Scientific Computing Using Matlab
Professor Vivek Aggarwal & Mani Mehra
Department of Mathematics
Indian Institute of Technology, Delhi/DTU
Lecture 14
Newton-Raphson Method For Fixed Solving Nonlinear System of Equations

Hello viewers, welcome back to the course. So, today we are going to start with lecture 14. So, in the previous lecture we have discussed the Newton-Raphson method and about its convergence.

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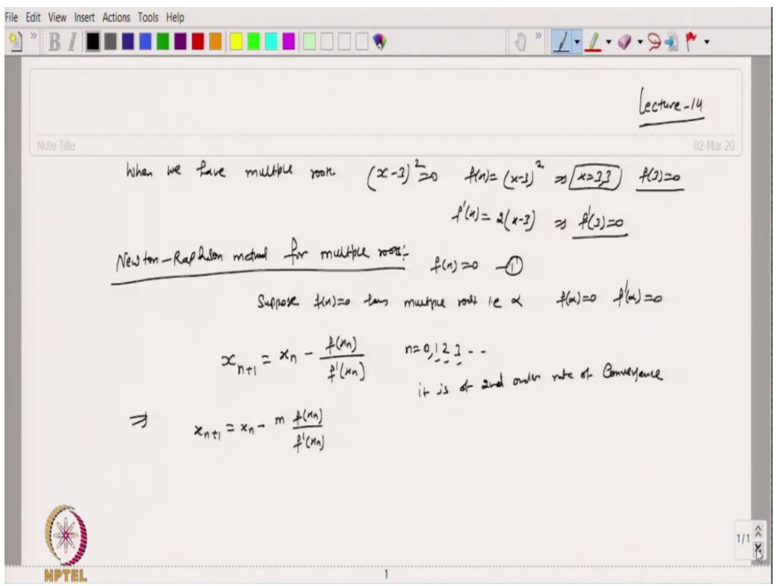

So in this lecture we will discuss further what will happen when we have multiple roots. So, in that case, like for example, I have an equation like $(x - 3)^2 = 0$, so in this case my function $f(x)=(x-q)^2$. So, it has the roots $x = 3, 3$. So, in this case we can say that this equation, equation $f(x) = 0$ has two roots and that is 3 3, but the roots are repeating and its multiplicity is two.

So, what will happen that suppose I want to find the root of such types of equations using my Newton-Raphson method, because if you see here, if I take the

$$f(x) = (x-3)^2$$
$$f'(x) = 2(x-3)$$

So, what is happening here, this is also 0 and this is also 0. So, we have a Newton-Raphson method for multiple roots.

So, how it works? So, the method is the same, we have an equation f(x) = 0 and suppose f(x)=0 has multiple roots and that is we take it α. So, it means that f(α)= 0 and f'(α)= 0 . So, in this case the Newton-Raphson method I know that my Newton-Raphson has and method is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \ldots .$$

So in this case, and we also know that it is of second order rate of convergence, second order rate of convergence. So what we will do is just to deal with the multiple roots, we have:

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3, \ldots$$

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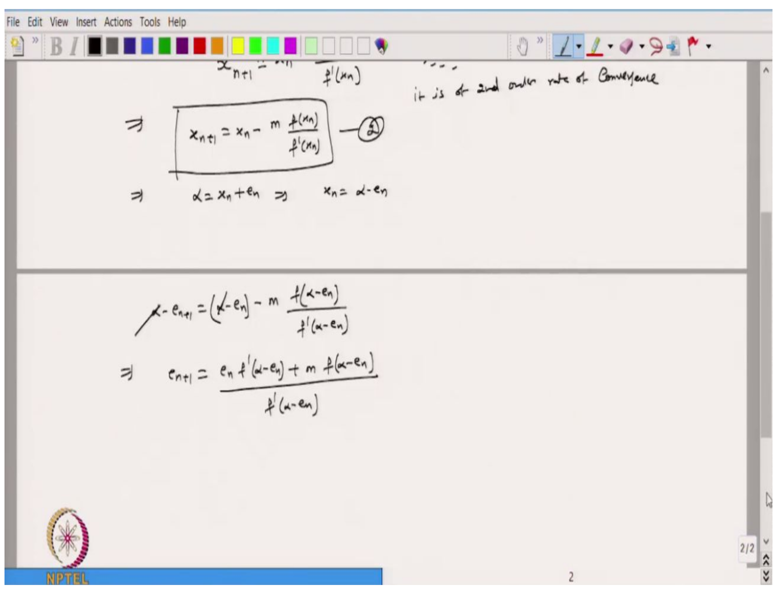

