Scientific Computing Using Matlab
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Lecture 13
Raphson Method for Solving Non-Linear Equations

Hello viewers. Welcome back to this course. So, today we will start with lecture 13. So in the previous lecture we have discussed the Regula-Falsi method, the Secant method and its Order of Convergence.

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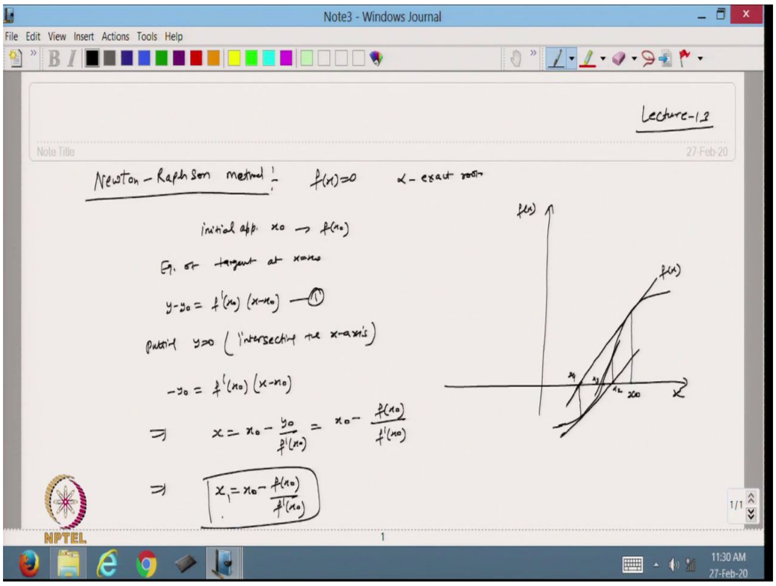

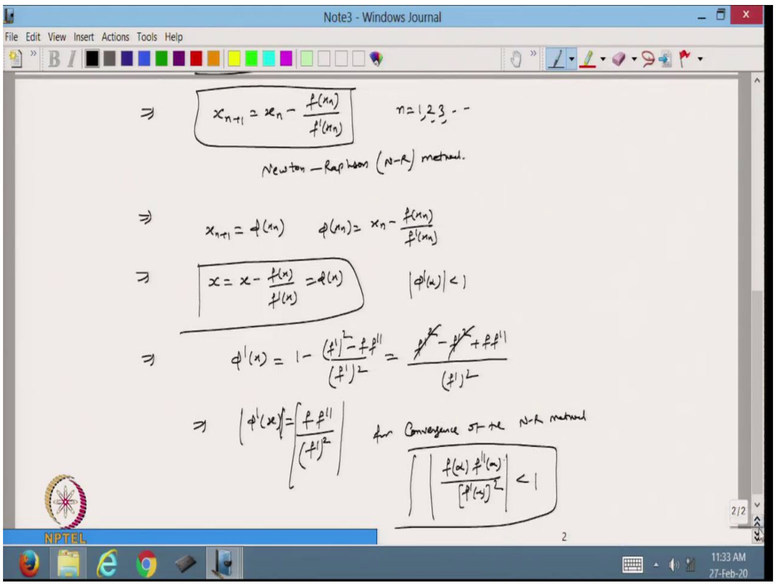

Now, we will start with another method that is very important: the Newton Raphson method. So, like till now we have discussed the methods which is mostly based on the two-initial guess. So, in the Newton Raphson method, how it works is that suppose I have an equation f(x)= 0 and I want to approximate its root so let α is the exact root. So, in this case what you do for example this is my x and this is my f(x) and so that is my supposed function so that is my f(x). Now, in this case what I do is that first I will start with the point and that point I call it x_0.

So, at x_0 I take the tangent line and I will see where this tangent line is cutting the x axis so wherever it is cutting the x axis I call it x_1. So, now after getting the x_1 so this is the point I will take the tangent line at this point and I will see where it cuts the x axis and I will call x_2. Now, at x_2 I will get the value of this function and then at this function I take the tangent and that I will get x_3.

