Introduction to Methods of Applied Mathematics
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Lecture No 9
Green's function

Welcome viewers, back to this course. So, today, we are going to discuss the lecture 9.

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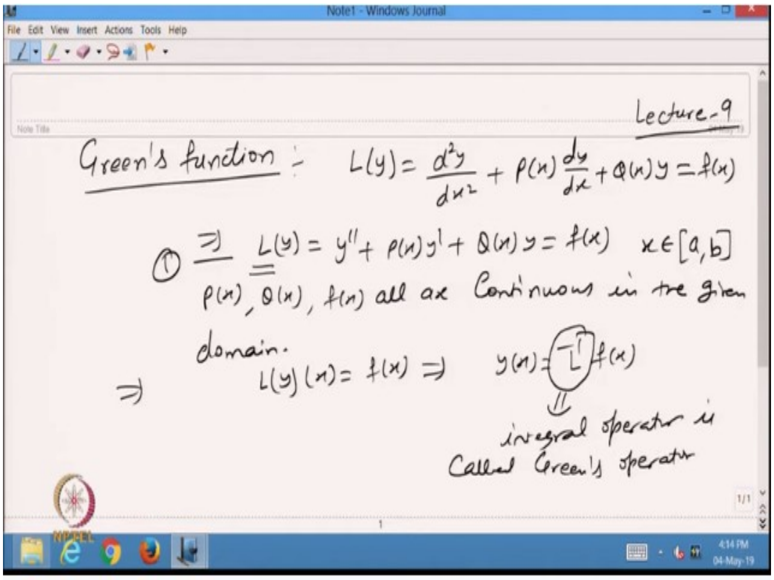

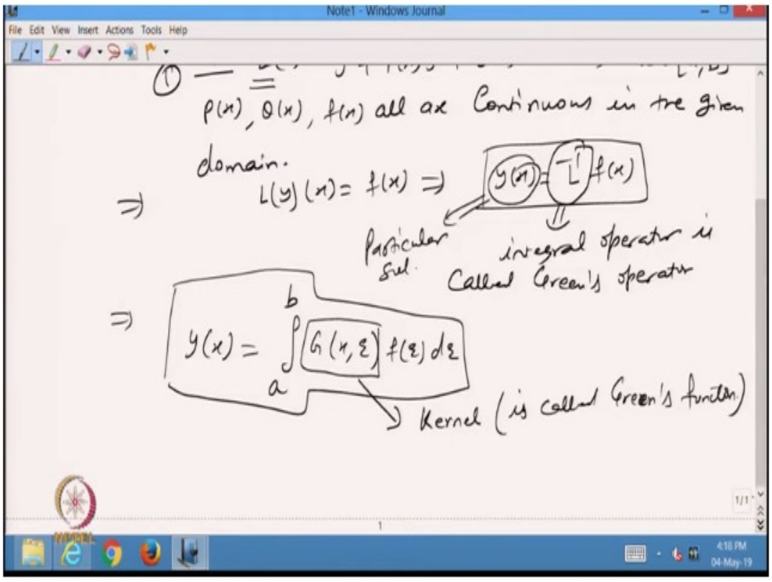

So, today, we are going to start with the new topic and that is called Green's function. So, this Green's function is basically used to find out the particular solution of any second order or any linear differential equation. So, as we have been doing this course, so in the last class also we have started with second order differential equation, so L is my differential operator. So, this is the differential operator we have defined, d square y over dx square + px dy by dx

+ Qxy = fx.

So, this equation, I know that this can also be written as L of y, because y is a function of x, so we can also write this equation as px y dash x + Qx y = fx. So in this case, we are taking the bounded domain, so x belongs to the interval, that is ab, the close interval. And we are assuming that that the Px and Qx and fx all are continuous in that given domain. So, now this differential equation, I call it equation number one.

So, this equation I can write as a Ly operating on a y = fx and if I want to solve this equation, I take the solution yx and I take L inverse fx, where this L inverse because it is the, I know that the L is a differential operator, so this inverse will be the integral operator, so this integral operator, is called Green's operator. So, this is the inverse of the given differential operator.

Now we are dealing with the linear second order differential equation. So this L inverse, so because if I am able to find this L inverse, then I can find the solution of this equation and then we are able to solve a differential equation. Because, so if you see this one, in this case, I am solving this equation. So, basically what I am going to do is that this my yx will be a particular solution.

Because we are solving the differential equation that is a non-homogeneous equation, and for that we know, that whatever the solution comes that is called a particular solution. So, in this case, we already know the different, different methods, when my function fx is some exponential function or a sin cos function or some other function, but, we have solved this one when the given differential equation has a constant coefficients.

And after solving the constant coefficient homogeneous equation, we are able to find the complimentary solution and based on the type of the function fx on the right hand side, we can apply the different, different methods to find out the particular solution. So, that we have already done and later on, we also solved the equation using a variation of parameter.

So, in that case, if I know the two linearly independent solution of the corresponding homogeneous differential equation, then using that two solution that is y1 and y2, we are able to find out the particular solution with the help of variation (of) parameter. So, the question comes that why we are going to start with this type of problem, this type of a new method.

So, in this case, this method Green's function is useful, because we know that, if I take the equation number one, my differential operator is fixed and if treat this equation as a initial problem or the boundary value problem, so, in that case also the initial condition are fixed or the boundary condition are fixed.

