Scientific Computing Using Matlab

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Lecture 11

Order of Convergence of an Iterative Method

Welcome back to this course. So, today we will start with lecture 11.
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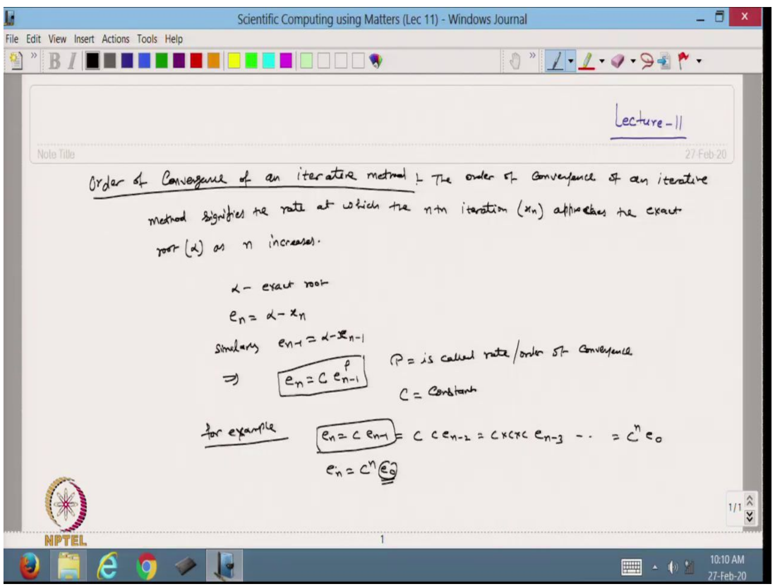

So, in lecture 11 we will talk about the order of convergence of an iterative method. So, that we can write down the order of convergence of an iterative method signifying the rate at which the example nth iteration may be represented by x_n approaches the exact root. So, maybe I call it alpha as n increases.

So, that gives me that because let α is the exact root and $e_n = \alpha - x_n$ is the error e_n and α is the exact solution and this is the approximation at the nth iteration nth step. So, that gives me the value of e_n. Similarly, I can define the error at the (n- 1)th step so that will be $e_{n-1} = \alpha - x_{n-1}$. So, from here I will define

$e_n = C\, e_{n-1}^p$

So, if I write like this one then it shows that whatever the error was there in the (n -1)th step I will take the power p multiplied by some constant that will be the error at the nth step. So, in this case the p is called rate or order of convergence and c is a constant so that we can find for the various methods. So, c is the constant and p is called the rate of convergence. For example, so in a method I get $e_n = C\, e_{n-1}^p$.

We can write as

$e_{n-1}=C\, e_{n-2}^p$, $e_{n-2}=C\, e_{n-3}^p$, $e_{n-3}=C\, e_{n-4}^p$,...

$e_n=C^n\, (e_0^p)^n$

So, it shows me that if I have an order of convergence that is linear means value of p is equal to 1 here so from here I can write that $e_n= C^n\, e_0$. So, it gives me that whatever the error was there at the initial step because in the initial step just we have approximated the root.

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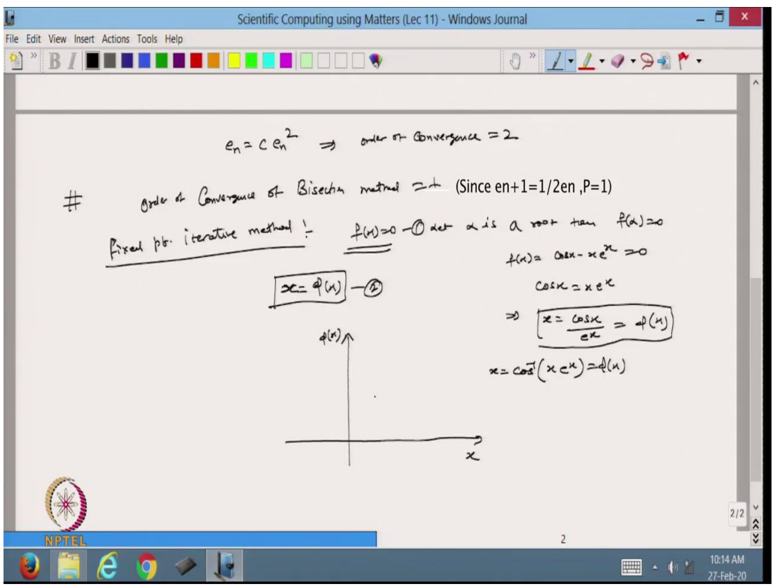