And based on this one that is the point of root and so with the agitation grows or moves on then our iteration point or x1, x2, x3 there you can see that they are heading toward the root of the equation so that is the concept behind the Newton Raphson method. So how it works so I start with the initial guess. Initial approximation that is my x_0 so based on this one I get my $f(x_0)$.

Then I will write the equation of the tangent at $x = x_0$

$$y - y_0 = f'(x_0)(x - x_0)$$

So now I want to see where it intersects the x axis so putting y is equal to 0 because intersecting the x axis so intersecting that x axis you will get the y equal to 0. So, putting here y equal to 0 I will get $-y_0 = f'(x_0)(x - x_0)$

So, based on this one I can get: $$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

So, based on this one I can make this process the iterative process and from here I can write:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \; n = 0, 1, 2, \ldots$$

so this is an iterative process and called the Newton Raphson methods N-R method. So, based on this one I give the value of the function and its derivative and then based on the iteration I will do my approximation for the given root so that is the iterative methods iterations for the Newton Raphson method. Now, based on this one I can say that this method can be written as $x_{n+1} = \phi(x_n)$

$$\phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$

So, based on this one I can say that this method is similar to my iterative method fixed point methods. Now, I know that for the convergence of the methods that we have discussed for the fixed point method the necessary and the sufficient condition was that the

$$|\phi'(x_n)| < 1$$

phi dash alpha modulus value should be less than 1.

$$\left|\phi'(x_n) = \frac{f(x_n)\, f''(x_n)}{f'^2(x_n)}\right| < 1$$

So, that is the condition we got so based on this one for the necessary and the sufficient condition for the convergence of the Newton Raphson method. So, in this case also we generally do know what is the value of α.

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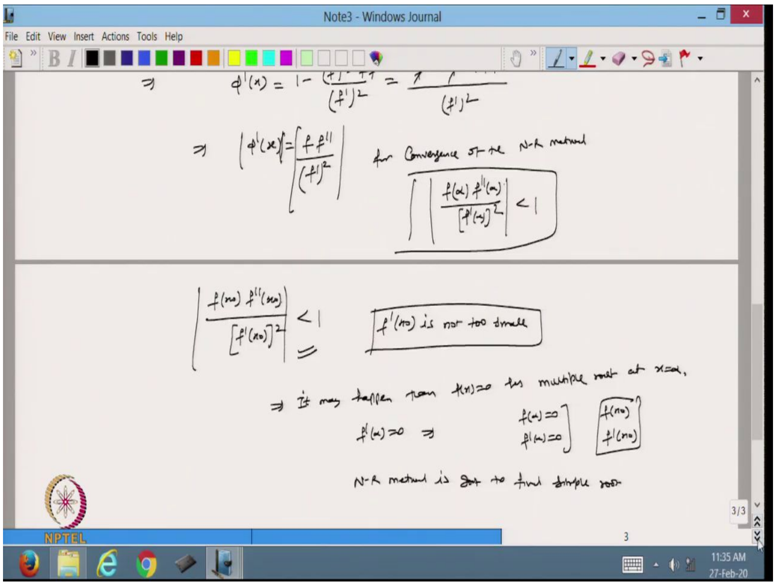