But what about if I change the right hand side, my right hand side function I change. So, if I go by the previous methods, then every time the function fx is changed, I have to change the strategy to find out the particular solution. So, just to get rid of that one, we want to apply the method that is called the Green's function. So, now, so my solution is yx. So, this operator actually we define like this one, so, that is defined as G and some, so call it g the function of x and Xi and then f of Xi d Xi and Xi is belonging from a to b.

So, if I do this one, this is the L inverse and operating on the function fx, so, we call it the fXi because this is our indexing we are taking. So, in this case, this is the solution of the given differential equation 1 and in this case, the Kernel, because this is, Gx is called a Kernel and this Kernel is called Green's function. So, the main thing is that to find out this Green's function for the given linear differential operator.

So, before that one, so we want to introduce another type of function that we call it the special function to deal with such problem.

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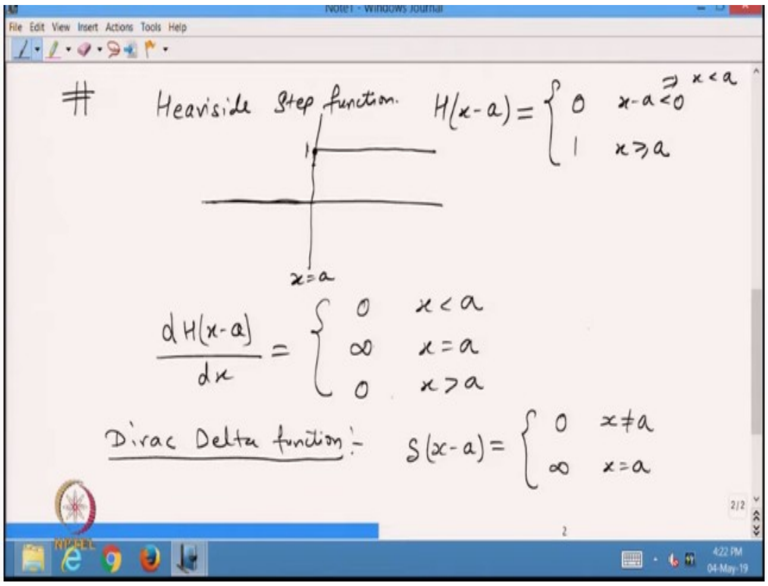

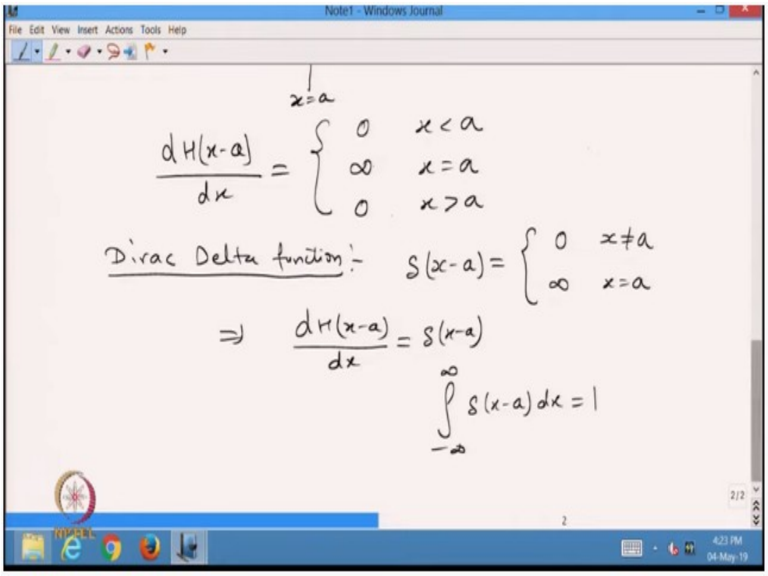

So, that is called, so I define a function that is useful for developing the theory of the Green's function and that is called step function or this is also called Heaviside Step function. So, what is that? So, we define a function, we call it H, x-a, where x is my variable and a some fixed value. So, I call it this, that this function has value 0, then x-a will be less than 0. And from here I can say that my x is less than a.

And otherwise value will be 1 when x is >= a. So, in this case if you want to plot this function, so, this function is look like this one. So, suppose this is my a, x = a. So, if you see that this function is 0 for the whole value. So, this is 0 function and at x = a, its value will be 1 and then its value will be 1 again. So, if you see this one, that this is a value 1. So now in this case, if you see, then this we have a value 0 and this value is 1, and here is a jump.

So it is also called the jump function and we have a jump of width 1 in this case. So this is called the Heaviside Step function. So, this function has only two values; either 0 or 1. So let us say, define with another function with the help of the Heaviside function. So, let us see that I want to take the derivative of this function. So I just want to calculate what will be the derivative of this function, d of H, x-a dx. So this one I want to define it.

If you see that at x=a, it has the jump discontinuity. So, but if we take the x less than a, this is a function parallel to x axis, so its derivative will be 0. After this x=a, greater than a, its value will be always the derivative will be zero, so this is parallel to x axis and here we have a jump. So here if I want to find out the slope of this one, so that is infinity. So in this case, this will be 0, when x is less than a, this will be infinity, when x = a, and this will be again 0 when x is greater than a.

So with the help of this function, this value, we define a function and that is called Dirac Delta function. So, this is by the physicist that is the Dirac. So, in this case we define a Dirac Delta function, so, I define a Dirac Delta function, delta x-a, this is equal to, if I call it that a 0, when x is not equal to, because if you say here x is less than a or x is greater than a, its value is 0. Otherwise, its value is infinity, when x = a.

