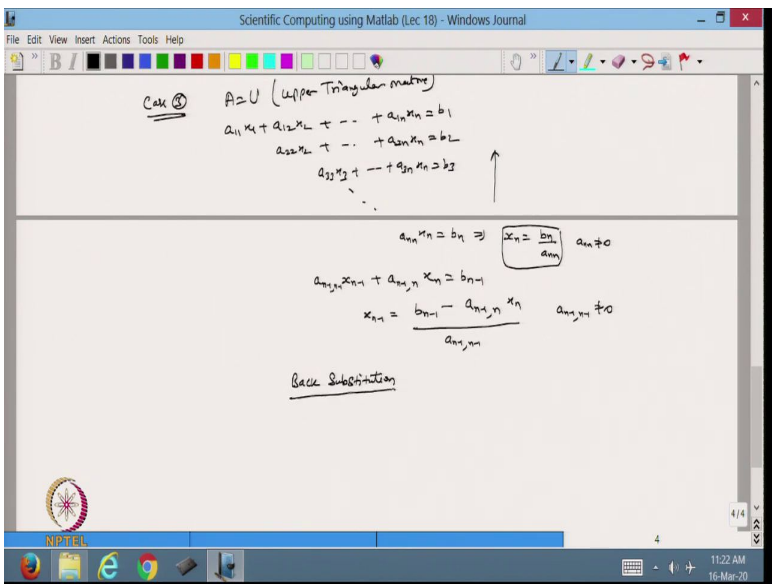

$$x_1 = \frac{\left(b_1 - \sum_{j=2}^{n} a_{1j}\ x_j\right)}{a_{11}},\ \ldots,\ x_{n-1} = \frac{\left(b_{n-1} - a_{n-1,n}\ x_n\right)}{a_{n-1,n-1}}\ ,\ x_n = \frac{b_n}{a_{nn}}.$$

So in this way we are able to find all the values of the variables x_1, x_2, x_3 and the direction of the variable will be from down to upward. So this process is called back substitution.

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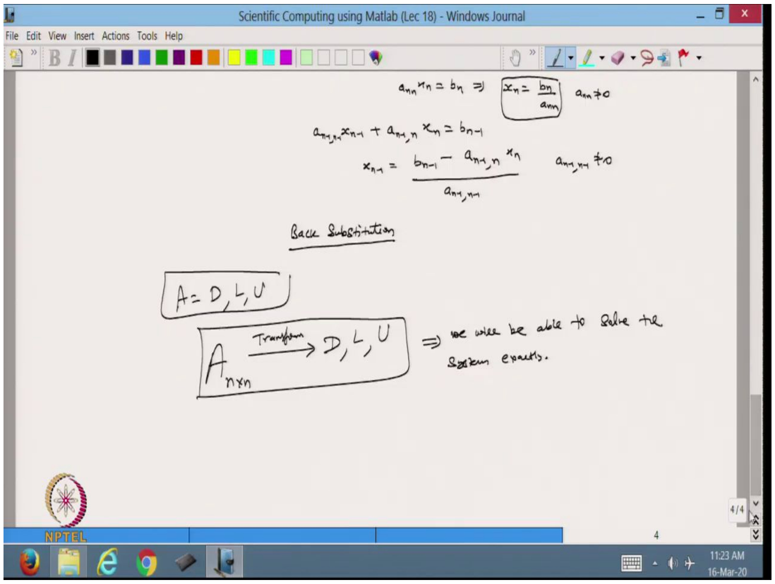

So now we have seen that if my matrix A is of the type D, L, U, then we are able to solve the system very easily. And then we are able to find the solution exactly. So our main purpose is now to transform the matrix, so this is my matrix that is n x n matrix. And what I will do, I will convert this matrix, transform this matrix either in diagonal form in lower triangular or in the upper triangular.

And then we will be able to solve the system very easily, so that is our purpose. Now to solve the system, so if I am able to do this one then we will be able to solve the system exactly. So this is our purpose now.

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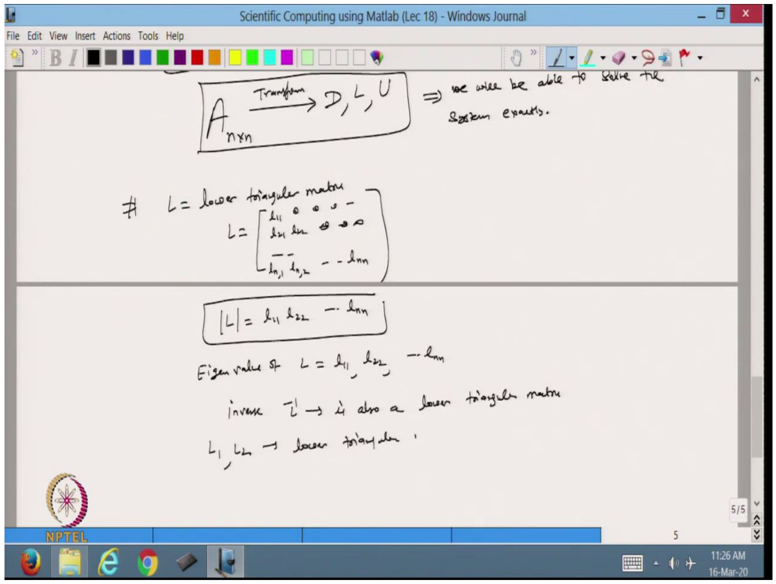

