Introduction to Methods of Applied Mathematics

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Lecture – 7
Factorisation of Second Order Differential Operator and Euler Cauchy Equation

Welcome viewers, welcome to lecture number 7. So, in the last lecture, we have discussed the variation of parameters to find the particular solution of a linear second order differential equation. So, in the last lecture I have discussed the problem.

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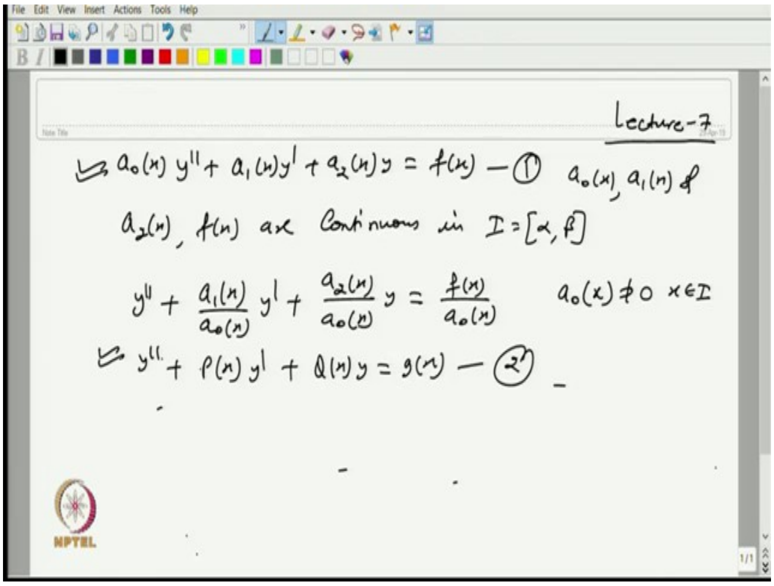

So, this is lecture 7. So, now, suppose I have a second order linear differential equation with a 0 x y double dash + a 1 x y dash + a 2 x y is equal to some function f x on the right hand side. So, this is the problem we have taken where a 0 x a 1 x and a 2 x f x are continuous in the interval I, that is the interval that we have taken from alpha to beta.

So, this number equation I can also write as y double dash + a 1 x over a 0 x y dash + a 2 x over a 0 x y is equal to f x divided by a 0 x provided that a 0 x is not equal to 0 when x belongs to the interval I. So, this equation I can also write in the form that, sometimes we also write this differential equation in the form of P x. So, this is P x, this is Q x, and this I give a new name that is g x.

So, sometimes we are dealing with equation number one and sometimes we are dealing with equation number 2. So, this is the general form of the differential equation and this is the standard form. Standard form means that the coefficient of highest power y double dash is equal to 1. So, in the last lecture we have discussed how to solve this type of equation or this type of equation using the method of variation of parameter.

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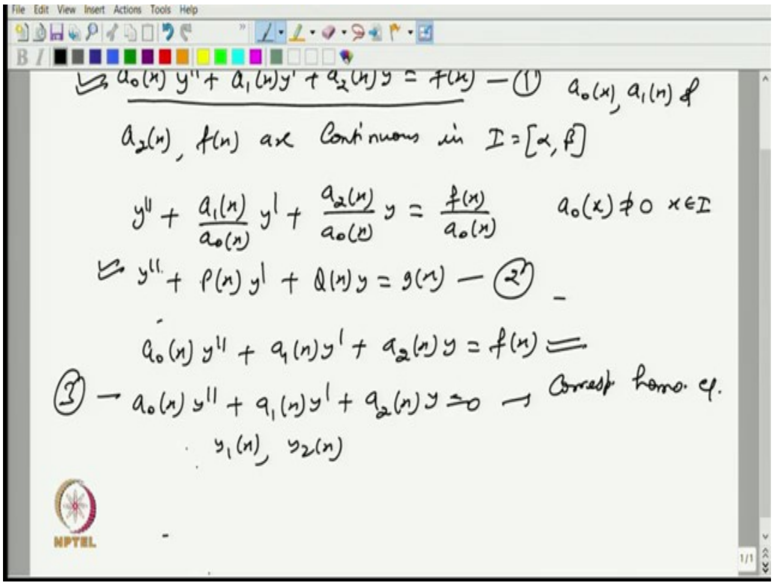