So, from here, I can say the Dirac Delta also can be written as the derivative of d, the Heaviside function, dx and that can be written as Dirac Delta. So this is a function and this function has a property that for the Dirac Delta, that if I take the integral from - infinity to infinity of this function x-a dx, then its integral is equal to 1.

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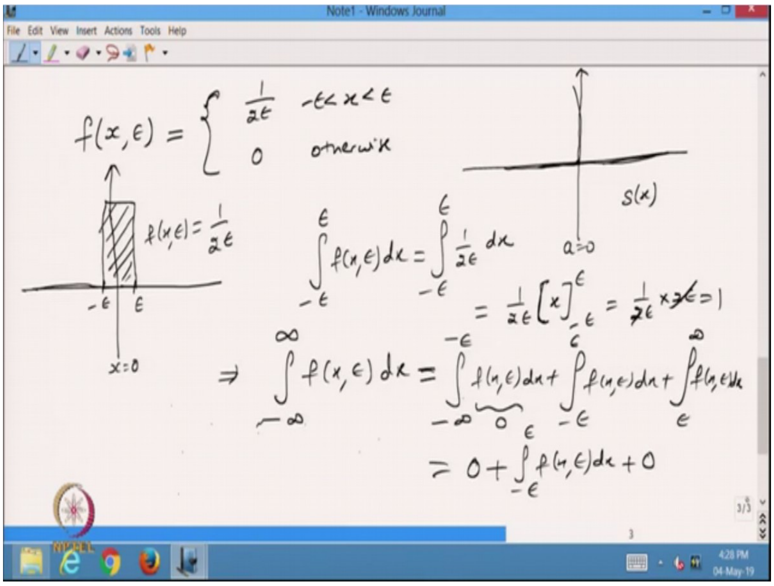