Or you will see in some cases we get $e_n = C\, e_n^2$. So, in that case I will say that this method is quadratic and the order of convergence is 2. So, the more the order of convergence the faster the method will be so that is called the order of convergence. Now, we start with the methods so in the Bisection method we have started. So, in the Bisection method you can see that at each step we are reducing the interval length by half.

So, the order of convergence of Bisection method is 1 because $e_n = \frac{e_{n-1}}{2}$. Now, we define another method and that is called the famous one fixed point iterative method. So, it will use the fixed-point theorem. So, for example I have my function f(x) = 0 so this is my equation and I want to find the root of this equation.

So, if α is the root then $f(\alpha) = 0$. So, in this case what I will do is that we will reduce this equation into this form so that I can write this equation as $x = \varphi(x)$. So, I will reduce this function into this form. For example, suppose I have a f(x) like in the previous Matlab code we have started with $f(x)=\cos(x) - x\, e^{x}$

I think that was the function we have taken so in this case what I will do I will write this function $x= \frac{\cos(x)}{e^{x}} =\varphi(x)$. So, this equation I can reduce into this form or maybe from here I can write
$x= \cos^{-1}(x)\ x\ e^{x} = \varphi(x)$.
So, I can always write my function in this form so this is my equation and from this equation I have written like this one.
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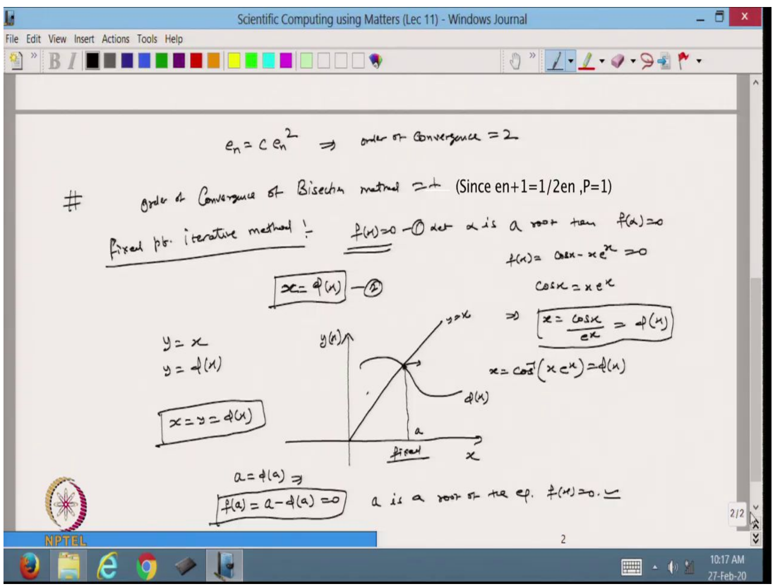

So, what is this that in this case now I have this equation so just to find the root of this equation I will apply the fixed point so for example this is my x axis and this is my y axis or maybe $\varphi(x)$, I can define or I can define my y axis so this is my y axis function of x. So, this is my x axis let my function is $\varphi(x)$ so I draw this function $\varphi(x)$ here so this is my $\varphi(x)$ now what I do?

I draw another function that is so now I have two functions $y = x$ and $y = \varphi(x)$. So, this is my y axis so this is my $\varphi(x)$ and then I will draw $y = x$ so this is equal to $y = x$ this is a straight line and now this is the point where it meets the x axis that is called the fixed point.

So, at the fixed point my x and y are the same so in this case I will get my $x = y$ and that $y = \varphi(x)$. So, basically this value of x where it meets the function $\varphi(x)$ is the fixed point and this is if you see so suppose this value is coming. Maybe I will call it so from here you can see that my a

$=\varphi(a)$.