But in the variation of parameter it was a condition that, suppose I want to solve this type of equation, 1, and so, suppose I am dealing with the equation a 0 x y double dash + a 1 x y dash + a 2 x y is equal to f x and I want to solve this differential equation for any function f x on the right hand side using the method of variation of parameter. But, we know that to solve this one I should be able to solve the differential equation a 1 x y dash a 2 x y is equal to 0.

So, this is the corresponding homogeneous equation. So, if I am able to solve the differential equation, this is the homogeneous question, I should write it as number 3, then if I solve this one I should be able to get two linearly independent solutions y 1 x and y 2 x, and once I know the y 1 x and y 2 x, then with the help of the variation of parameter I can find the solution for this equation.

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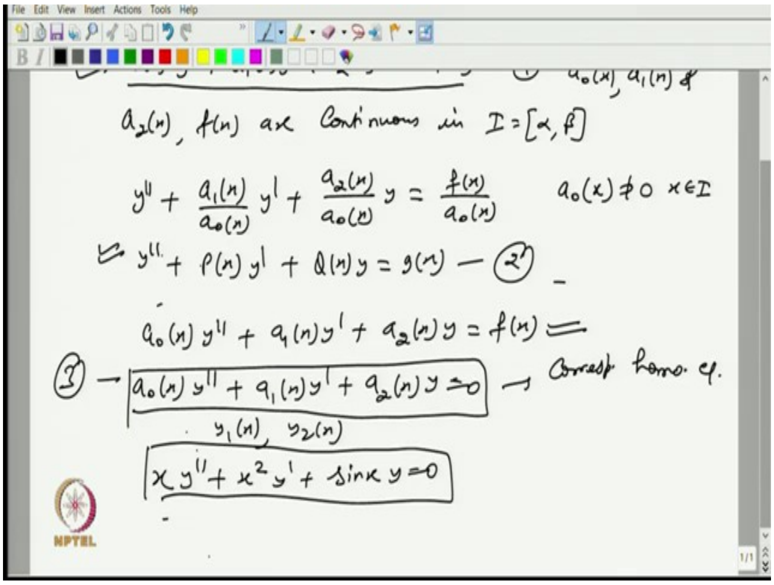

So, in this case it means that the real challenge is to solve this type of equation. So, this is the homogeneous equation, but in this case we know that a 0 x, a 1 x, and a 2 x is function of x and it may be the constant. If it is a constant function, then, we know that with the help of the characteristic equation I can find the roots of the equation. And then we can find out the linearly independent solution y 1 x and y 2 x, that we have done. But, what about if I do not have this function as a function of x.

Like suppose, I have a differential equation like x y double dash + x square y dash +, suppose, sin x y is equal to 0. So, suppose if I am able to solve, find out the solution for this type of equation, only then I would be able to solve, if instead of 0 I take any function f x using the variation of parameters. So, variation of parameter is good for solving, to find out the particular solution, but for that we should have the linearly independent solution for the corresponding homogeneous equation.