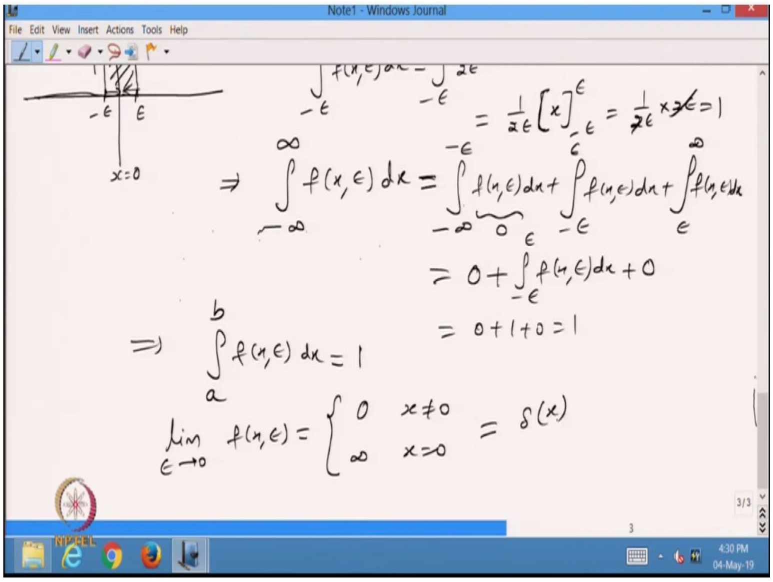

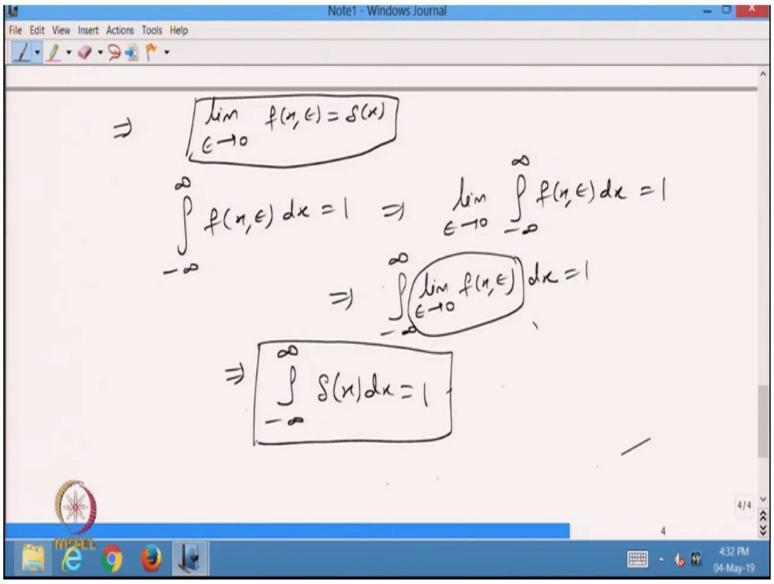

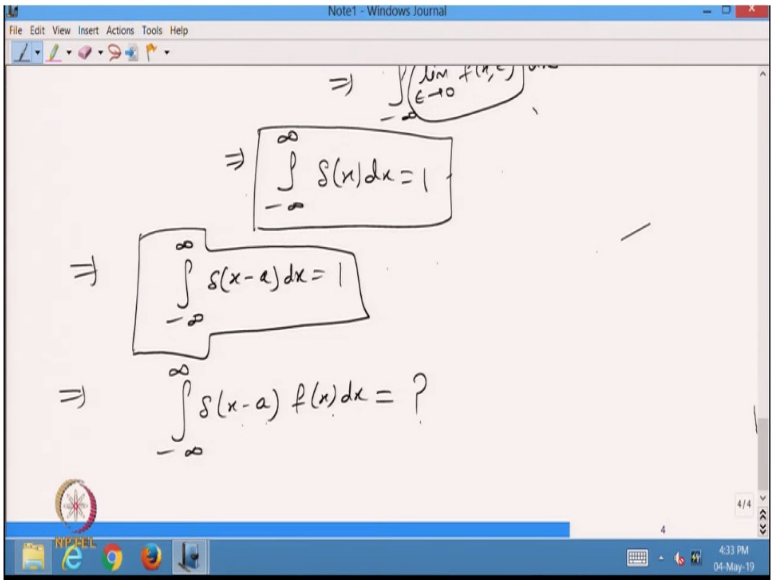

So, if you just want to define this function, because this function if you want to draw, then you will see that, suppose I take this function, this is my function and let a = 0. So, in this case a = 0 and so, the function is 0 everywhere, this is zero. Only at this point, its value is unbounded, it is going very high. So, it is unbounded, so this function has 0 value everywhere except, I have taken a = 0.

So, in this case my Dirac Delta function will be delta x. So, this is the function we can draw, which will be about the Dirac Delta function. Now, the thing is that we want to prove this property. So, for this one, because this is a special type of function, I define this function as if a new function f that is x and I define a new parameter epsilon, so, I define this function as another function.

So, this function I take as, so this is I take a interval of width, and this is x equals 0, so, in that I define a function that this new function whose width, so this function I am defining, this function, so, my fx in this case I take the width that is one over, try to find this function like this. Now, it is just if I start taking the epsilon tends to 0, then this function turns to infinity. So, in this case, this is the, width of the function and that is the height of the function.

So, I define this function at this one. So, I define the function fx = 1 over, when my x belongs to - pi to pi and otherwise its value will be zero when, otherwise, so this is the function. Now, if I take this function and I want to integrate this function from - fx epsilon dx, then if you see this one that the integral I want to define this function, so this function is a positive function, so, basically I want to find the area of this function under this curve.

So, from here I can say that - epsilon to epsilon 1 over 2 epsilon dx, other words value zero. So, this will be again **2** 1 over 2 epsilon and then this will be x, so it will be 2 epsilon into 2 epsilon. So, its value will be 1. Because in this case, I have the width of that length 2 epsilon and the function is 1 over 2 epsilon. So, if I multiply, that value becomes 1. The same thing I can apply for this function, suppose I apply this one for - infinity to infinity, dx. So, this one I have already defined.

So, if you see this function, I can write in this form, from - infinity to - epsilon, fx epsilon dx + - epsilon to + epsilon and then from epsilon to infinity, fx epsilon dx. So, in this case, the value of this function 0, from - infinity to epsilon, this value is 0, this value is 0 here. So, from here I can say that this will be 0 + from - epsilon to epsilon f, x epsilon dx + 0.

And this value we already know that this integral is 0 + 1 + 0, so this will be 1. It means that if I integrate this function for interval - infinity to infinity, its value is also equal to 1. Similarly, I can do for finite interval from a to b. So, in this case value, the same way I can define and its value will be again 1. Now, if you see, from the function I have defined, if I take the limit epsilon tends to 0, f of x epsilon.

So, if I apply this limit, then you can see that this width thing is going to decrease and becoming zero, then the height of the function, is going to increase and it going to become the infinity. So, if I put the limit epsilon tends to 0, this one, then I can say that when x is not equal to a, so, in this case I define the function, this value. So, this value will be 0 when x is not equal to 0 and become infinity when x is equal 0.

So, this function, if you see, this is the function. So, when the epsilon tends to zero, we are heading towards the x = 0, its value will be infinity, in other words value will be 0. So, basically from here, I can say that, this is equal to a Dirac Delta function. So, from here I can say that, so from here, one thing is there that the limit epsilon tends to 0 f of x epsilon is equal to Dirac Delta function. So, this is the definition of the Dirac Delta.

Now, I want to define, that what about, I know that from - infinity to infinity my function fx epsilon dx the value is equal to 1, so, just now we have done that. So, now what I do, I take the limit, so, limit epsilon tends to 0 from - infinity to infinity f of x epsilon dx and this is equal to 1.

So, by the Leibnitz test, I can take this interval, this limit inside and this will be, so, this one will be - infinity to infinity then the limit fx epsilon dx and that is 1 and just know I define that this is equal to the delta function. So, from here I can say that if - infinity to infinity it is the delta function, x dx that is equal to 1.

So, this is what we want to go. So, in this case I have taken the value a = 0 and the same way I can define for the shift function. So, the same way I can define, that - infinity to infinity delta function - a dx, that is also equal to 1. So, with the help of this one, so this is we are able to do.

Now the next thing I want to prove that is will be useful in future that what about this one, if I take the interval from - infinity to infinity, x - a and then I define fx a dx, so this will be what? In this case, what we are doing that we are taking the Dirac Delta function and multiply by function fx. And I want to see that what will be the value this one.

