Introduction to Methods of Applied Mathematics
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Lecture – 8
Power Series Solution of General Differential Equation

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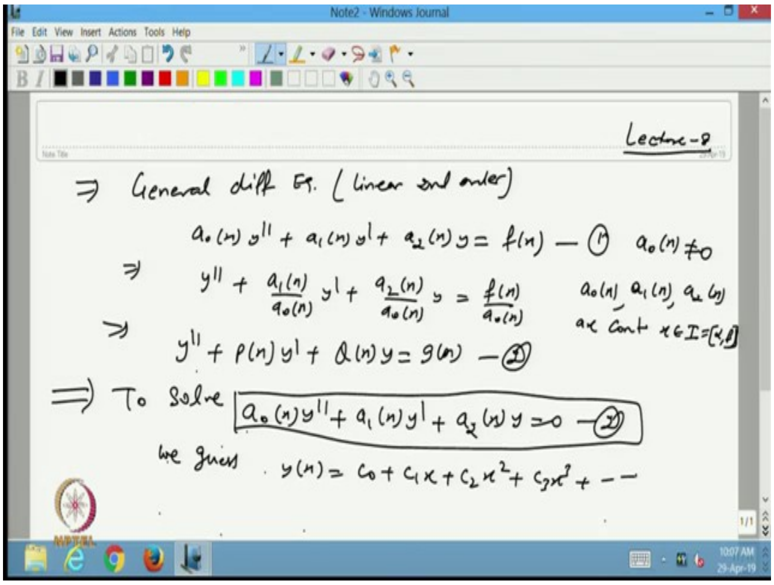

Welcome viewers, back to this course. So, today we are going to discuss lecture number eight. In the last lecture, we have discussed how we can factorise the given second order linear differential equation with the help of Riccati equation, and then we have discussed the Euler Cauchy form.

So, today, we are going to discuss the next part and now I just want to move further and I want to solve the differential equation, a general differential equation, that is linear second order of the form A 0 x y double dash + a 1 x y dash + a 2 x y = some function f x. And this equation I just give it a name, 1.

And I know that this can be further written as y double dash + a 1 x over a 0 x y dash + a 2 x y = some f x over a 0 x provided that my a 0 x, a 1 x, and a 2 x are continuous functions for x belongs to the interval that interval we are taking is alpha beta. Then, we can further write this equation as y double dash + some function P x y dash + Q x y = some g x.

So, this equation I know in the standard form. Now, so I know that if I am able to solve, so in this case, this is also taken that my a 0 x is never equal to be 0. Okay? So, my main purpose is now to solve the equation a 0 x y double dash + a 1 x y dash + a 2 x y = 0. So, this is the homogeneous part I want to solve.

I know that this homogeneous part that is equation number 2 can be solved using the help of factorization, and then we reduce it to the Riccati equation and once I know the solution of the Riccati equation, then we can find out the solution of this equation number 2. But you know that the Riccati equation is a highly nonlinear equation.

So, sometimes we are unable to find the solution for the Riccati equation. So, if we are unable to solve the Riccati equation, we cannot solve the given equation number 2 in this case. And other equation we are able to solve only is the Euler Cauchy form. So, what will happen if we have a differential equation which is not of this form, either Euler form and neither I can find out the Euler Cauchy form nor I am able to find the solution for the corresponding Riccati equation.

So, in that case, how I can solve this differential equation? And if you see that in the starting we have discussed that a differential equation in this case is a y solution multiplied by some function of x + the derivative of that function and then multiplied by another function of x and the second derivative of the function + another function of x.

So, in this case, we are multiplying the solution and all its derivative with some function of x. So, then, this comes to our mind that, because this happens only if we consider that let we guess that my y x is a polynomial. So, I just write it as C 0 + C 1 x + C 2 x square C 3 x cube and so on.

And in this case, I do not know whether my this polynomial is of finite degree or infinite degree because, in this case, if it is of finite degree, suppose I take this as a degree of 10, so in that case what will happen if I take one derivative, its degree will reduce to nine and second derivative it will reduce to eight. Then a 0 x and a 1 x depending on the values of these functions we will be able to get the solution for the equation number 2 or not, okay. So, in this case, we can expect that this polynomial may be of finite degree or it will be infinite.

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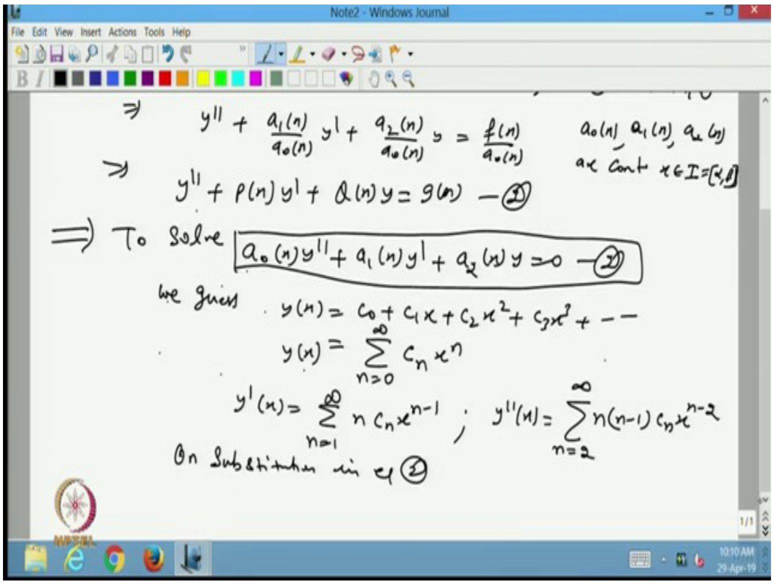

So, now, this one can be written as, I can write as n is from 0 to infinity, C n x raised to power n. So, let my solution is this one, okay, and then I take the derivative of this polynomial, or the series, we call it series also, y dash x. So, in this case, it will be n C n x raised to power n – 1. So, in this case, I have the n and n equals to 0 does not contribute anything in this series.