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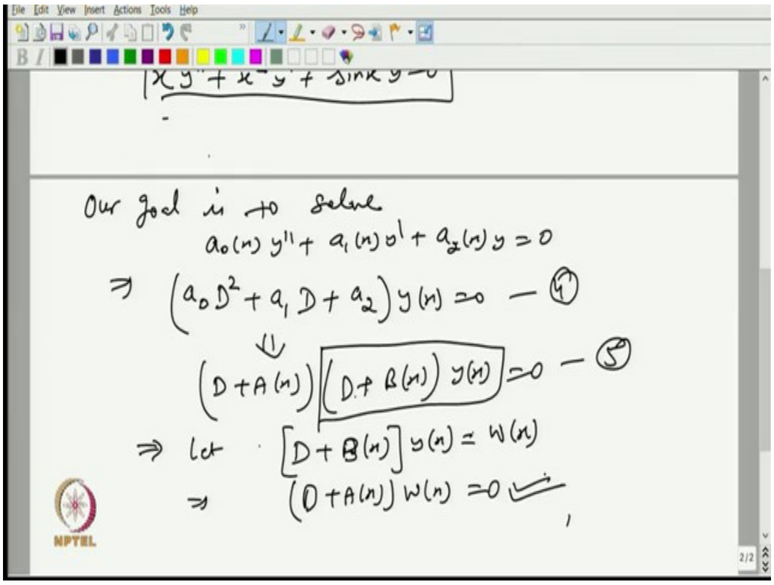

So, in this case, it means that now our main goal is to solve this equation, a 2 x y is equal to 0. So, this equation I can write in this form. So, I just write a 0, but it is understood that it is a function of x, I can it as D square + a 1 D, because I am writing this one in the differential operator form, a 2, and this one I write as a y, y is a function of x is equal to 0.

Now, if suppose I am able to factorise this differential operator in the form of D + sum function of A x, D + some function of D x, y x equal to 0. It means somehow I am able factorise this operator into this form, then with the help of this one now I can solve this differential equation, then what I do, I will take this one. So, let, I will consider that D + B x, y x, is equal to, suppose I choose some W x and then this equation becomes D + A x W x equal to 0.

So, this equation I should write it as number 4 and this number 5. So, the equation number 5 can be solved by converting this one into the two first order linear differential equations. So, in this case, what I have done is that D + B x y x, I consider that as W x, and then putting the W x here I will get D + A x W x equal to 0. So, I will do that. Now, in this case first I will solve this equation, this is a first order equation.

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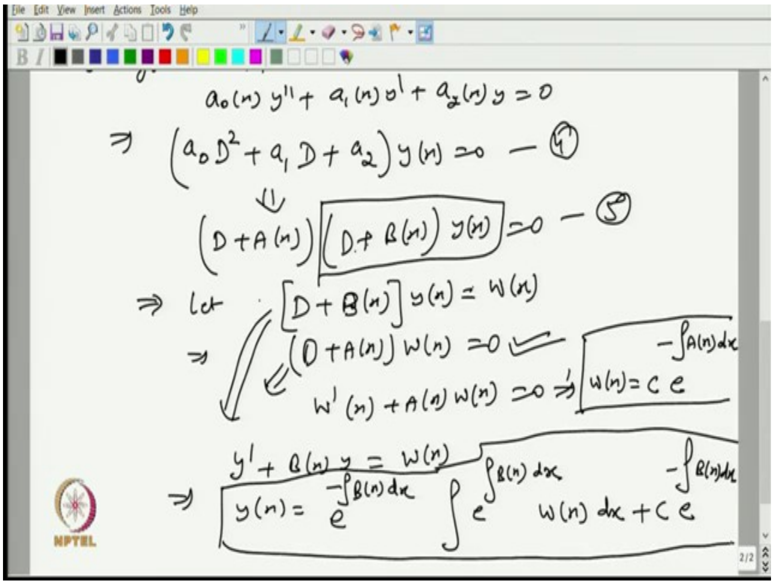

So, from this equation, this equation becomes W x + A x Wx equal to 0. So, this is the first order homogeneous equation and I know that from here my W x can be written as some constant C e raised to power – A x d x. So, this is my solution.

So, once I am able to solve or find out this solution W x I will put this W x here in this equation from here. Once I know the value of W x I will find out, so this is my y dash + B x y is equal to W x. So, W x is known to me, then I know that the solution of this equation, so y will be, by the help of the integrating factor, the solution in this case will be e – B x d x, and then multiplying e B x d x into W x d x + constant times B x d x. So, that will be my solution. So, this will be my solution in this case.