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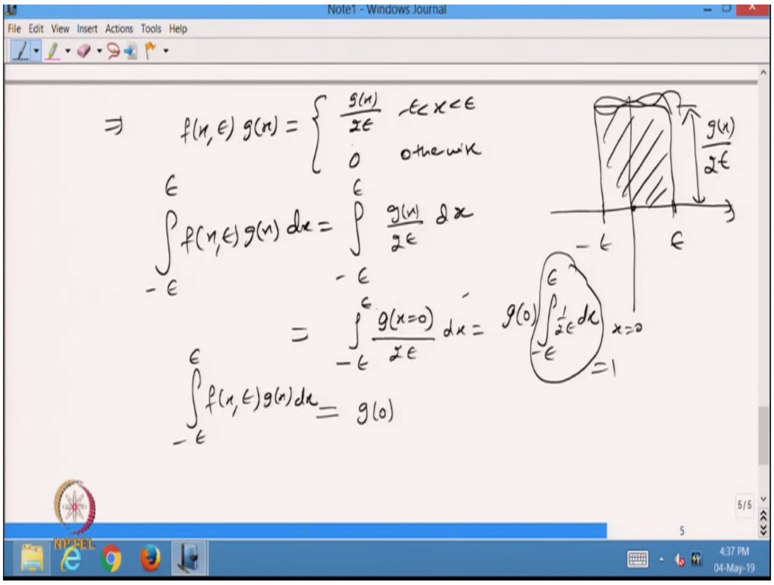

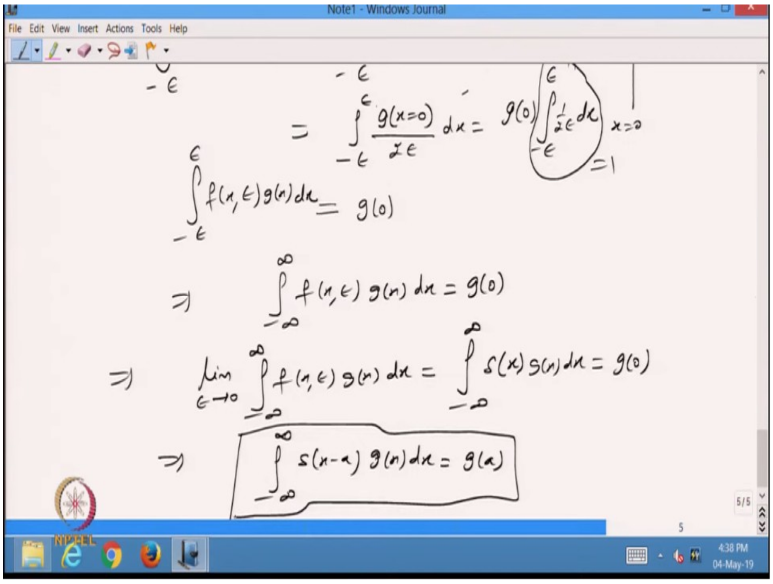

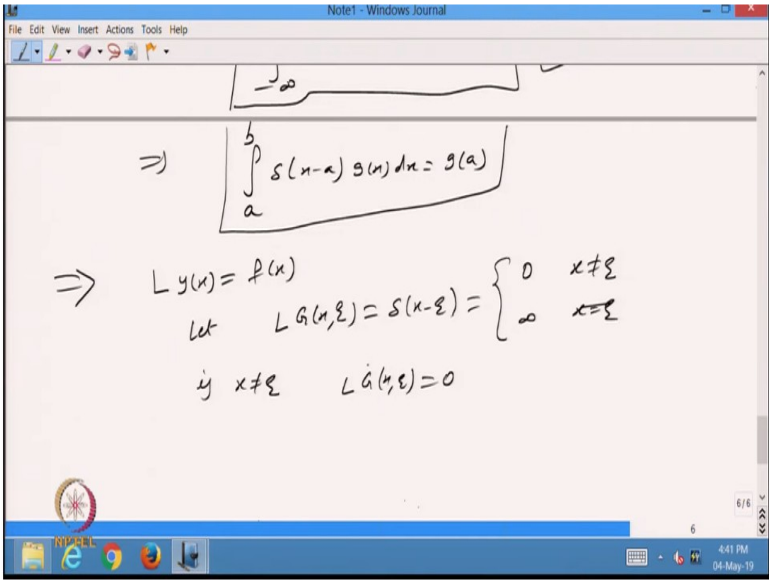

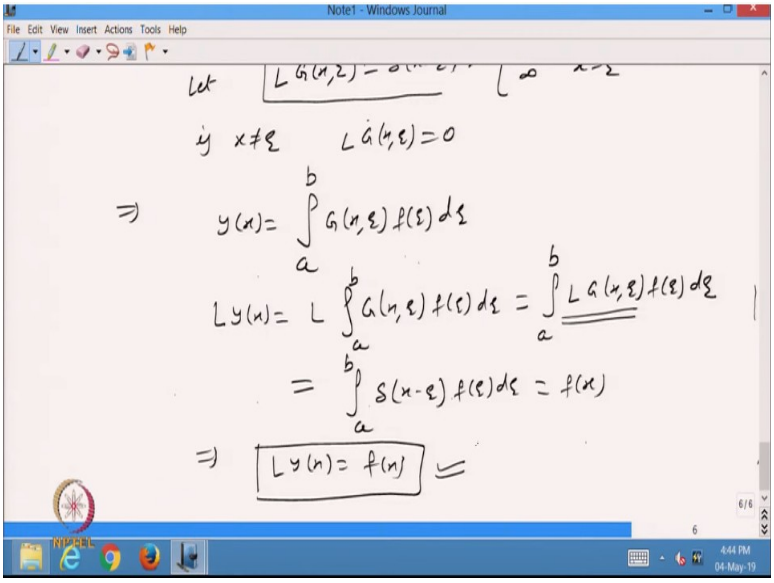

Now, before that I just want to prove that taking the same function again that we have defined. So, in the last one I have defined, just take, I just want to write this as a G function. Now, I want to define that what will be function fx epsilon into gx. So, if you see this will be gx, 1 over 2 epsilon when x is from - epsilon to epsilon and this will be zero otherwise, right. So, in this case, so, this is my function.