So, I will take it from 1 to infinity, because when the polynomial, this is a polynomial in x, and if it has infinite terms, then I can call this as a series also. I will just take further derivative, second derivative, and in that case I will get n and – 1 C n x raised to power n – 2. So, in this case, n = 0 does not make any sense, n = 1 does not contribute. So, it will start from n = 2. So, I can write from here that n = two to infinity.

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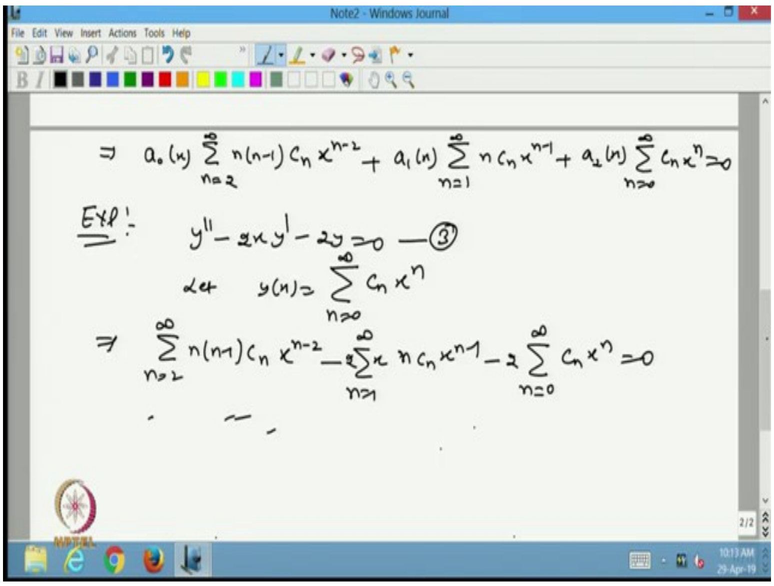

So, suppose, I substitute all this one in the equation number 2, so on substitution in equation 2 I will get, so from here I will get my a 0 summation n from 2 to infinity n n – 1 C n x n – 2 + a 1 x, summation n from 1 to infinity, n C n x power n – 1 + a 2 x summation C n x n, n from 0 to infinity = 0. Now I do not know what is the function a 0 x, a 1 x, or a 2 x because it may be the function of x also, it may be the constant also.

So, depending upon what is the value of this a 0, a 1, a 2 x then we further can solve this differential equation with the help of this series, and then we can find the solution for this equation. So, let us take one example and try to find out how this series solution can be useful in finding the solution of the equation. So, I will just take a very simple example. Suppose I take a differential equation y double dash – 2 x y dash – 2 y = 0.

So, in this case, this differential equation has constant coefficient here, but here it is function of x. So, I do not know how to solve this type of differential equation. So, then, we substitute that, let my y x, because I consider that y may be the solution, may be the polynomial of infinite degree such that that is a solution of this equation.

So, I will consider that let y x is the solution, n from 0 to infinity c n x n and then I can take the derivative of this one. So, from here, my equation, this equation I can call it number 3. So, this can be written as summation n from 2 to infinity n n – 1 c n x n – 2 – summation 2 x n c n x n – 1 and from 1 to infinity – 2 summation n from 0 to infinity c n x n = 0. So, I just substituted whatever we got after taking the derivative and then I will go further.

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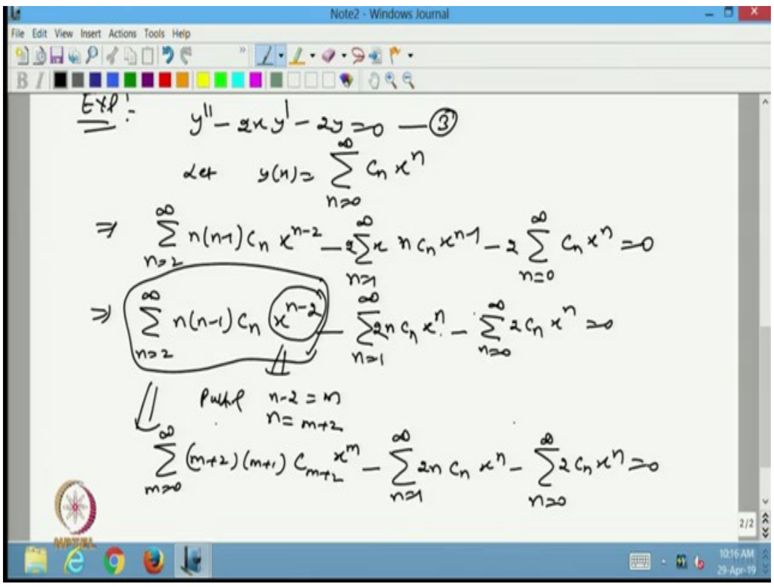

This x I can take from here. So, this equation becomes n from 2 to infinity n – 2. So, this x will multiply by this. So, it will reduce in this form. So, I will get n from 1 to infinity and c n two times x to the power n – summation n from 0 to infinity 2 C n x n = 0. Now, in this case, if you see that here it is x raised to power n, it is also x raised to power n, but here it is x raised to power n – 2.

So, first thing is that we have to make the power of x same form in all the terms. So, this is the first term we want to solve. So, in this case, I want to change this one into this form. So, what I will do, here I am putting n – 2 = m, m I am putting. So, from here I will get n = m + 2. So, this factor, the first term, this will reduce to, now, if n is starting from 2, m will start from 0.

So, from here, I can add that m is starting from 0 to infinity n is m, so I can write it as m + 2, it will become m + 1, C it becomes n + 2 and this becomes x raised to power m. And then, further, so this is the same thing, same equation, so we can write this as n = 1 to infinity 2 n C n, n from 0 to infinity 2 C n x n = 0.