Now, and I also know that my root $\alpha = x_n + e_n$, it means e_n is the error at the nth step. And from here, I know that my $x_n = \alpha - e_n$. So just I want to if I want to apply this method, now I want to see what will be the

$$\alpha - e_{n+1} = (\alpha - e_n) - \frac{m f(\alpha - e_n)}{f'(\alpha - e_n)}$$

rate of convergence, so, let us call it 1, equation number 2, not 1 this can be taken as 2. Now, the same way we have done for the Newton-Raphson.

$$e_{n+1} = \frac{\left(e_n\ f'(\alpha-e_n) + m f(\alpha-e_n)\ \right)}{f'(\alpha-e_n)}$$

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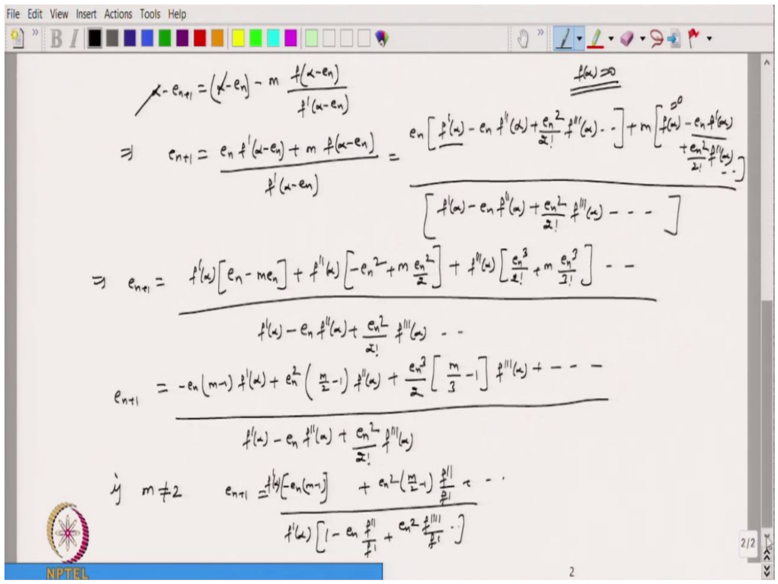

Now from here, because in the same way we can do that my function is differentiable, so I can apply my Taylor's expansion.

So, this way we can define my Taylor expansion. So, from here I can write my e_{n+1} is equal to Now, I will have the terms. So, I know that my f (α)= 0. So, suppose the α is the root of the equation and I do not know about its multiplicity whether the root is multiple or not. So, in that case, now I can collect the terms corresponding to f '(α).

So, if I collect the term corresponding to f'(α) , I will get

$$e_{n+1} = \frac{-e_n(m-1)\ f'(\alpha)+\left(\frac{m}{2}-1\right)e_n^2\ f''(\alpha)-\frac{1}{2}\left(\frac{m}{3}-1\right)e_n^3\ f'''(\alpha)+\ldots}{f'(\alpha)-e_n f''(\alpha)+\frac{e_n^2}{2} f'''(\alpha)-\ldots}$$

Then from here I can take my e^2_n common, I will get.

Now, from here you can see that the, the everything depends upon that what is the value of m. So, in this case, I will just take the common value of f'(α) from here and f'(α) from here and then you can see that this will be, so now, everything depends upon what is the value of m.