So, I am able to solve the equation number 4 into the sequence of two first order equations. First, I will solve this equation that I can solve because it is linear homogeneous, and then with the help of this one, I want to find the solution Y x. But the question is how to solve this one. So, let us consider that I have the differential equation a 0, so what I do now, just for the simplicity I will just try to solve the standard form.

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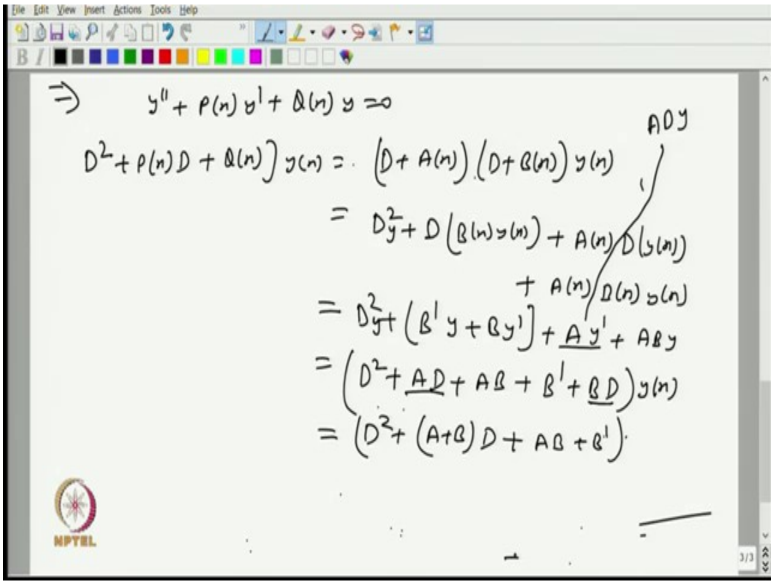

The standard form was my y dash + some P x y dash + Q x y equal to 0. So, this equation I write just in the form of D square + P x D + Q x + y x. If I am able to write this one as D + some A x D + B x y x, if I am able to do this one, then I am able to solve this differential equation. So, this one is, this is the factor, so I just want to find out if I multiply this operator.

I operate this operator on y x and then operate this one on this operator, I will get this operator D square + D on B x y x + A x, then D of y x + A x B x y x. So, this is, we are able to write. From here I can write this as D square +. So, this is a differential I am putting on this one. So, I will apply the product rule.

So, this will become B dash y + B y dash. So, this one becomes this one +, or you can write it as A, just it is a function of x but just for the calculation becomes easy I am writing it as A, then y dash A B y. So, from here, if I am able to find out this one, so this becomes D square + I can write it as AD, this factor I can write it as AD, + this is my AB +, I can write it as B dash +, then I can write it as BD.

The whole equation I can write as this one because D square y, so this is also y, so D square y + AD will come from here. So, this is AD because this one can be written as ADy. So, this is AD, AB is coming from here, B dash is this one, and B is this one. So, from here, I can write it as D square + A + B, that is D. So, I am taking the D common from here and here, + AB + B dash y x.

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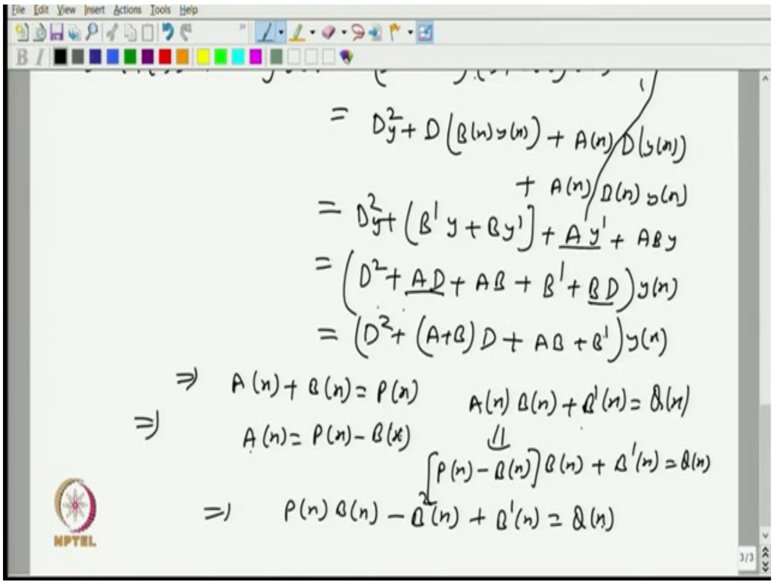