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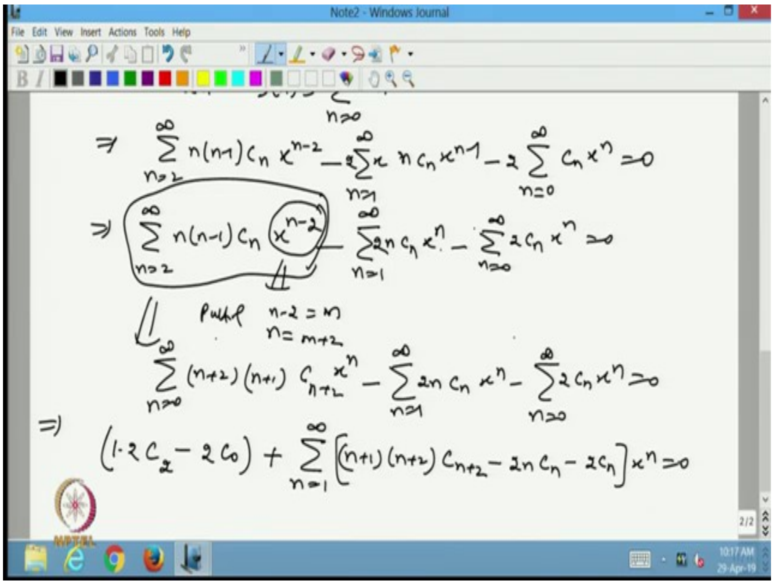

Now, this is the m, this is just the indices. So, index we can change as, so this equation after putting this one, I can change my m to n back and then I can write this equation again in the form of n. So, I can further make it as n, it becomes n + 2, it becomes n + 1, and this become C n + 2 and x raised to power n.

So, now, I have the same power x raised to power n in all the terms of the equation. Now the only thing is that, so in this case, the first term n is starting from 0, in the second term, n is starting from 1, and in the third term it is starting from 0. So, I will write the equations corresponding to n = 0 first.

So, if I put n = 0, I will get from here 1 into 2 C 2 x n. So, 0 here is a constant, n and 0 coming from here also. So, this will be 2 C 0. So, this is the terms corresponding to n = 0. And then, if I start from n = 1, I can take all the terms together. So, it will become n + 1, n + 2, C n + 2 – 2n, from here I am writing 2n, C n – 2 C n x raised to power n = 0, okay.

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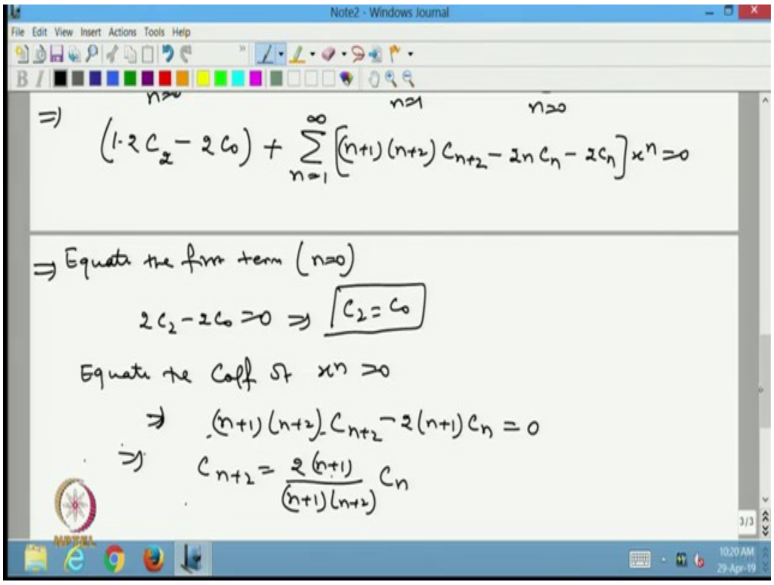

So, from this one, now, from here, what we do is that, now this whole series = 0, then what we do, to find out all the coefficients we equate the similar power of x = 0, and then we will try to find all the coefficients. So, once I equate the first term corresponding to n = 0. So, that equation we have to equate the first one. So, from here I will get that 2 C 2 – 2 C 0 and that is equal to 0.

So, from here I will get my C 2 = C 0. So, I get my coefficient C 2 in the form of C 0. Now, instead of equating each term what I can do is that I can put or equate the coefficient of x raised to power n = 0 because here we are able to take collect all the terms corresponding to the coefficient x raised to power n.

So, what I do is that I will equate the coefficient of x raised to power n = 0 on both side. So, from here I will get n + 1 n + 2 C n + 2 –. So, this is my 2 n and this is my 2. So, from here I can write as – 2 n + 1 C n = 0, which further can be reduced to this. From here I can write my C n + 2 = 2 n + 1 and then I can divide by this factor n + 2 C n.

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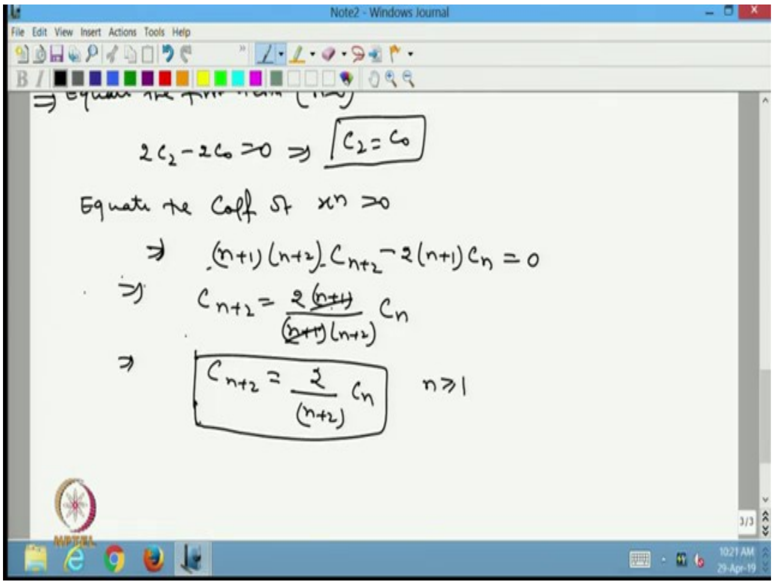

So, from here further I can write it as, so this will cancel out, so I will get C n + 2 = 2 times divided by n + 2 C n where n is starting from 1. So, that is greater than or equal to 1. So, from here, I will get this relation in which if I know the value of C n I can find the value of C n + 2.