Now, if $m \neq 2$, so in this case what will happen? So, if this is an $m \neq 2$, then from here, we get first order of convergence.

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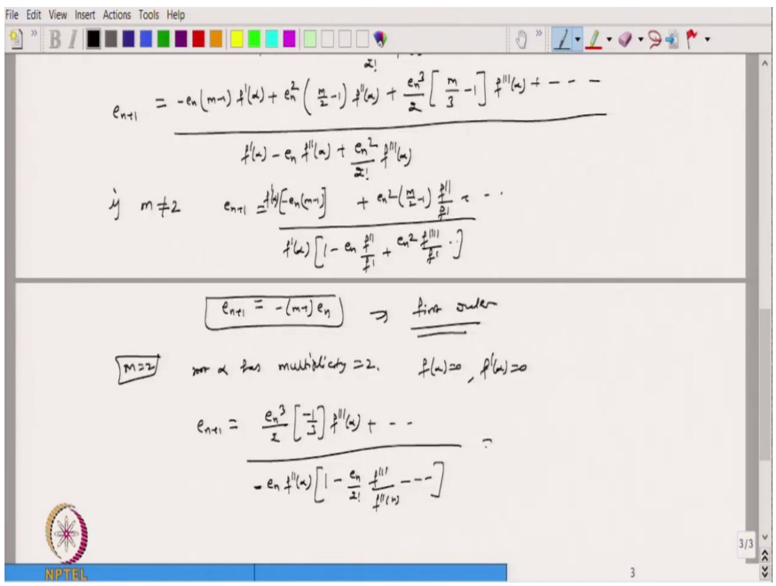

So, in that case, you will see that my en plus 1 will be if I cancel out this one and ignore the higher power of e_n. If I ignore the second order powers of e_n. So, in this case, if you will see that this method becomes first order. So, when m is not equal to 2, the method will be the first order. Now, what will happen if I have $m = 2$? $m = 2$ means the root has multiplicity that is equal to 2.

It means my root $f(\alpha) = 0$ and $f'(\alpha) = 0$. So, in that case what will happen? So, in that case I can write (ignoring higher power terms of e_n)

$$e_{n+1} = \frac{-\frac{1}{2}\left(\frac{m}{3}-1\right)e_n^3\ f'''(\alpha)}{-e_n f''(\alpha)+\left(1-\frac{e_n f'''(\alpha)}{2\ f''(\alpha)}\right)}$$

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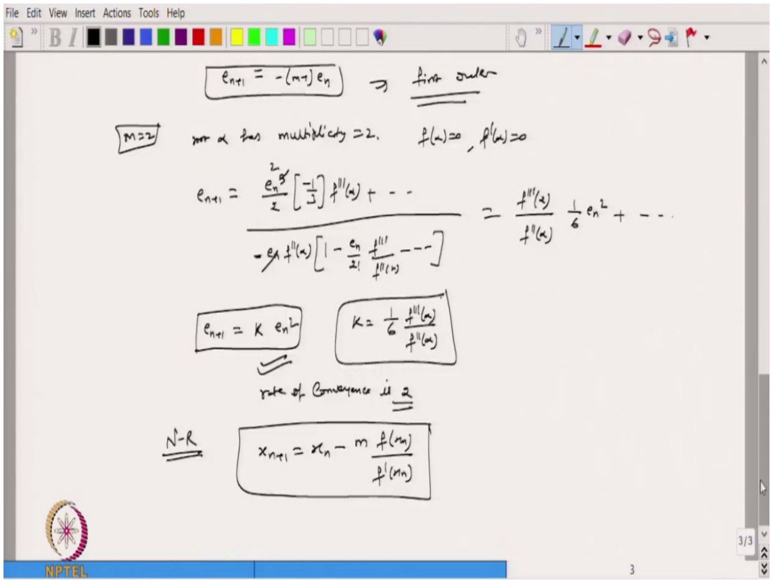