So, this is my x axis and this is the axis I am taking y axis. Now, we have defined a function, so, this is - epsilon and this is + epsilon. So, in this case, now I have defined another function, so this function is gx divided by 2 epsilon. So, this can be another function. So, in this case, I just want to define that what is the area under this curve.

So, I want to find out what will be the value if I take the integral from - epsilon to epsilon f of x epsilon dx gx into dx. So, we know that, this will be, so, this function we have defined, so, this value will be gx over 2 epsilon dx. Now, the gx is a function so, this function can be of this type, this can be of this type, but this is a continuous function in the given domain.

So, it is integrable function. So, what we do that, I approximate this function, the height of this function, the value at this point. So, I can approximate this function as - epsilon to + epsilon and I approximate this function is G at x = 0. Because I defined the function, I am taking the interval about x = 0.

So, I just take the value of the function, approximate value, as g at x = 0 divided by 2 epsilon dx. So, from here, I will get the value and this is - epsilon to epsilon, 1 over 2 epsilon dx and from here, this value I already know, that this is equal to 1, so from here, I can say that this will be equal to g of 0. So, f of x epsilon gx dx.

So, this will be the value of the function here. Similarly, I can define this one, for - infinity to infinity fx epsilon gx dx. So, in this case also, I can define this value as 0. Now, I just define the value of the Dirac Delta function. So, I also know that if I put the limit, epsilon tends to zero for this integral, gx dx, so this will be again, so that becomes the Dirac Delta function that is x into gx dx, and this value is again it will be equal to g of 0, because I have defined the Dirac Delta function at zero.

So, from here, I can say that if I take the integral from - infinity to infinity, Dirac Delta x - a into gx dx, so, in this case this value will be equal to ga. So, that is very one of the important integral of (or) the properties of the Green's function, so that is we are going to use a lot for defining the Green's function. Similarly, the same property I can define that from A to B.

Over this finite interval, the Dirac Delta and gx dx, so that can be written as again the ga. So, this is for the final domain. So, now, after doing this one, so, let us start that how we can define the green function for the given linear differential operator. So, the L, the differential operator I know so, that is operating on the function yx, this yx is the function is a continuous, basically this belongs to c to ab and that is giving me equal to fx.

Now, I wanted to find that, let L of, I define the function G, that is the Green's function is equal to Dirac Delta. So, this is the function I am defining. And this function is defining, so, let us define this operator that operating on the Green's function some G x epsilon or instead of epsilon I can put the Xi, because we are dealing with the Xi here and that is equal to, I put x Dirac Delta function, x - Xi. So, now, I know that this function can be written as, it would be zero when x is not equal to Xi and this will be infinity when x = Xi.

So, in this case, we know that if x is not equal to Xi, then my, equal to 0. Now, if I have this one and what we have defined that with the help of the Green's function, I can define my, so this will be fx, so this will be f of Xi d Xi. So, that is the solution we have defined. So, let us see what will happen here, I operate this one with the L, so L of yx will be L defining on this, operating on this, integral. So, I can take this operator inside with the help of Leibnitz theorem.

So, from here I can write, so this will be L of G xXi fXi dXi because a and b are the constants, so other term will be 0 and this will be again a to b and LG, we have just defined, that is equal to the Dirac Delta function. So, this will be x - Xi, fXi dXi. So, this one we have defined in the previous slide, then that this will be equal to f of x, because I am applying the delta x - Xi and we are taking the integral with the respect to Xi.

In previous one, we are taking the here, so we are taking the integration with respect to x, like this one. So here we are taking the integration with respect to x and we get the value ga, here. The same thing is coming here, so, in this case, I am taking the integration with respect as Xi, so I will get the fx. And what is this, so from here, I will get that my L of yx = fx. And that is the given differential equation we are solving.

So, from here one thing is clear that we start with this problem that taking the LG operating on the Green function, and that is equal to Dirac Delta function. If I apply this one, based on this one, I am able to solve my differential operator or the differential equation. It means that if I am able to solve this equation, then I am able to solve, with the help of Green's function I am able to solve this equation and from here I can find out the particular solution of the given equation.

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$$\Rightarrow \quad \boxed{L y(x) = f(x)}$$

How to find $G(x,\xi)$ i.e Green's function for the given diff. operator L ?

$$\Rightarrow \quad L = \frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)$$

$$\Rightarrow \quad L\, G(x,\xi) = 0 \quad x \neq \xi$$
$$= \delta(x,\xi)$$

$$\Rightarrow \quad \boxed{L\, G(x,\xi) = 0} \quad x \neq \xi \qquad a < x < b$$
$$= \delta(x,\xi)$$

$$\Rightarrow \quad L\, y(x) = 0 \Rightarrow y_c(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$\Rightarrow \quad G(x,\xi) = \begin{cases} c_1 y_1(x) + c_2 y_2(x) & a < x < \xi \\ d_1 y_1(x) + d_2 y_2(x) & x > \xi \end{cases}$$