This is my value of the function at a so from here I can say that this a is a root of the equation $f(x) = 0$. So, based on this iterative method the fixed point method we are able to find the root of the given equation. So, based on this one now how we can find this root using the iterative process. So, this fixed-point iteration means that now I can write my equation from the above.
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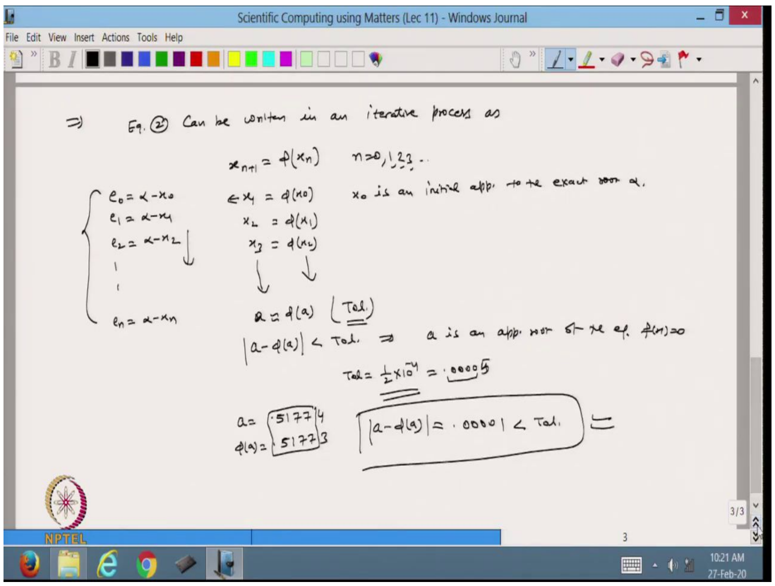

So, this is my equation 2. So, equation 2 can be written in an iterative process as $x_{n+1} = \varphi(x_n)$ where n = 0, 1, 2, 3 and so on. So, in this case what I will do I will start with my $f(x_0)$ so that is our initial approximation. So, x_0 is an initial approximation to the exact root that is α. So, from here I will get my value of x_1. Now, what I will do I will put the value of x_1, I will get the value of x_2.

So, that is my second approximation I will use this x_2 I will get x_3 so this is a third approximation. So, based on this one I will iterate my or improve my iteration, improve my approximation value of the root and after sometime where I need to stop. I will find out that my value of a is approximately coming the same as the value of $\varphi(a)$. So that depends upon my tolerance for how much tolerance I am giving.

So, in this case what is happening is my value of $|a - \varphi(a)| < tol$. So, once it is reached then I will stop this one and from there I will say that a is an approximate root of the equation $f(x) = 0$ and based on this one I can find my error also. So, based on this one here from here you can see that I can find my error $e_0 = \alpha - x_0$, $e_1 = \alpha - x_1$, $e_2 = \alpha - x_2$ and so on.

So, en will be alpha minus xn so whatever the error I am getting after the nth iteration that will be en. So, in this way we will see from the computer programming that this error is reducing and then ultimately after doing some iteration well reaching the required number of solutions that how much accuracy is needed in the solution because in the iterative process, we will never get the exact solution.

So, the exact solution and always we will get the approximate solution so this approximation can be improved based on what is the value of the tolerance because from here if I give the value of the tolerance = ½ x 10^{-4} so it gives me the accuracy 0.00005. So, from here I can say that whatever the solution I want a and φ(a) that should be the same up to 4 digit and less than so value should be less than this one.

So, suppose my a is coming in some case suppose my a = 0.51774 so my φ(a) = 0.51773. So, in this case my a - φ(a)= 0.00001 and it is less than the tolerance so it will stop. So, in this case I can say that my approximation is correct up to 4 digits. So, that is the way we can find accuracy.