So, from here, now I can take this numerator and I can apply my binomial expansion. So, from here if I take this one, then I can cancel out by this term, we get

$$e_{n+1} = \frac{-\frac{1}{2}\left(\frac{m}{3}-1\right)e_n^3\ f'''(\alpha)}{-e_n f''(\alpha)} \left(1 - \frac{e_n f'''(\alpha)}{2\ f''(\alpha)}\right)^{-1}$$

So, from here you can see that my least power will be e_n^2. And from here I can write

$$e_{n+1} = \frac{-f'''(\alpha)}{6f''(\alpha)}\, e_n^2$$

$$e_{n+1} = K\, e_n^2\ , \quad K = \frac{-f'''(\alpha)}{6f''(\alpha)}$$

And so it means that just to keep the Newton- Raphson method of the second order, I put m is equal to 2 and in that case I am getting the second order convergence.

So, rate of convergence is 2. So, based on this one, I can say that in the future I can apply my Newton-Raphson method, Newton-Raphson method always in the form

$$\alpha - e_{n+1} = (\alpha - e_n) - \frac{mf(\alpha-e_n)}{f'(\alpha-e_n)}\ .$$

Because, when it has a multiplicity only 1, then I will put m is equal to 1 and in that case it will be the Newton-Raphson method to find out the simple root and otherwise I will put the multiplicity here and then I will put these methods to find the multiple root.

So, this is all about how we can apply the Newton-Raphson method for multiple roots.

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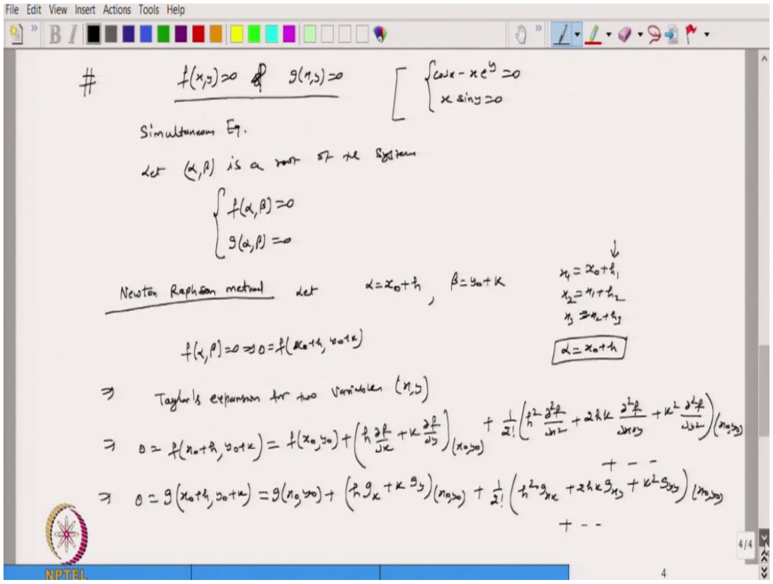

Now we will go further. And then we will suppose we want to have a system of equations like Suppose I have a

$$f(x, y) = 0 \;\&\; g(x, y) = 0.$$

For example, I have

$$f(x, y) = \cos(x) - x\, e^{y} = 0$$
$$g(x, y) = x \sin(y) = 0.$$

So, in this case suppose I have two equations and then I want to find the roots of this equation.

So, that is called a simultaneous equation. So, how to find out the roots of the simultaneous equation? So, I will find α, β is a root of the system, so I have the system, so that my this $f(\alpha,\beta) = 0$ and $g(\alpha,\beta) = 0$. So, this is the value of the equation value of the system at (α,β) the root of the system then, so how can we apply the Newton-Raphson method?

So, I want to apply Newton-Raphson method for the system of equations. So, in this case what we will do, let, so from here I can write that let $\alpha = x_0 + h$ it means that x_0 is my initial guess plus some error and I get my solution that is α or I can say $\beta = y_0 + k$. So, this is what I am considering because if you know that in the iterative methods, if I put $x_0 + h_1$, then I will get x_1.