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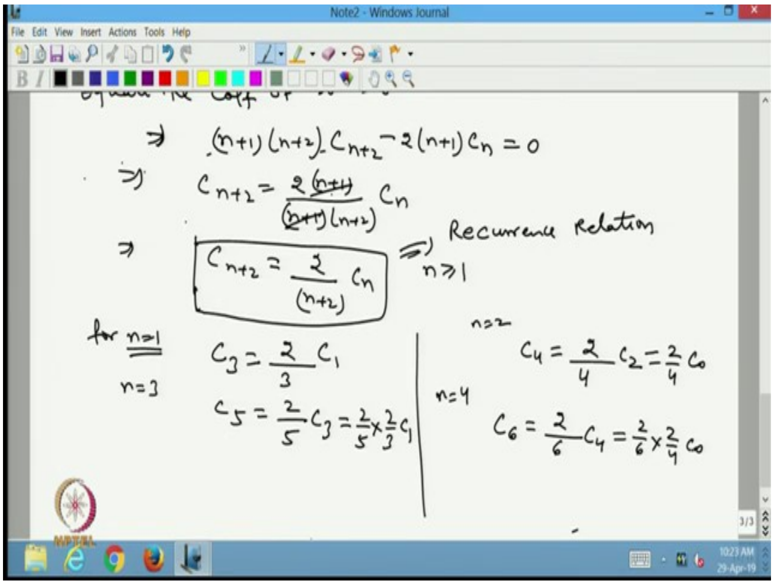

So, this type of equation is called the recurrence relation. So, in the recurrence relation if I know one coefficient, I can write another coefficient in the form of that coefficient. So, I know that my C 0 = C 2. So, for n = 1 I can write my C 3 as 2 over, so this is n = 1, so this will be 3 C 1. Next, for n = 2 I will get C 4 = 2 times it is for 4 C 2. But C 2 I know that that is also is equal to C 0. So, from here I can write as 2 by 4 C 0. Then, n = 3 I will get C 5 = 2 by 5 C 3.

But C 3 is given to me in the form of C 1. So, from here I can write 2 by 5 multiplied by 2 by 3 C 1. Then, I can find out for n = 4, so it will be C 6 will be 2 divided by 6 C 4 and C 4 is given to in the form of C 0, so from here I can write 2 by 6 into 2 by 4 C 0. So, in this way I can find all this one, coefficient for odd power I will get the coefficient in terms of C 1 and for even n we will get all the coefficient in terms of C 0.

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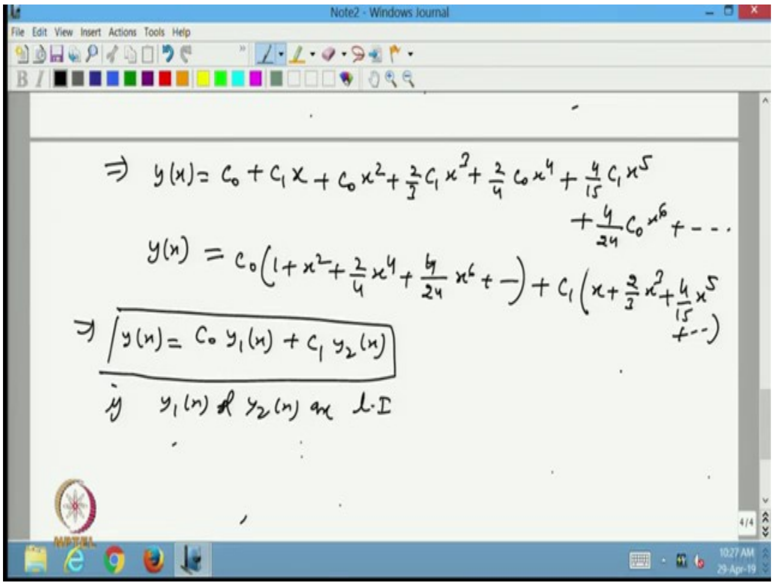

So, once I am able to write this one, that I am able to find out the solution in this form, then from here I can write my solution y x. So, y x is C 0 + C 1 x +, C 2 is already C 0, so I can write it as C 0 x square. Then C 3, so C 3 I know in the form of C 1, so I can write it here 2 by 3 C 1 x cube. Then C 4 in terms of C 0, so I can write it as 2 by 4 C 0 x raised to power 4.

Now C 5 is also in terms of C 1. So, I can write it as 4 by 15 C 1 x raised to the power 5, and then further I can write it as C 6, it will be 4 by 24 C 0 x 6 and so on. Now, from here I can solve further and I will collect the term corresponding to the coefficient C 0 and C 1. So, if I collect the terms corresponding to C 0, I will get 1 + x square.

So, this is x square + 2 by 4 x power 4 + 4 by 24 x raised to power 6 and so on, + I then collecting the terms corresponding to C 1. So, this will be x + 2 by 3 x cube + 4 by 15 x 5 and so on. So, in this case, I will get the solution y x. Y x is the general solution of the equation.

So, once I know the general solution of the equation, then I can write this as y x =, now C 0 and C 1 are the arbitrary constants, so I can write it as C 0 and then I will consider this as a function that is y 1 x + C 1 and that is y 2 x. So, this is the general solution that we know that contains two arbitrary constants and that constants I can take as C 0 and C 1, and I consider this y 1 x is one solution and y 2 x is another solution.

So, this one is the general solution. If, the question is that y 1 x and y 2 x are linearly independent, because here it is just an infinite series and here also it is infinite series and I want to check whether this infinite series and this infinite series are linearly independent or not.