Now, if I compare this equation with the left, which implies that my A + B should be equal to P. So, I can write now in the form of x. So, I can write from here that my A x + B x becomes equal to P x, and then my A x B x + B dash x is equal to Q x. So, from here I can write my A x is equal P x – Bx. Okay?

So, from here I can find my A x and putting this in this equation I will get P x – B x, this one I am substituting here, into B x + B dash x is equal to Q x, so this I will get. So, now, if I further simplify this one, this equation becomes P x B x – B squared x from here, then B dash x is equal to Q x, okay?

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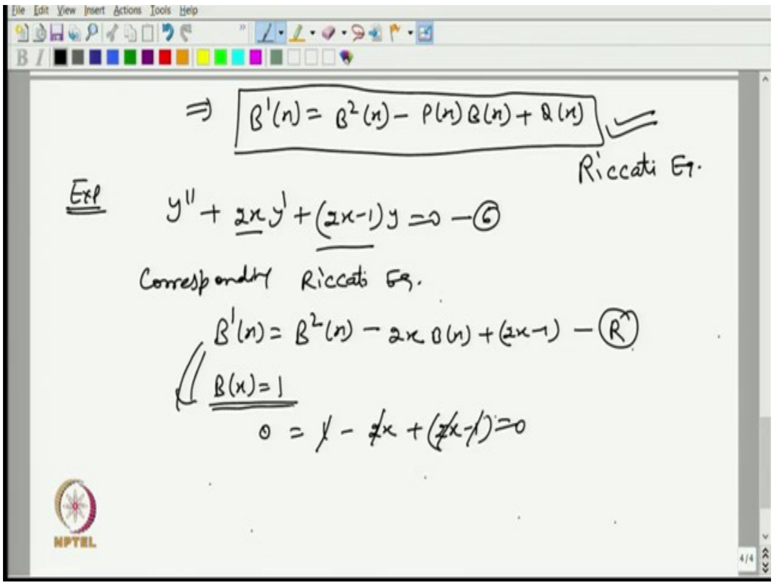

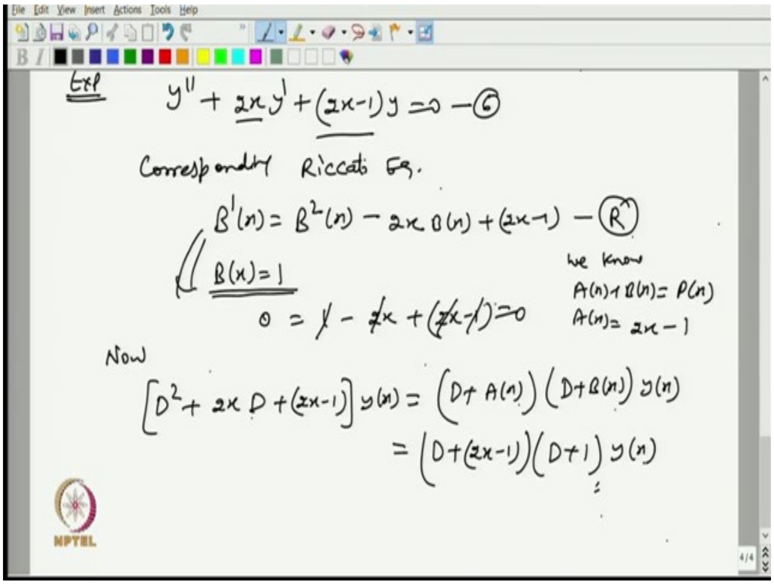

Now, from here, if I simplify this one, I just take B dash x on the left hand side. On the right hand side, I will get B squared x – P x B x + Q x. So, this I will get, and if you remember that this is one of the most famous differential equations. It is non-linear differential equation we have solved either in lecture 1 or 2.