Then I put x1 plus h2, then I get x2, then I, if I put x2 plus h3, then I get x3, and so on. So, what I am doing is that I am considering all these h1, h2, h3, and maybe I am taking the maximum of this error, or the submission of this error, and I am considering that I get my $\alpha = x_0 + h$. So I am making this parameter this error as independent of the iteration. So that is why I am considering this $x_0 + h$.

Similarly, I am considering beta is equal to y naught plus K. Now from here, I know that alpha beta is equal to 0, because this is the root. So from here I can write f of x naught plus h and y naught plus K so that is equal to 0. Now, this function is a well defined function whose partial derivative exists. So, I can apply my Taylor expansion for several, for I can say for two variables that are x and y.

$$f(x_0+h, y_0+k) = f(x_0, y_0) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)(x_0, y_0) + \frac{1}{2!}\left(h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy}\right)(x_0, y_0) + \ldots$$

$$g(x_0+h, y_0+k) = g(x_0, y_0) + \left(h\frac{\partial g}{\partial x} + k\frac{\partial g}{\partial y}\right)(x_0, y_0) + \frac{1}{2!}\left(h^2 g_{xx} + 2hk\, g_{xy} + k^2 g_{yy}\right)(x_0, y_0) + \ldots$$

Now, from here I can find these two equations by neglecting the higher order terms as:

$$\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)(x_0, y_0) = -\,f(x_0, y_0)$$
$$\left(h\frac{\partial g}{\partial x} + k\frac{\partial g}{\partial y}\right)(x_0, y_0) = -\,g(x_0, y_0)$$

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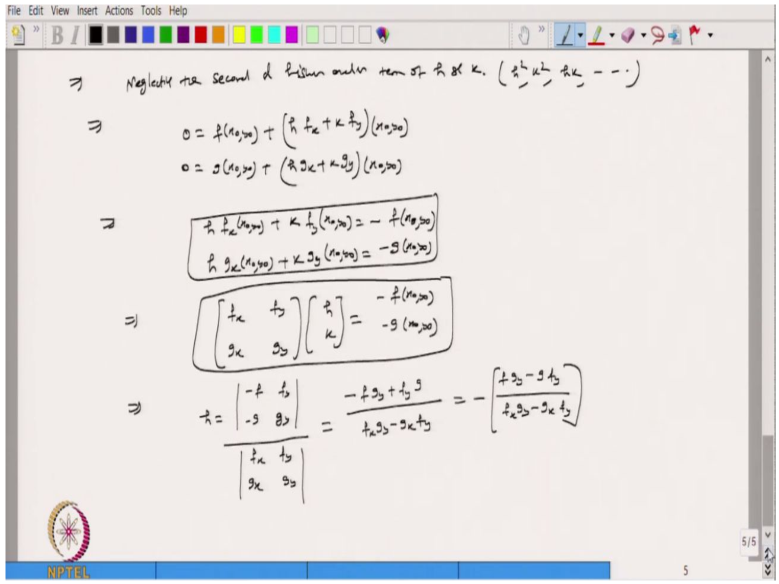

We can write in the matrix form as:

$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} -f(x_0 , y_0) \\ -g(x_0 , y_0) \end{bmatrix}$$

So using Cremer's law, we get

$$h = -\frac{(f\ g_y - f_y\ g)}{f_x\ g_y - f_y\ g_x}$$

$$k = -\frac{(g\ f_x - f\ g_x)}{f_x\ g_y - f_y\ g_x}$$

So, based on this one, I can solve this system equation using Cramer's rule because I am considering that this matrix is a non singular matrix. So, based on this I can find my h and k.