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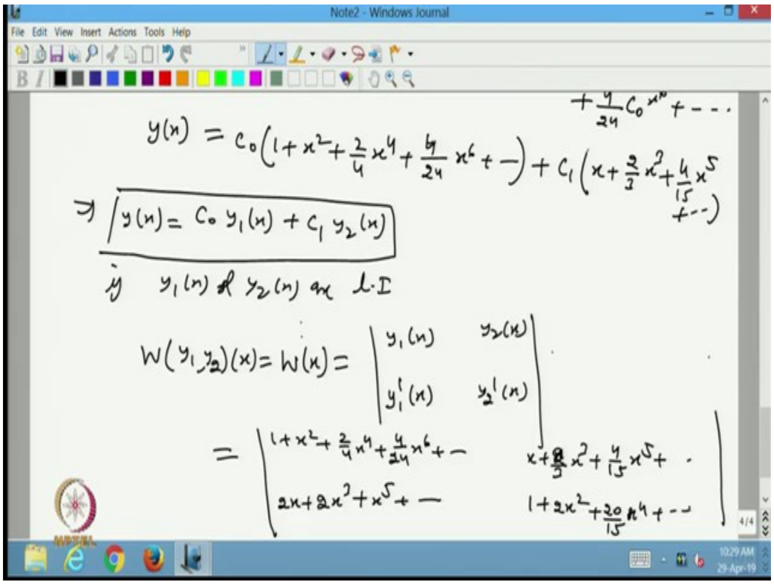

So, how we can verify or check whether these two functions y 1 x and y 2 x are linearly independent or not, then we go further, go back to that Wronskian and I know that the corresponding Wronskian corresponding to y 1 and y 2 x this can be written as Wx. So, this I know that I can write y 1 x, y 2 x, and this is y 1 dash x and this is y 2 dash x. So, I will try to find the Wronskian for this corresponding solution.

So, from here, I can write this Wronskian as 1 + x square + 2 by 4 x power 4 + 4 by 24 x power 6 and so on. y 2 is this one, so this I will take as x + 2 by 3 x cube + 4 by 15 x 5 and so on. Then, I am taking the derivative of this one. So, this will be 2 x +, 4 will cancel out, it will be 2 x cube +, so 6 will cancel out, so it will be x 5 and so on.

And this will be 1 +, 3 will cancel out, it will be 2 x square, 5 will cancel out, it will be 20 by 15 x power 4 and so on. Now, if I want to take the determinant of this matrix, then it is very difficult because if I take the derivative of this one, so this polynomial and this polynomial will be multiplied – the multiplication of this polynomial. But here, we are going to use the concept of the Wronskian at one point that we have discussed in the previous lectures.

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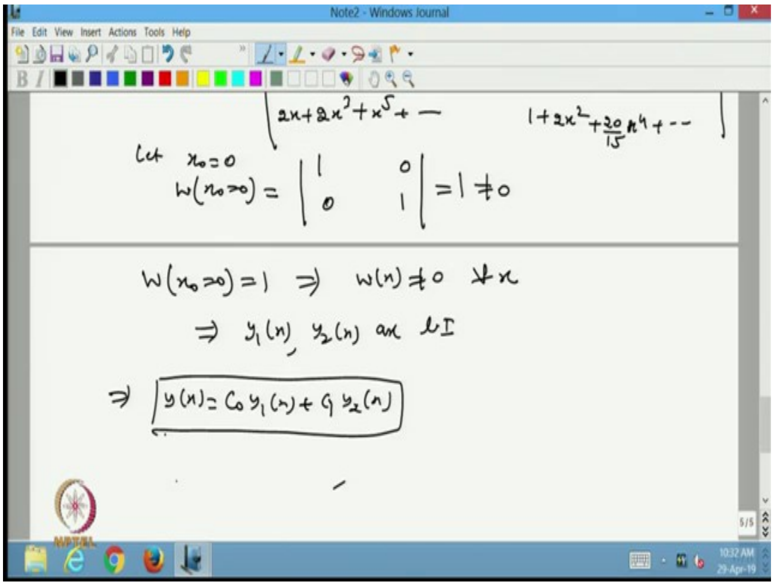

So, now, let, I take x 0 = 0 and I try to find the Wronskian at x 0. So, if I take the value x 0 = 0 and try to find the Wronskian here, so it will reduce to 1 0 0 1, just putting x = 0 in the given polynomial and value would be 1 which is never 0. And we know from the previous lectures that, so in this case, my Wronskian at x 0 = 0 is 1, and from my previous knowledge of Wronskian I can say that this Wronskian is never 0 for all values of x.

So, once if I know that the Wronskian is never 0 for all values of x, so from here I can say that my solution that y 1 x and y 2 x are linearly independent. So, if they are the linearly independent solution, then we can say that there is a general solution. So, in this case, you have seen that the solution y 1 x and y 2 x are the polynomials of not a finite degree. So, this I can say that, from here, I can say that my infinite series, my y x is C 0 y 1 x + C 1 y 2 x and this is in a series, infinite series, for that one.

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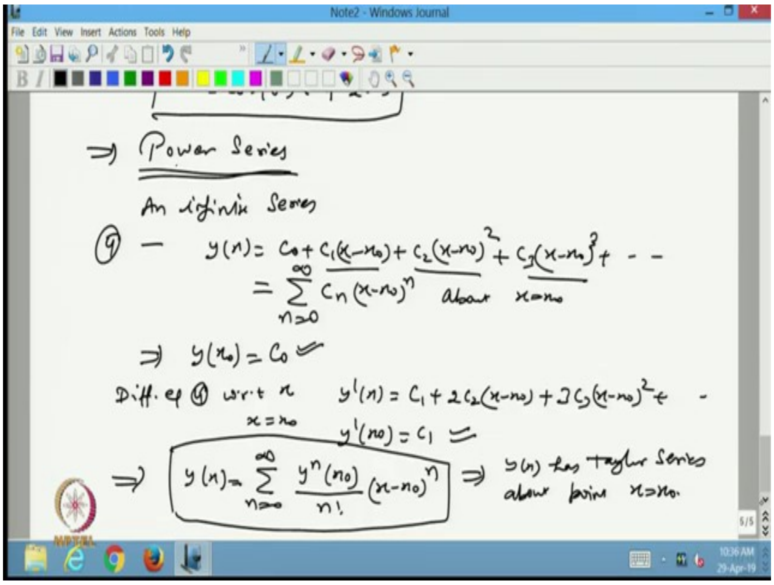

So, from here, I can start a new concept that a solution of the form of power series. So, this we call it as a power series solution. So, from here I can say that an infinite series of a function y x = C 0 + C 1 x – x 0 + C 2 x – x 0 square C 3 x – x 0 cube and so on. So, this one I can write as C n x – x 0 power n, n = from 0 to infinity. This is called the power series of the function y x about x = x 0.