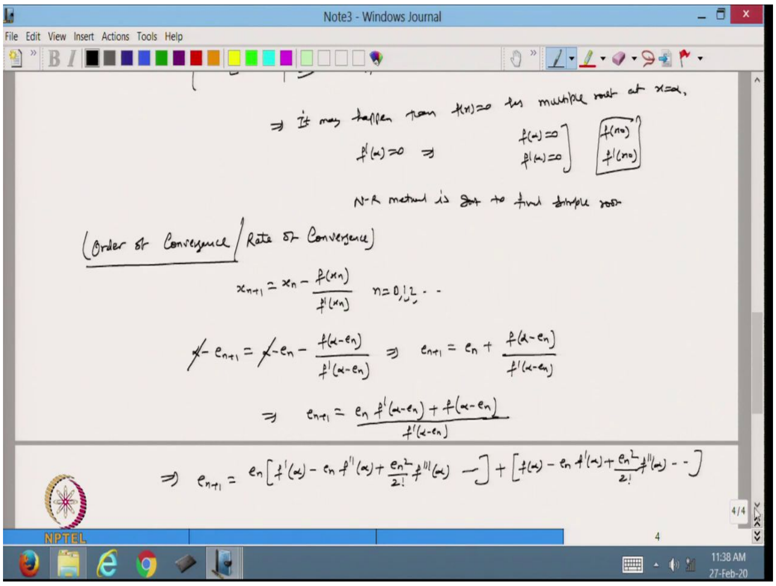

So, we will start the process with $\left|\dfrac{f(x_0)\, f''(x_0)}{f'^2(x_0)}\right| < 1$.

So, we have to take that function or the value of the function such that my $f'(x_0)$ is not too small. So, that is also one of the conditions and how it is possible so from here I can say that in the initial lecture I told you about the simple root and the multiple root. So, it may happen that $f(x) = 0$ has multiple roots at $x = \alpha$. So, in that case $f'(\alpha) = 0$.

So, in this case my $f(\alpha) = 0$, $f'(\alpha) = 0$ and suppose I start with the initial $f(x_0)$ and that is very small $f'(\alpha)$ is also very small. So, in that case this condition may violate so this method is not good to find the roots for which is of the multiple type. So, Newton Raphson method is good to find a simple root for the multiple we have to make some changes in the Newton Raphson method.

So, this Newton Raphson method is good to find the simple root. Now, let us find out its order of convergence also. So, let us write down the order of convergence or the order of convergence or rate of convergence. So, in this case I have the equation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \; n = 0, 1, 2, \ldots$$

I can write this equation as

$$(\alpha - e_{n+1}) = (\alpha - e_n) - \frac{f(\alpha - e_n)}{f'(\alpha - e_n)}$$

$$e_{n+1} = e_n + \frac{f(\alpha - e_n)}{f'(\alpha - e_n)}$$

From here I can expand this one with the Taylor expansion

$$e_{n+1} = \frac{(e_n f'(\alpha - e_n) + f(\alpha - e_n))}{f'(\alpha - e_n)}$$

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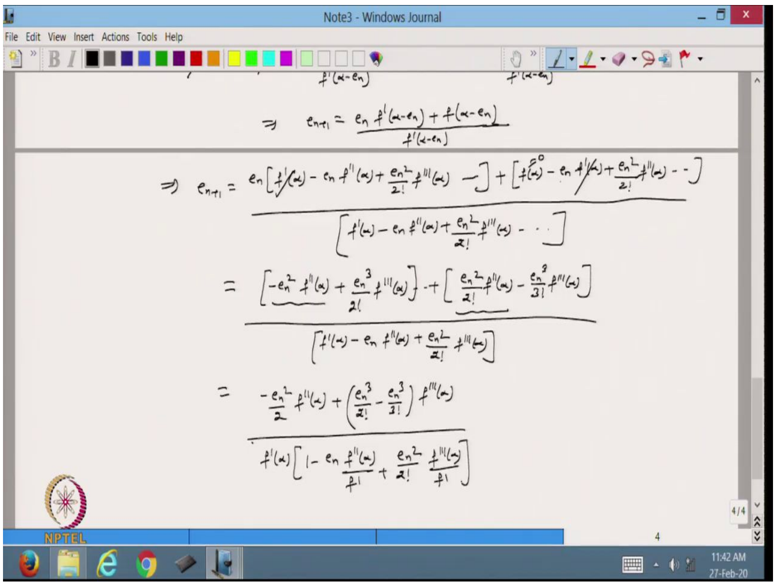