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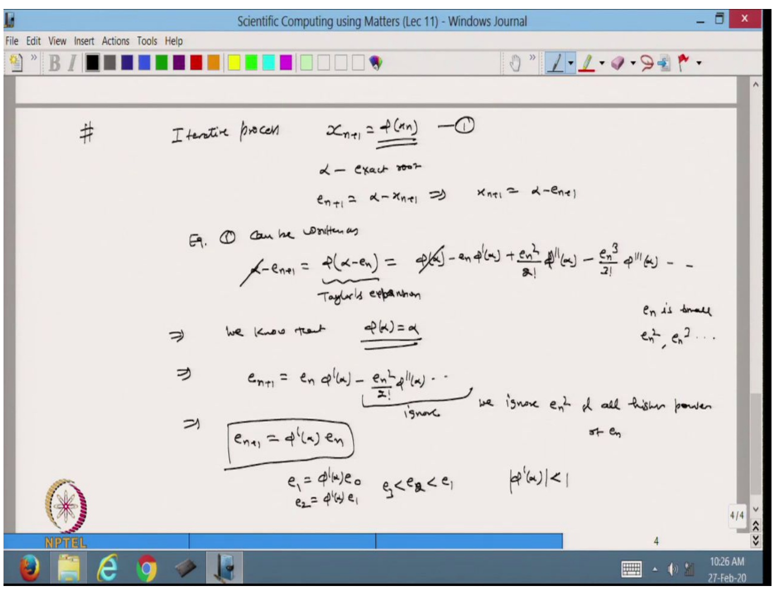

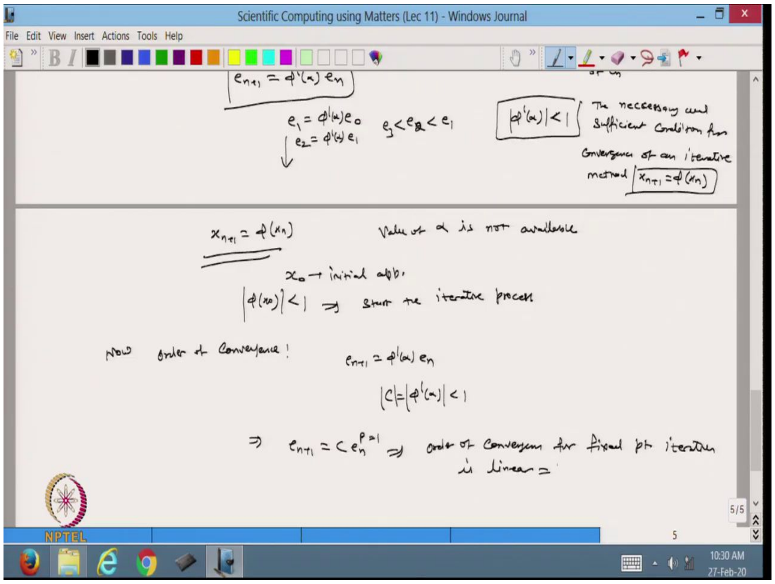

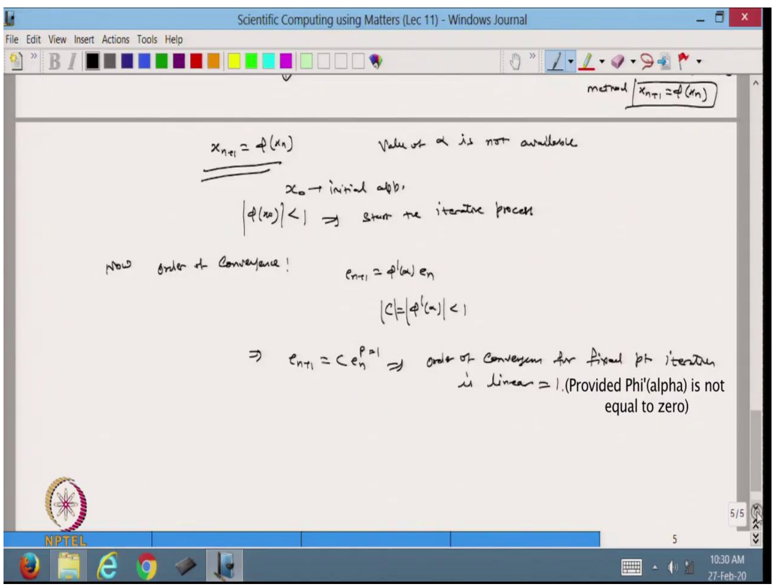

Now, we want to find the that so I start with the now this is my plus 1 is equal to so suppose I take the iterative process this is my iterative process $x_{n+1} = \varphi(x_n)$. Now, this is my iterative process: let α is the exact root. Now, I want to see what will happen so e_{n+1} is the error I can write $\alpha - x_{n+1}$.