So, this is the power series of function y x about a point that is x 0. So, this is the power series of the function y x about the point x = x 0. So, this is the power series of this one and so, y x is C 0 + C 1, that is already there. Now, I want to find what is the value of C 0, C 1, C 2, C 3, because these are the coefficients and the constants and that we need to find with the help of y x. So, let us suppose I want to find my C 0. So, what I do is that, I just take it, give the name 4.

So, suppose I want to find the value of C 0 because this function has a power series, this one on the right hand side. I will try to find the value of the y at x 0. If you see the value of the x 0, all these terms will become 0, and from here I will get my C 0. Further, if I want to find the value of C 1, what I do is differentiate equation 4 with respect to x. So, what I will get, I will y dash x = C 1.

So, from here I will get the C 1 only, and then from here 2 C 2 x – x 0 + 3 C 3 x – x 0 square and so on. Now, what I do is that putting x = x 0, I will get y dash at x 0 = C 1. So, from here I am able to find the value of C 1. From here I am able to find the value of C 0, and if you move further, then to find the value of C 2 I will take one more differentiation of this equation and then I will get the value of C 2, C 3, and so on.

So, from here we see that I can write my equation as y x = summation n from 0 to infinity and the C n is in this case it is just y x 0, it is y dash x 0. So, from here I can write that it will be n^{th} derivative at x 0 divided by n factorial and this will be x – x 0 power n. So, if we take the help of the function and all its derivative to find the coefficients, then the coefficients will be this form and in that sense I will say that y x has Taylor series about point x = x 0. So, in that case, I can say that this power series becomes the Taylor series.

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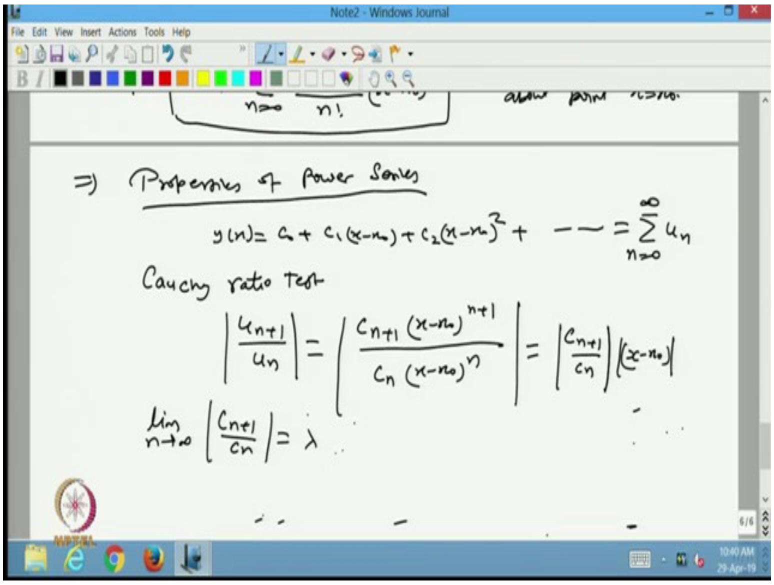

So, now, there are some properties of power series. The first thing is that I know that function y x has the power series, that is C 0, C 1 x, C 2 x square, and C 0 + C 1 x – x 0 + C 2 x – x 0 square and so on. So, first thing is that I want to find out how I can check that this is a convergent or not. So, in that case, I will apply the Cauchy ratio test to find out if the series is convergent and what will be the radius of convergence of that series.

So, in this case, I will apply the Cauchy ratio test. So, Cauchy ratio test, I will take C. So, if this series can be written as summation U n, n from 0 to infinity, then I know that with the help of ratio test it will be U n + 1 over U n. So, from here I can write, it will be C n + 1 x – x 0 and + 1 divided by C n x – x 0 modulus value because we know that the ratio test we can apply only for the positive values.

So, in this case, I am taking the absolute value. So, that will become C n + 1 over C n. So, this is the ratio the coefficient and this will become x – x 0 power 1, so this one. So, now, assume that let limit n tends to infinity C n + 1 over C n becomes lambda. So, this is a convergent series of coefficient and its value is equal to lambda.

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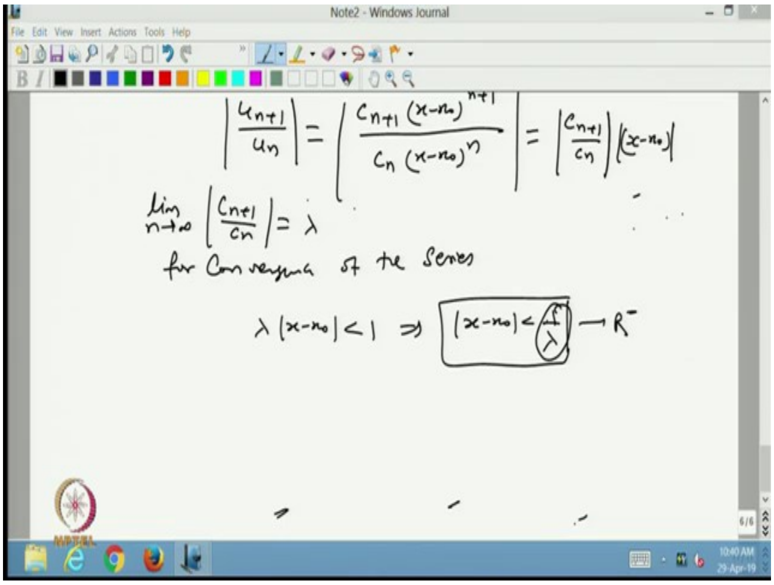

So, from here, for convergence of the series, in that case my value, this will be lambda x – x 0 less than 1 because I know that if ratio is less than or equal to 1 as n tends to infinity, then it will be convergent. So, from here I can say that my x – x 0 will be less than 1 over lambda. So, this factor I call it as R.