⇒ Properties: (1) $G(x,\xi)$ is continuous at $x = \xi$

$$\Rightarrow \quad c_1 y_1(\xi) + c_2 y_2(\xi) = d_1 y_1(\xi) + d_2 y_2(\xi)$$

$$\Rightarrow \quad (c_1 - d_1) y_1(\xi) + (c_2 - d_2) y_2(\xi) = 0$$

$$\Rightarrow \quad (c_1 - d_1) y_1(\xi) + (c_2 - d_2) y_2(\xi) = 0 \quad — (2)$$

(2) $\frac{\partial G}{\partial x}$ has a jump discontinuity of magnitude 1 at $x = \xi$.

$$\left.\frac{\partial G}{\partial x}\right|_{x>\xi} - \left.\frac{\partial G}{\partial x}\right|_{x<\xi} = 1$$

$$\Rightarrow \quad d_1 y_1'(\xi) + d_2 y_2'(\xi) - c_1 y_1'(\xi) - c_2 y_2'(\xi) = 1$$

$$\Rightarrow \quad (d_1 - c_1) y_1'(\xi) + (d_2 - c_2) y_2'(\xi) = 1 \quad — (3)$$

$$\Rightarrow \quad d_1 y_1'(\xi) + d_2 y_2'(\xi) - c_1 y_1'(\xi) - c_2 y_2'(\xi) = 1$$

$$\Rightarrow \quad (d_1 - c_1) y_1'(\xi) + (d_2 - c_2) y_2'(\xi) = 1 \qquad \text{--- (3)}$$

either IVP $\left(y(x_0) = \alpha,\ y'(x_0) = \beta \right)$

BVP $\left(y(x_0) = \alpha,\ y(x_1) = \beta \right)$

$\Rightarrow$ from eq. (2) & (3)

$$\Rightarrow \quad (c_1 - d_1) y_1 + (c_2 - d_2) y_2 = 0$$

$$(d_1 - c_1) y_1' + (d_2 - c_2) y_2' = 1$$

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$$(d_1 - c_1) y_1' + (d_2 - c_2) y_2' = 1$$

$$\Rightarrow \quad (d_1 - c_1) y_1 + (d_2 - c_2) y_2 = 0$$

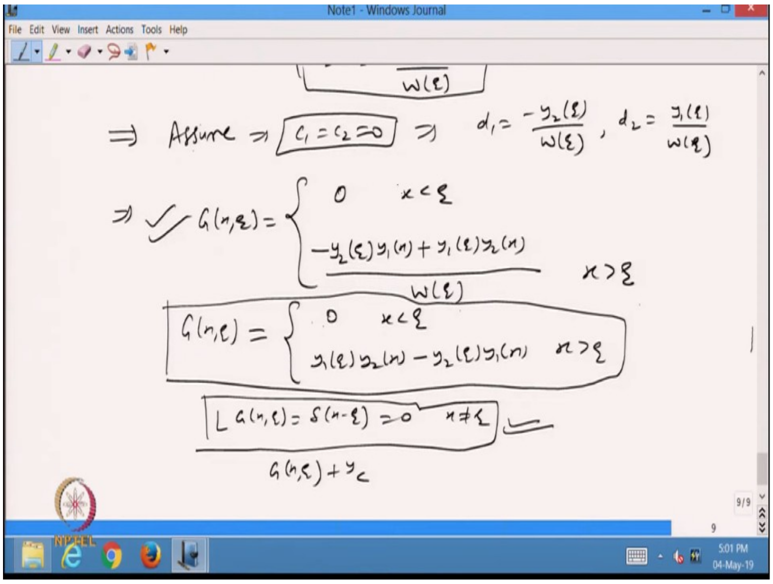
$$(d_1 - c_1) y_1' + (d_2 - c_2) y_2' = 1$$

$\Rightarrow$ $y_1(x)$, $y_2(x)$ are l.I $\quad W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$

Cremer's rule

$$\Rightarrow \quad d_1 - c_1 = \frac{-y_2(\xi)}{W(\xi)}$$

$$\Rightarrow \quad d_2 - c_2 = \frac{y_1(\xi)}{W(\xi)}$$



Now, the main thing is that, now the question is that how to find that is the Green's function for the given differential operator L? So, that is our main goal, that how we can find the Green's function for the given operator. So, now, you know that my operator L is d square over dx squared + px dy dx + Qx. So, this one what I do is that, if you see that L of applying on the Green's function that is equal to 0, basically, when x is not equal Xi, because this is equal to Dirac function.

So, I just want to solve this equation. So, this equation if you want to solve, I know that when x is not equals Xi, it will be 0, otherwise infinity. So I want to solve this equation for x is not equal to Xi. So, let us see what will happen. So, if you see this one, so, this is the equivalent to the corresponding homogeneous differential equation. And if I am able to solve these differential equation, then I know that from here, so this equation is equivalent to solving the corresponding homogenous equation.

And I know that this equation if I solve, then then I can write the complimentary solution of this equation. If I am able to find two linearly independent solution, and I can write that the c1 y1x + c2 y2x, so this I am able to do this one. So, once we are able to find this value, so based on this one, I can define my Green's function. So, in this case, I just define my Green function. So, this will be I just defined c1 y1x + c2 y2x, so, because x is not equal to Xi, so in this case, I am taking my x is lying between a and b.

So, Xi is also lying between a and b. So, this one I can say that, this is when x is less than Xi, and another one I take d1 y1x + d2 y2x and this is a case, when x is greater than Xi. Because x = Xi, so this will be infinity so that we are not interested. So, this is the corresponding Green's function we are able to define. Now, the question is that, I have c1, c2 and d1, d2, these are the four constants we need to find out to define the given Green's function.