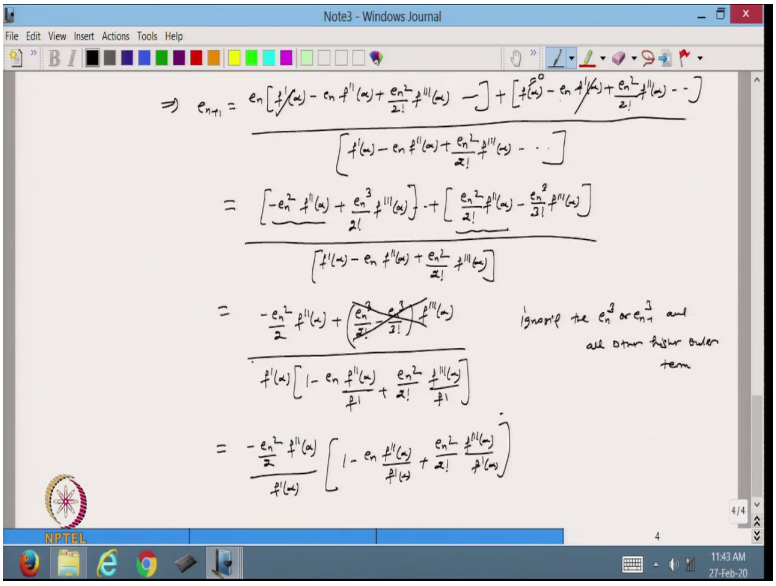

$$e_{n+1} = \frac{\left(-e_n^2 f''(\alpha) + \frac{e_n^3}{2} f'''(\alpha) + \ldots\right)}{f'(\alpha) - e_n f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha)}$$

And next is I can write as:

$$e_{n+1} = \left(-e_n^2 f''(\alpha) + \frac{e_n^3}{2} f'''(\alpha) + \ldots\right)\left(f'(\alpha) - e_n f''(\alpha) + \frac{e_n^2}{2!} f'''(\alpha)\right)^{-1}$$

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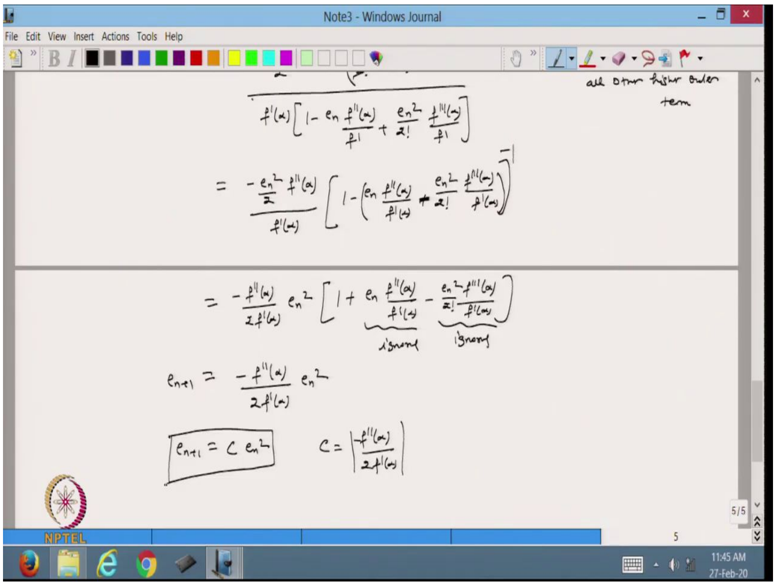