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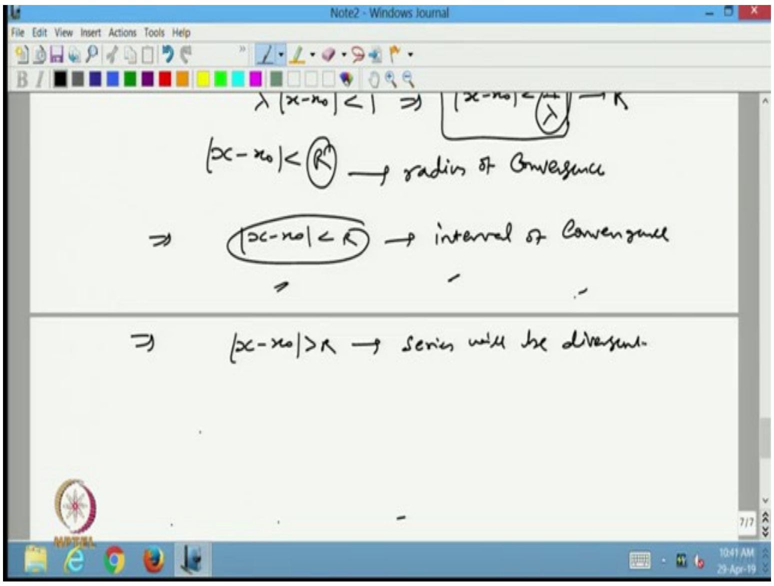

So, I can say that x – x 0 modulus value is less than R. So, this R is called radius of convergence of this power series and this x – x 0, the whole value is called the interval of convergence of the power series, and I can say that x in the region x – x 0 is greater than R, then the series will be divergent. So, this is the radius of convergence we found.

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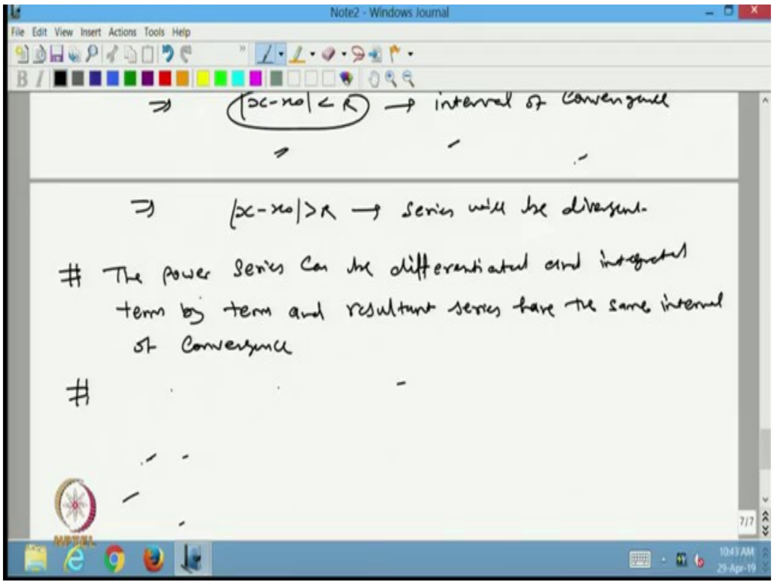

Then, the next property is that the power series can be differentiated and integrated term by term and the resultant series have the same interval of convergence because whenever we take the power series of function then it can be differentiated term by term and it can be integrated term by term and the resultant series will have the same interval of convergence.

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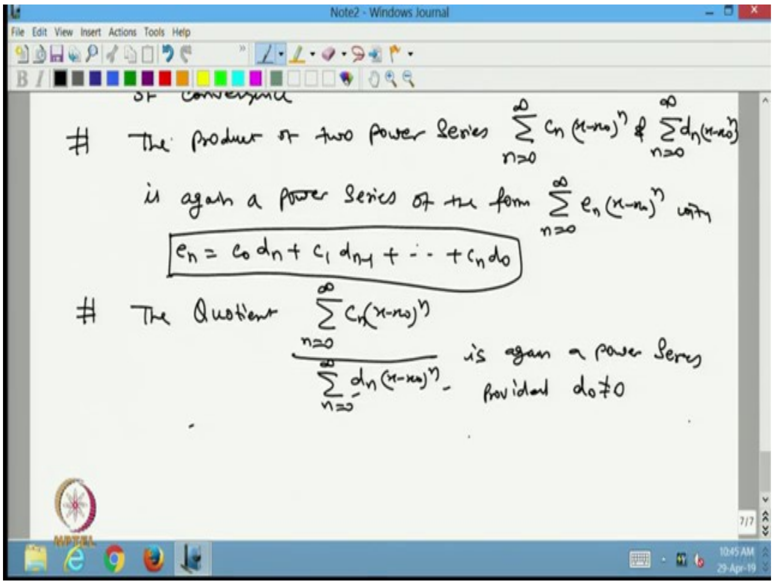

Next is that the product of two power series, for example, I have two series and from 0 to infinity C n x – x 0 power n and summation n from 0 to infinity d n x – x 0 power n. So, this is the two series I am taking, then the product of the two series is again a power series of the form n from 0 to infinity e n x – x 0 power n.

The e n term will be C 0 d n + C 1 d n – 1 and so on + C n d 0. I will multiply the coefficients. So, that will be again the power series. And the next one is that the quotient, that is, C n x – x 0 to power n divided by d n x – x 0 power n is again a power series provided d 0 is never equal to 0. So, if I take the ratio of the two power series, then it is again a power series with d 0 is not equal to 0.