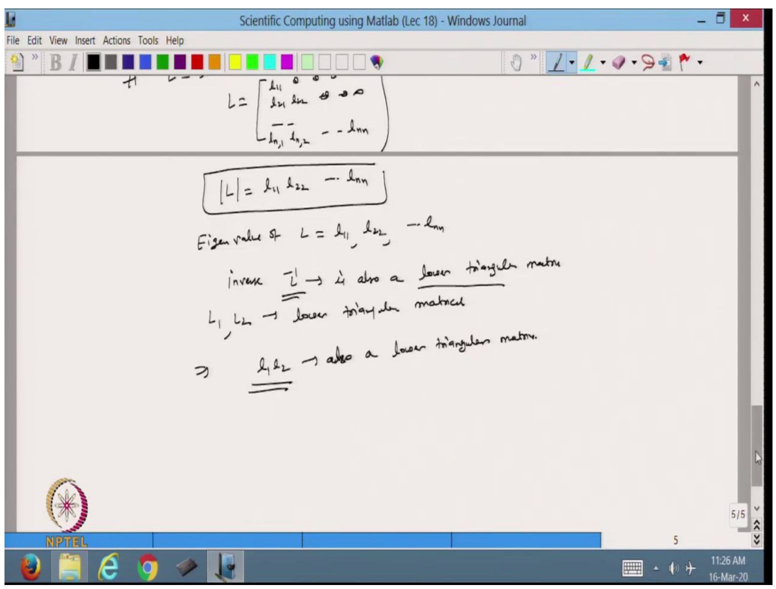

So now before we are starting with the process, so let us define some, discuss the some properties of the lower triangular diagonal or upper triangular matrix. So in the lower triangular matrix now let us start with the L is equal to:

$$L = \begin{bmatrix} l_{11} & \dots & 0 \\ l_{21} & l_{22} & 0 \\ \vdots & \ddots & \vdots \\ l_{n1} & \dots & l_{nn} \end{bmatrix}$$

So in the lower triangular matrix the determinant

$$|L| = l_{11} \times l_{22} \times l_{33} \dots \times l_{nn}$$

Now product of the diagonal elements, also Eigenvalues of L again will be diagonal elements. Inverse, L inverse will be again, so inverse of a lower triangular matrix is also a lower triangular matrix. Now suppose I have two lower triangular matrices L_1 and L_2, both are lower triangular matrices then L_1 L_2 is also a lower triangular matrix.

So the product of the two lower triangular matrices is also triangular matrix, inverse of a lower triangular matrix is also a lower triangular matrix, the determinant is the product of the diagonal elements and the Eigenvalues is also the diagonal elements. So this is the some properties of the lower triangular matrix.

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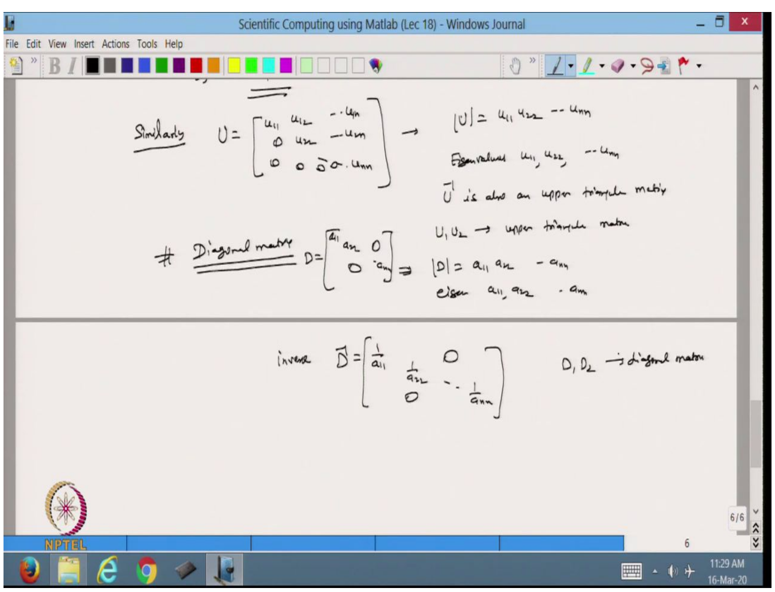

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ \vdots & u_{22} & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

So this is the upper triangular matrix, so in this case also if I take U,

$$|U| = u_{11} \times u_{22} \times u_{33} \dots \times u_{nn}$$

determinant of U that is equal to the product of diagonal elements. Eigenvalues of U will be u_{11}, u_{22}, … , u_{nn}; so these will be the Eigenvalues.