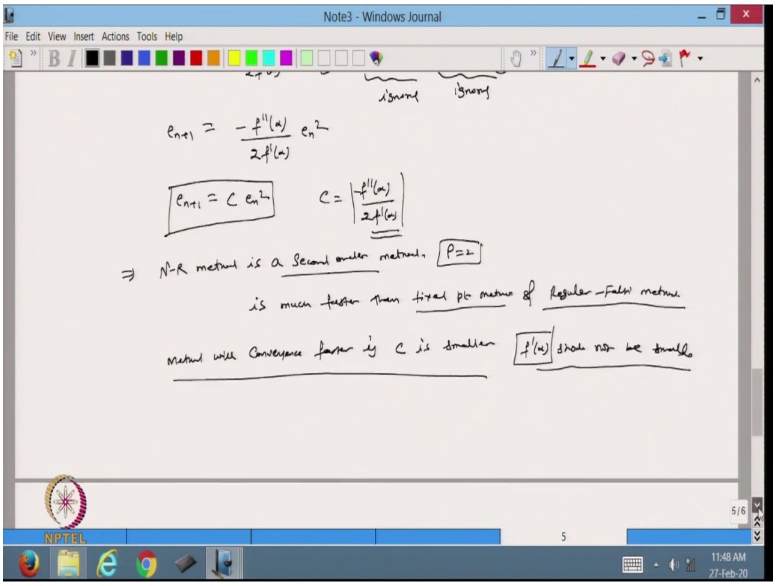

So, I have written this form and based on this form so now it will be minus sign so from here I can take this term as a common and putting this minus sign here. So, I can write

$$e_{n+1} = -\frac{e_n^2}{2!}\,\frac{f''(\alpha)}{f'(\alpha)}$$

$$e_{n+1} = -\frac{e_n^2}{2!}\,\frac{f''(\alpha)}{f'(\alpha)} = C\,e_n^2$$

with $C = -\dfrac{f''(\alpha)}{2f'(\alpha)}$

So, that is the absolute value we take so based on this one I can write this Newton Raphson method can be written in this form and from here I can say that the Newton Raphson method is a second order or we can say that in this case my p = 2.

So, based on this p, I can say that this method is much faster than the fixed point method and Regula-Falsi. So, based on this one I can say that okay I should get much faster convergence when I apply the Newton Raphson as compared to the fixed-point method or the Regula-Falsi method, but based on this one we can also see that if we want to be sure that our convergence, the Newton Raphson method should converge.

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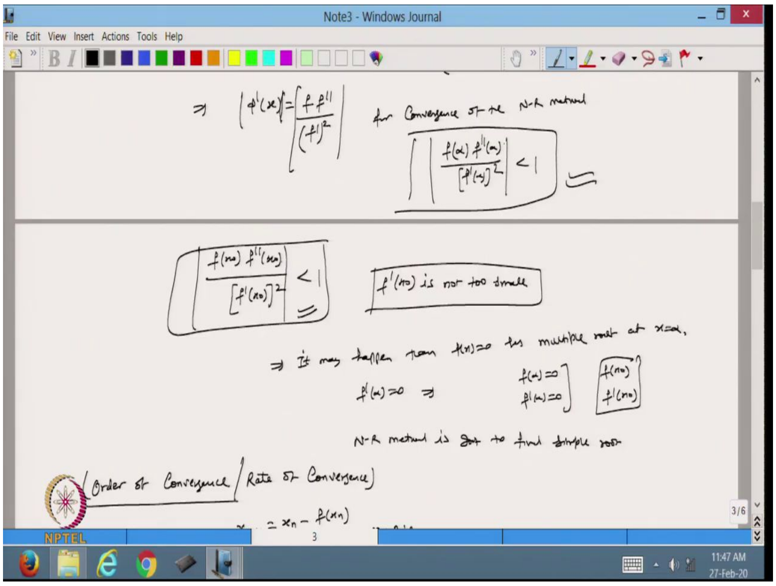