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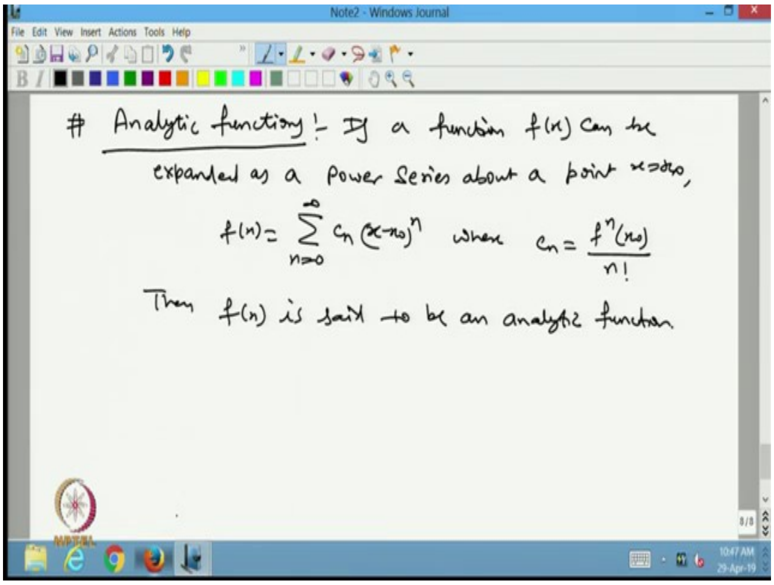

Next is that, now, after doing this one we will try to give a new concept that is called analytic functions. So, what is the meaning of analytic function. So, if a function f x can be expanded as a power series about a point that is x = x 0, so that we can write as my function f x can be written as summation n from 0 to infinity Cn x – x 0 power n where my C n is the n^{th} derivative at x 0 divided by n factorial.

Then, the function f x is said to be an analytic function because in this case we know that this function and all its derivatives exist because we are able to find the coefficient C ns only when with the help of the derivatives as we have seen that I can find the C 0 by putting the value of x = x 0 in the equation.

Then C 1 I can put taking the derivative C 2 in the next derivative. So, in this case my function f x is said to be analytical in another form I can say, that it said to be analytic function if all its derivative which is at the point x = x 0. So, this is the definition of the analytic function.

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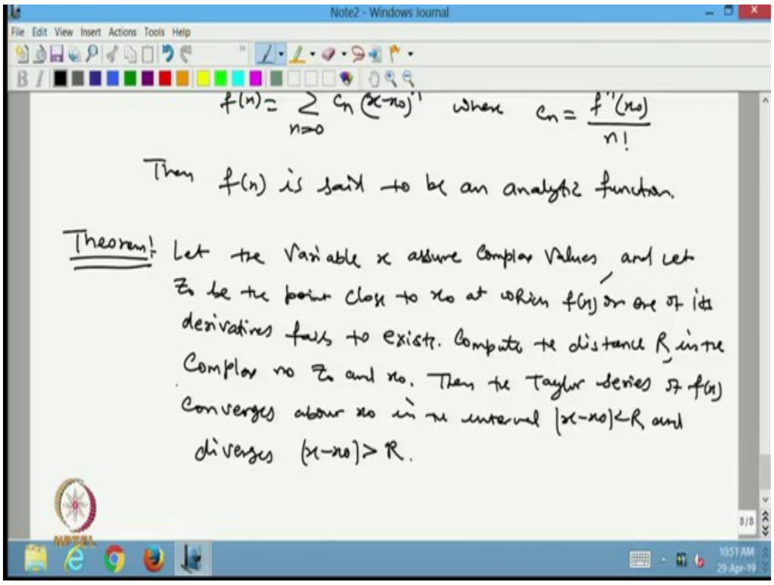

Now, one theorem we want to discuss is that, let the variable, so in this case you can see that this function is a real-valued function, it is defined on the x and it is a real-valued function. So, let the variable x assume complex values and let Z 0 be the point close to point x 0 at which function f x or one of its derivatives fails to exist. Then, what do you do? You just compute the distance R in the complex number Z 0 and x 0.

Then the corresponding Taylor series of f x converges about x 0 in the interval x – x 0 less than R and diverges for x – x 0 is great than R. So, in this case, what we have done is that we have the power series and in that power series suppose that at some value of the x equals x 0 or some complex number the function f x fails or all its derivatives fail, then the given power series has the convergence in this region and it will diverge for the region x – x 0 greater than R. Okay?

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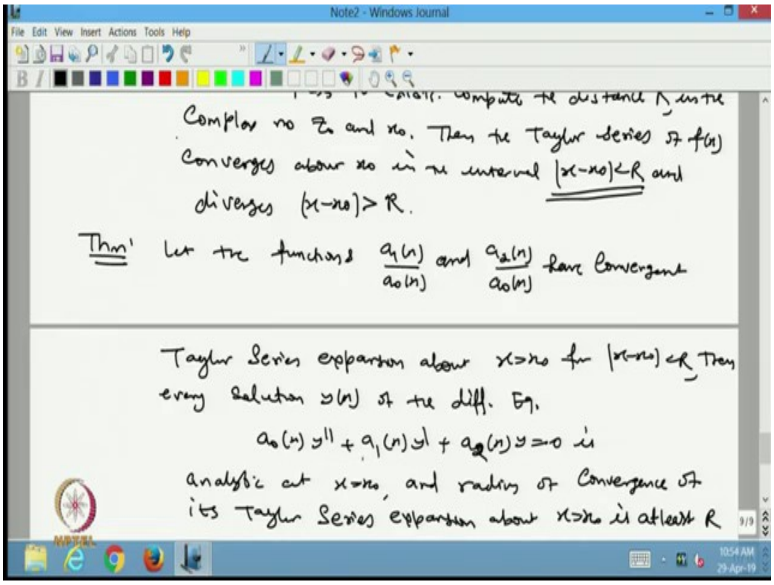

So, one more theorem it would be useful for the definition. Let the function a 1 x over a 0 x and a 2 x over a 0 x have convergent Taylor series expansion about x = x 0 for the region x – x 0 is less than R, then every solution y x of the differential equation a 0 x y double dash + a 1 x y dash + a 0 x a 2 x y = 0 is analytic at x = x 0 and radius of convergence of its Taylor series expansion about x = x 0 is at least R.

So, this theorem says that, because in this case if I divide by a 0 x, so if a 1 x a 0 x and a 2 x divided by a 0 x have the convergent Taylor expansion about x = x 0 and the radius of convergence is R then every solution of the differential equation y x is analytical at x = x 0 and the radius of convergence will be at least R, so that is lecture number eight.

So, thanks for viewing this. Today in the class we have discussed how, instead of having the exponential solution how we can have the solution that is infinite series for the given differential equation. So, in the next class we will go further about the series solution. Thank you very much.