So, from here I can write that $x_{n+1} = \alpha - e_{n+1}$. So, equation 1 can be written as

$\alpha - e_{n+1} = \varphi(x_n)$

$\alpha - x_{n+1} = \varphi(\alpha) - \varphi(x_n)$

$e_{n+1} = \varphi(\alpha) - \varphi(\alpha - e_n)$

Now, phi is a continuous function as well as differentiable so I can apply Taylor's expansion for this one.

$$e_{n+1} = \phi(\alpha) - \left\{ \phi(\alpha) - e_n \phi'(\alpha) + \frac{e_n^2}{2!}\phi''(\alpha) - \frac{e_n^3}{3!}\phi'''(\alpha) + - \dots \right\}$$

e_n is small and e_n^2 is much smaller than e_n. So we ignore the higher power of e_n .

We get $e_{n+1} = \phi'(\alpha)\, e_n$

Now we can write as: $e_{n+1} = \left[\phi'(\alpha)\right]^n e_0$

And then I multiply by phi dash alpha and then I get e1 now I want that the error at the next step e1 and then phi dash alpha I will get e2. So, I want that my e2 should be much less than e1 then e3. So, this is only possible when the value of phi dash alpha is less than 1 because if it is greater than 1 the error will increase as we keep on doing the iterations, but if the value of f dash phi is less than 1. Other than I can say that if I go for the more iteration the error will decrease as the iteration will grow.

So, in this case alpha is less than equal to 1 so that is we call it the necessary and sufficient condition for convergence of an iterative method x equal to phi x plus 1 so that is my iterative method xn plus 1 is equal to phi xn. So, that should be that is the sufficient and necessary condition.

It means that if I start with an iterative process that is my xn plus 1 and that is equal to phi xn then what I will do that first I should know the value of alpha, but this value of alpha is not available most of the time. So, in that case what you will do because I know I will start with always x0 that is the initial approximation. I want to start an iterative process so I will start with phi x0.

And I will choose my phi such that the value of this is less than 1 because in this case my initial approximation I can choose and my phi I will choose in such way that the value of this is less than 1 and then using this one I will start the iterative process. So, based on this one I can start my iterative process. Now, order of convergence so what about the order of convergence so just now we have seen that en plus 1 is equal to phi dash alpha en.

So, I can choose my c is equal to phi dash alpha and based on this one and this value definitely will be less than 1. So from here I can say that the c is less than 1 so based on this one I can write that en plus 1 is equal to c en and p so p is equal to 1. So from here I can say that the order of convergence for fixed point iteration is linear and that is equal to 1 in this case so it is coming 1. So, the order of convergence for the iterative process fixed point iteration is 1. So linear I can say that these methods are linear in this case.

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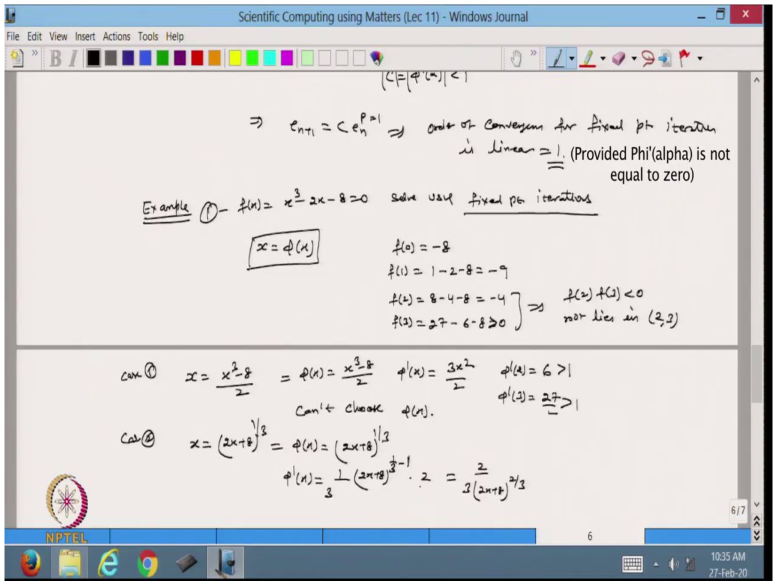