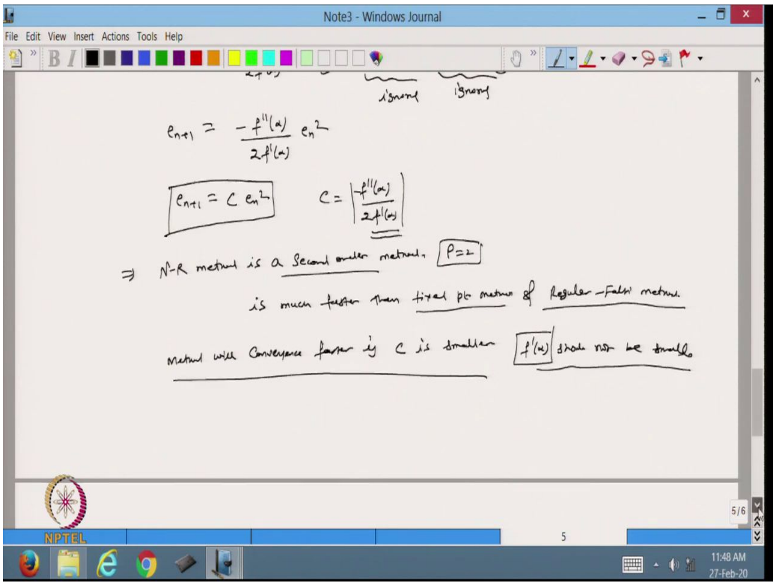

Then we have to apply this condition also that if this condition is satisfied only then my Newton Raphson method will converge and if it converges then definitely faster order. So, based on this one I can say so the method will converge faster if c is smaller because as small the c is the error will be reduced in the next step and the method will be much faster.

So, error will be c will be smaller than this value will be smaller. So, in this case also we are my c depends upon the f'(α). So, my f'(α) should not be small because if my f'(α) will be small then the c will be level very large and even in that case the condition sufficient condition for convergence is not satisfied so in that case my Newton Raphson method may not converge.

And if it is supposed to converge then the rate of convergence is not much faster because the c will be very large. So, in both the cases we have to rely upon the f'(α) so this is all about the Newton Raphson method. Now, we have discussed the various methods so let us see how we can find out the order of convergence when we are dealing with the codes.

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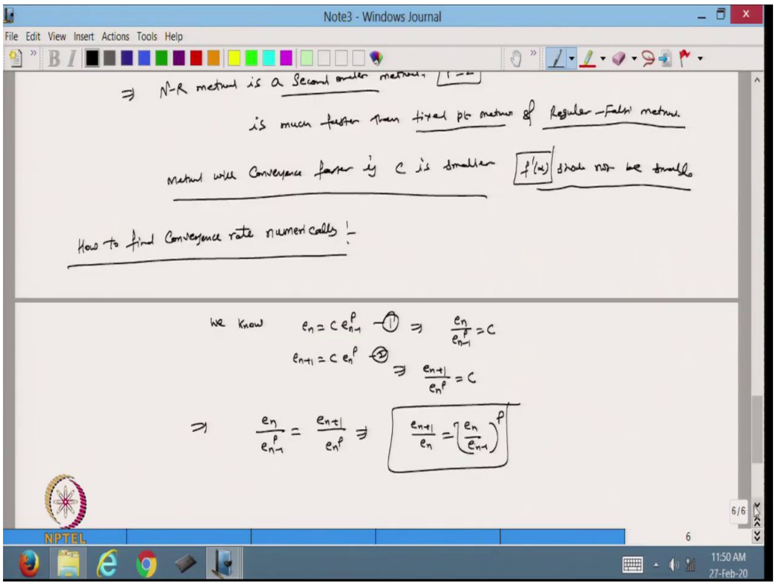