So, this one we can define with the help of the following properties of the Green's function. So, from here, the properties we are going to use for find out this interval. The first property is, so this is the properties. The first one is that the Green function Gx Xi is continuous at x equal Xi. So, in this case, if this is continuous at x = Xi, from here I can say that the c1 y1 Xi + c2 y2 Xi = d1y1 Xi + d2y2 Xi and from here, I can say that c1-d1y1 Xi + c2-d2y2 Xi that is equal to 0.

So this is the first equation we are dealing with, so this I call it equation number 2. So this is the first property, the second property is that delg by delx has a jump discontinuity of magnitude 1 at x = Xi. So, this is another property. So, if you see from here, I can define that delg over delx for x greater than Xi - delg by delx x less than Xi, so, this is defined at x = Xi, so this is equal to 1. So, from here, we can say that d1y1 dash Xi + d2y2 dash Xi – c1y1 Xi c2y2 dash Xi that is equal to 1.

So from here, I can say that d1–c1y1 Xi + d2-c2y2 Xi that is equal to 1. So, this is another equation we are going to have. So, this is the question number 3. So, now, you can see that I have question number two and this is the question number three. So, we have two equations, and we have four variables, c1, c2, d,1 d2 so, that we have to find out. So, in this case, I need two more equation to find out the value of all the c1, c2, d1, d2, so that value is basically.

So those values or those conditions I needed, so those condition will be coming from either the initial value problem, that is we have some y at x0 equal 0 and y dash at x0 = 0. So, this is the initial value program I am defining or I will define the boundary value problem. So, boundary value problem I will define like y at x0 is equal to, so some value, that I call it alpha and y at some x1 that is beta. So this also I can define.

It may not be 0 always, it can be non-zero also. So, I just define this as alpha and this is equal to beta. So, this is my boundary value problem and this is my initial value problem. So, if I solving the initial value problem and trying to find the Green function for the initial problem, then I will take the help of this one and if I am solving the boundary value problem, then I will take the help of this one.

So, based on this one, we will able to get two more condition and after having the four condition that is equation number 2, equation number 3, and the 2 condition from here, we are able to find out all the value of c1, c2, d1, d2 and then we are able to define the Green function for the given linear differential operator. So now, let us do one example based on this one, because if you see, then the same thing we have done for with the help of variation of parameter also.

So, in this case also I can find out, now from here, so, from equation number two and three, c1-d1 this is y1x, so I just call it y1 because we have to write, c2-d2 y2 that is 0 and then I will be defining this one. So, this is again I am defining, so, that will be d1-c1 y1 dash + d2-c2 y2 dash = 1. So, from here I can define, so, this one I just, because this value, if you see, so this one I just I can write this function also like this one, just changing the position of, so I can write this as d1-c1 and d2-c2. c2.

So, now, I have two equations, now I know that, so from here I know that my y1x and y2x are linearly independent. So, if they are linearly independent, then I know that the corresponding Wronskian, so, this Wronskian we have defined. So, this one y1 y2 y1 dash and y2 dash this will be not equal to 0. In this case if this is not equal to 0 then, this will be linearly independent and with the help of the Cremer's rule, I can find the solution of this one.

So, that solution if I find out so, this will be, we have already done this one. So, d1-c1 I can define, so this will be equal –y2 over the Wronskian at this one, x = Xi. So, this one I already know and from here another equation if I want to write, then d2-c2 will be y1 Xi over the Wronskian at x=y. So this is the value of the variable d1-c1 and d2-c2, that we already we have done for the, when we will taking doing the variation of parameter.

So, with the help of this one, now I can define, now, in this case, I can assume that this condition is either initial value problem or the boundary value problem. So, just in this case, I assume that that my c1 and c2, they are 0, so let us assume that one. So, in this case, if I assuming this one, then I can say my d1 will be - y2 Xi over Wronskian and d2 will be y1 Xi and that is Wronskian and this will be. So from here, I can define my Green function as Xi.

So, I have taken my c1 and c2 = 0, just I am taking this one, so in that case, this will be is equal to 0 when x is less than Xi and this will be equal to y2 Xi y1x + y1 Xi y2x divided by the Wronskian when is x is greater than Xi. So, from here, I can say that this will be equal to 0, when x is less than Xi and this will I can define as y1 Xi y2x – y2 Xi y1x when x is greater than Xi. So, this is my Green Function.

So in this case, because I know that I am also solving this equation, so this equation basically 0 when x is not equal to Xi, so this equation does not have a unique solution, because if, I have Gx Xi is one of the solution, then Gx Xi + yc, that is the compliment solution that is again the solution of this equation. So, this equation does not have a unique solution. So, in this case Gxy depending upon that I have chosen this case c1 and c2 = 0, I got this Green function.

If I have chosen c1 = 0 and d1 = 0, then this green function would have been the another one. We choose c1 = 0 and d2 = 0 then we get another Green function. So, this Green function is not the unique solution for this equation. So, in this case, if we assume c1 and c2 = 0 then this is the corresponding green function for the given second order linear differential equation.

So, in this class, we have discussed about the Heaviside Step function and then we have developed that, what do you mean by that Dirac Delta function. So, Dirac Delta function are very useful to define the Green function and then we have found that how we can find the Green functions for the given linear differential operator. So, in the next class will go further, and we will try to find out the Green function for some various type of linear differential equation. Thank you very much.