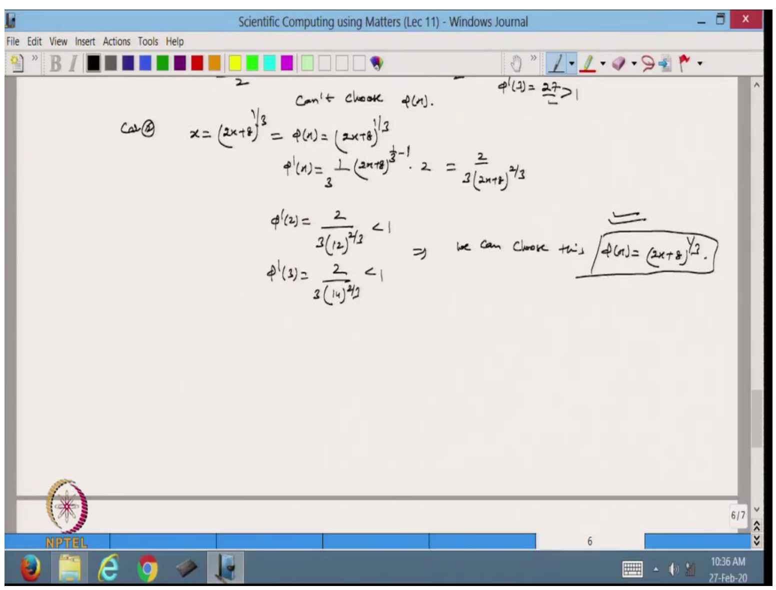

Now, the question comes that how we can choose the phi so let us do an example let me take f(x) is equal to x^3 - 2x- 8 = 0. So, this is the cubic I have and I want to find the root of this equation should I do that? Now I want to solve this one using fixed point iterations.

So, fixed point iteration I can solve this with a bisection also, but I want to solve with the fixed point iteration. So, I need to convert this one into the form of $x = \Phi(x)$. Now, I can just for checking I will get f0 if I put that is minus 8, f1 if you see this is 1 minus 2 minus 8 is equal to minus 9 then I will find out what is f2 so it will be 8 minus 4 minus 8 is equal to minus 4 f3 27 into 6 minus 8 so this is positive.

So, from here I can say that my f of 2 into f of 3 is negative so root lies in 2, 3 so root lies here. Now, I want to find the roots using my iterative process. Now what I do is that I can write from here like this one so this equation number 1 can be written as x is equal to $(x^3-8)/2$ so this the case 1. So, I can write this equation as x^3 this one I can take on the right hand side so $(x^3-8)/2$.

So, in this case my $\Phi(x) = (x^3-8)/2$. Now from here if I find what is my I $\Phi'(x)$ so it will be $3x^2/2$ that is it. Now, from here if I see that if I put the 2 here so the phi dash at 2 will be 6 so greater than 1 phi 3 will be 9, 27 by 2 so that is also greater than 1. So, in this case I cannot choose phi x so this phi x we cannot choose I will take the case 2.

So, in case 2 what I will do is that I can choose my x is equal to maybe I will take $\Phi(x)= (2x + 8)^{1/3}$. So, in this case my $\Phi'(x)$ will be what. $\Phi'(x)=2 / 3(2x+8)^{2/3}$ So, this will be my phi dash x. So now in this case I will just check what will happen.

Now, my $\Phi'(2)<1$. My $\Phi'(3) <1$. So, this value is less than 1 and this value is also less than 1. So, from here we say that we can choose this phi x. So, $\Phi(x)$ I can choose here that is $(2x + 8)^{1/3}$ so based on this one if I do the calculation I can take this value $\Phi(x)$ so that is the $\Phi(x)$ we are getting.

So, based on this one I can take different value phi so once I get this value phi I will stick to this one and then I will do this. So, in this lecture we have discussed the fixed-point iteration and based on the fixed-point iteration we have tried to find the necessary and sufficient condition so that my iterative method should converge and then we also showed that it has a linear convergence. So, in the next lecture we will continue from this one. So, thanks for watching. Thanks very much.