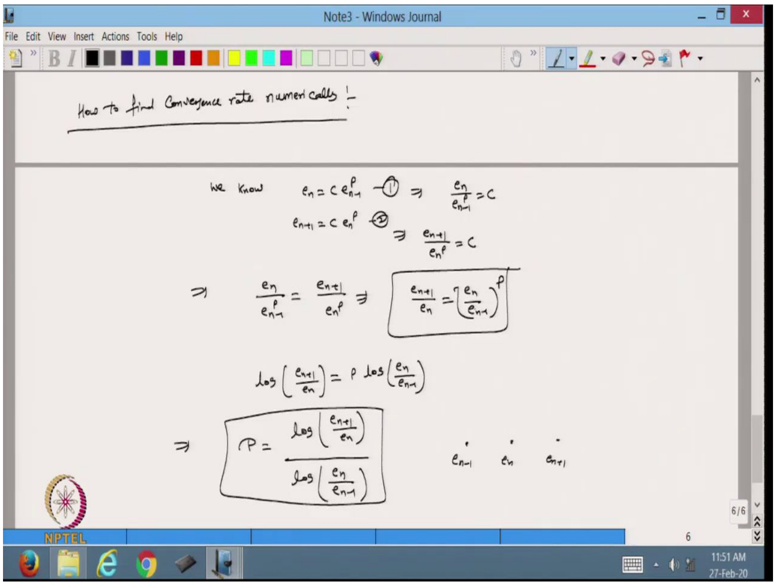

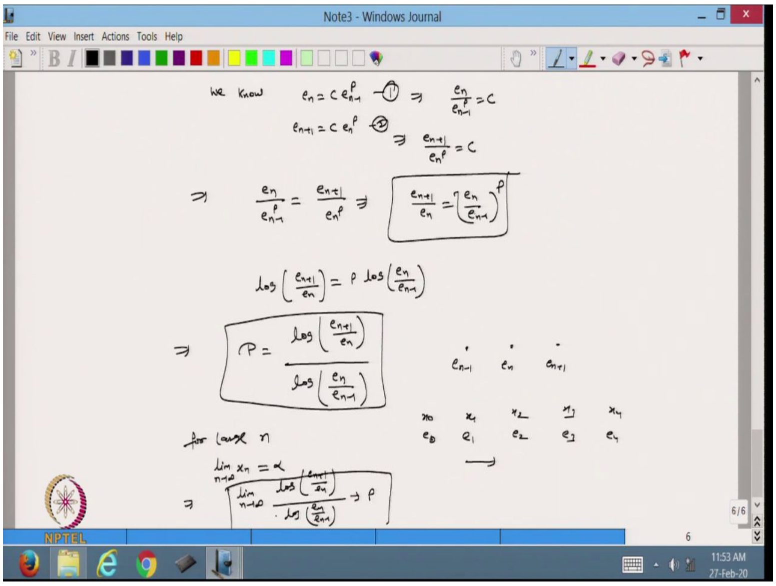

So this is how to find convergence rate numerically. Like suppose we made the code for Newton Raphson for the fixed point for the Regula-Falsi so in all the cases we will want to see how we can find the rate of convergence numerically. So, how we will do:

$$e_{n+1} = C\, e_n^{\,p}$$
$$e_n = C\, e_{n-1}^{\,p}$$

that we know that:

$$\frac{e_{n+1}}{e_n^{\,p}} = C = \frac{e_n}{e_{n-1}^{\,p}}$$

$$\frac{e_{n+1}}{e_n} = \left(\frac{e_n}{e_{n-1}}\right)^p$$

$$p = \frac{\ln\left(\frac{e_{n+1}}{e_n}\right)}{\ln\left(\frac{e_n}{e_{n-1}}\right)}$$

So, suppose I have three points so that is the error I am getting at the e_{n-1} step then at e_n and then e_{n+1} . So, at three points if I know the error then based on this error I can find the value p, but now the question is that suppose I take the Newton Raphson method and then if I see that starting from the initial point x_0, x_1, x_2, x_3, x_4 I will get the error e_0, e_1,e_2, e_3, e_4.

So, if I put this value should I get the value p always 2 in the case of Newton Raphson so that is not the case because as we start growing the iteration and we know that our method is converging to the root. So as we grow with the iterations so after some iteration you will see that for large n the number of iterations so I can say that my xn will grow so in that case I know that limit n times to infinity my x_n will converge to α.

So, in that case we can say that limit n times to infinity with this value log en plus 1 over en divided by log en, en minus 1 that will converge to p. So, it is not going to give the value always the same value p, but for the large value of n or we can say that this converges to p in the limit n times to infinity. So that is a way we can find the order of convergence numerically. So, that is the end of this lecture.

So, in this lecture today we have discussed the Newton Raphson method, its necessary and sufficient condition for the convergence and then we have discussed its order of convergence and that is the second order method. So, in the next lecture we will do some Matlab coding to try to write the codes for Newton Raphson , fixed point ,and Regula-Falsi. And then we will see whether the order of convergence we are getting theoretically matches numerically or not. So, that will be done in the next lecture. Thanks very much. Thanks a lot.