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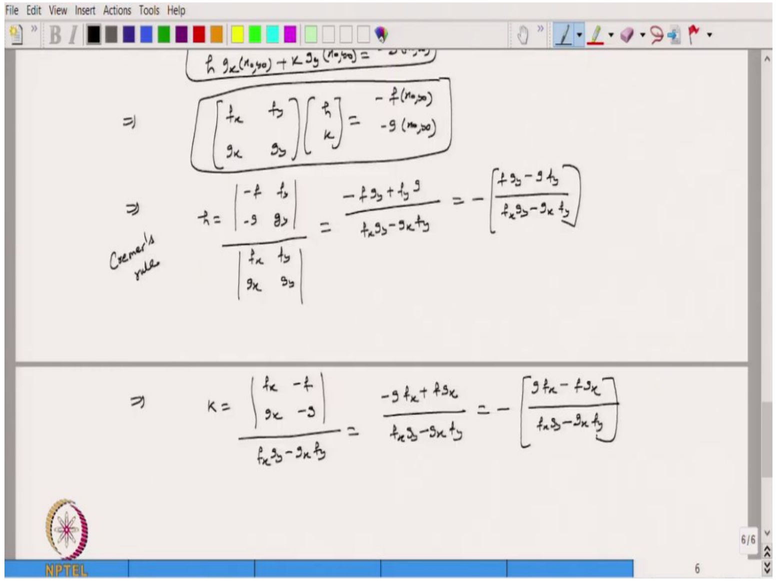

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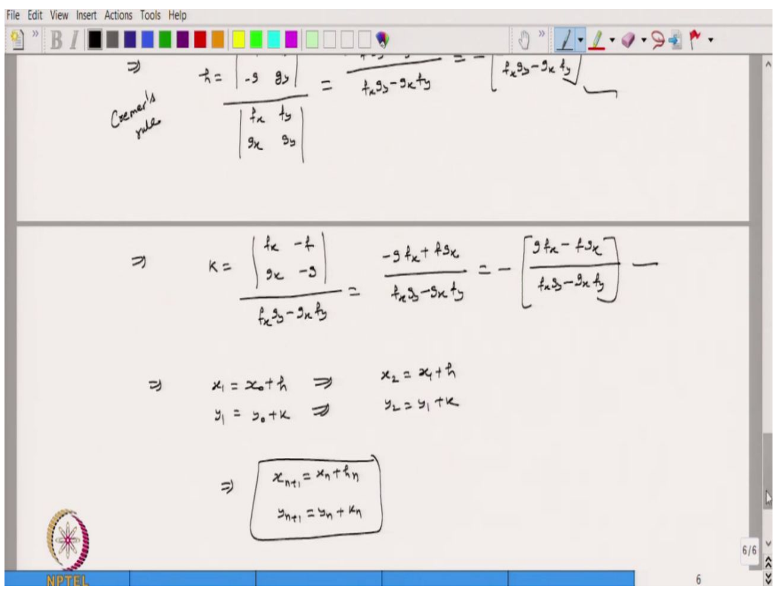

So, based on this one, now, I can find my h and k from here. Now, as I told you that what is going to happen in the real way that we start with the x_0 and then suppose based on the x naught i will get the value of this h and I will get y naught and this is the value of the k I am getting.

So, I after doing all this calculation, we get

$$x_1 = x_0 + h$$
$$y_1 = y_0 + k$$

So, this way we can have an iterative method for finding the root using the help of h and K. So, from here I can write that my

$$x_{n+1} = x_n + h_n$$
$$y_{n+1} = y_n + k_n\ , \quad where\ h_n\ and\ k_n\ are\ calculated\ at\ (x_n\ ,\ y_n)$$

So, based on this one, I can improve my value of the root with the help of this and that is the method.

So, with that iterative method, I am able to find the roots for the system of linear equations. See, the system of linear or nonlinear case does not matter. But we have a system of equations and we can find the roots using the Newton-Raphson method. Now based on this one, so this is all about finding the root of the system of equations.

So, based on the Newton-Raphson methods, one now we are able to find that what will be the rate of convergence when we have a multiplicity more than one for a root and then we have extended this that how suppose we have a system of nonlinear equation and we want to find the roots. So, based on the Newton-Raphson method and by this expression, we are able to find the root of the system equation. So, maybe in the next lecture, we will go forward from this. So, thanks for viewing, thanks very much.