Also the same way U inverse is also an upper triangular matrix. If I take the product of U_1U_2, then it is also upper triangular. So from here you can see that it is very easy to work with lower triangular or the upper triangular matrix, even if you take the, because we have already discussed that D,

$$D = \begin{bmatrix} a_{11} & \dots & 0 \\ \vdots & a_{22} \quad \ddots & \vdots \\ 0 & \dots & a_{nn} \end{bmatrix}$$

the case, third case, the diagonal matrix, so from here you can see that the diagonal matrix.

So it can be treated as a lower triangular matrix or the upper triangular matrix.

$$|D| = a_{11} \times a_{22} \times \dots \times a_{nn}$$

Eigenvalues are the same as a_{11}, a_{22}, a_{nn}. So the same way and the inverse of the diagonal elements,

$$D^{-1} = \begin{bmatrix} a_{11}^{-1} & \dots & 0 \\ \vdots & a_{22}^{-1} \quad \ddots & \vdots \\ 0 & \dots & a_{nn}^{-1} \end{bmatrix}$$

So that will be the inverse of the diagonal matrix, so that is again the diagonal matrix.

And D_1D_2 is again the diagonal matrix. So D_1 and D_2 I am taking two diagonal matrices and multiplying that one, so that is also another property of the diagonal matrix. So we can, from here we can see the lower triangular, upper triangular matrices are very easy to work with. So our next purpose is to transform any matrix into either of these forms.

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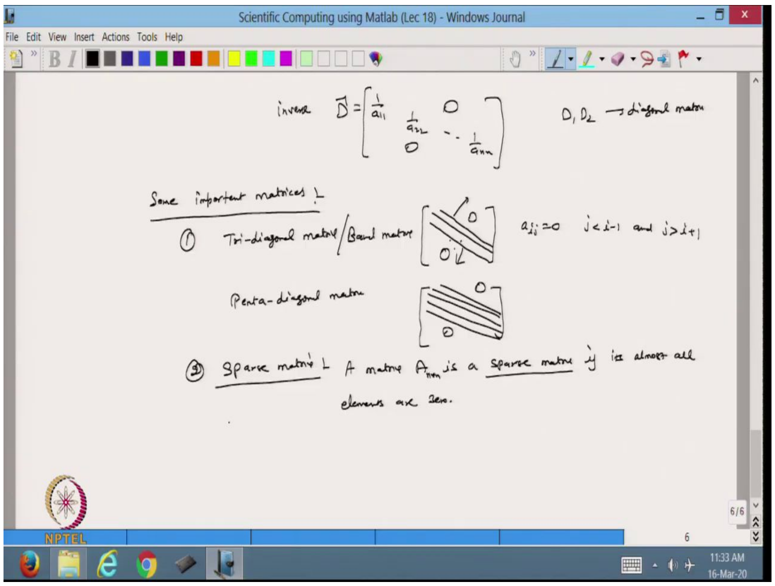

Now we, few more properties of the matrices we want to define. So let us define some important matrices. The first one is we call it tridiagonal matrix, so tridiagonal matrix is a matrix in which we have nonzero elements at the main diagonal, super diagonal and the sub-diagonal and all the other values are 0. So this is called the tridiagonal matrix. So the tridiagonal matrix we can define as $a_{ij} = 0, j < i-1$ and $j > i+1$.

So this is tridiagonal matrix, similarly I can define penta-diagonal, so in the penta-diagonal matrix the same way we have main diagonal, sub-diagonals, super diagonals and all other values are 0. So in this case this will be the penta-diagonal matrix. The second one is, so this matrix is also called the band matrix because we have 3 bands here, we have 5 bands here, so that is the band in matrix.

Now we define the sparse matrix, so a matrix A, a matrix A so that is n x n I am writing is a sparse matrix if almost all elements are 0. So it contains only a very small number of elements which are nonzero, otherwise all the elements are 0, so that is called the sparse matrix. So basically this is a band matrix, so you can also call it a sparse matrix. It is a, suppose I have a 10 by 10 matrix and this is a tri-diagonal, so that can also be called a sparse matrix.

So the matrix is called a sparse matrix if almost all the elements are 0, except the few ones. So this is the definition, so we will stop today here. So today we have started with some matrices, lower triangular, upper triangular and some properties of that one. So in the next lecture we will continue with this one. So thanks for viewing, thanks very much.