So, this we call it the Riccati equation. So, this is a famous Riccati equation because this is a B dash + B square – P x B x + Q x. So, now, this is a nonlinear equation. So, the only thing is that if I am able to solve this Riccati equation, it means that the corresponding operator, the differential operator, this one, we are able to factorise and then we are able to solve this differential equation.

So, this is how the Riccati equation is important and this is associated with solving the homogeneous differential equation. So, let us do one example, how we can solve this equation. So, let us just take one example. Suppose I want to find the general solution for the equation 2x y dash + 2x – 1 y = 0. So, I want to solve this differential equation and I want to find the solution for this homogeneous equation.

Now what I do is that I want to write the corresponding Riccati equation. So, this I just write number, what is the number, it is 5. So, let us take it equation number 6. Now, I write the corresponding Riccati equation. So, this Riccati equation I can write as B dash x = B square x –, so Px is this one, so it will be 2x B x, and Q x is this one, + 2x – 1. So, this is the Riccati equation I have. So, I just call it Riccati R.

Now, if you see this one I just, because I have taken this equation, if you see that, if I take B x = 1, then let us check whether B x equal to 1 is the solution of this Riccati equation or not. So, just putting in this equation, B dash become 0 is equal to 1 – 2x + 2x – 1 because I am choosing B x = 1, so derivative will be 0 = 1 – 2x + 2x – 1. So, this will cancel out and this will cancel out. So, this is equal to 0. So, I can say that this Riccati equation has a solution B x equal to 1.

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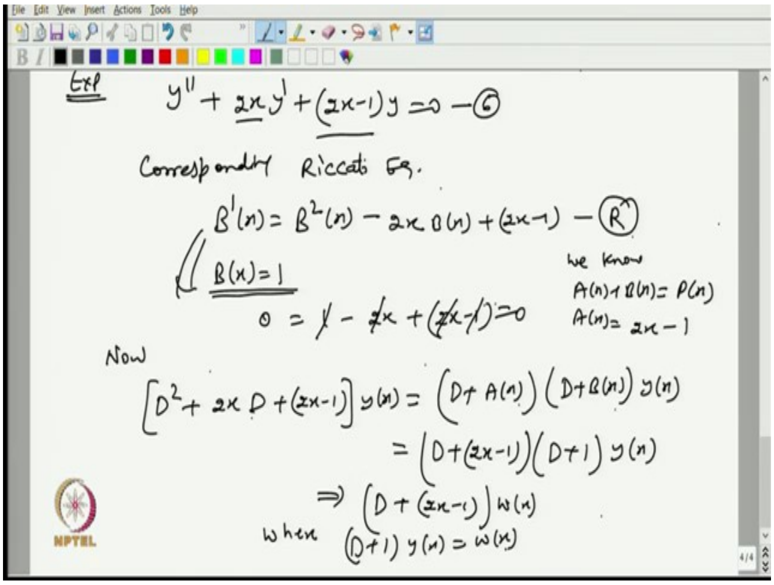

So, my equation D square + 2x D + 2x – 1, y x becomes D + A x, D + B x y x, so this becomes, where my B x is, so B x is this one. Now, we also know that A x + B x = P x, that we already know. So, I know the value of a B x, so my A x in this case will be 2x – 1, because B x is 1. So I can find the A x that is 2x – 1.

So, from here I can write that my differential operator can be factorised in the form of 2x – 1, D + 1, so B x is 1, y x. So, once I am able to factorise my differential operator then from here what we write is that D + 2x – 1, I just write it as W x where D + 1 y x = W x. Okay?

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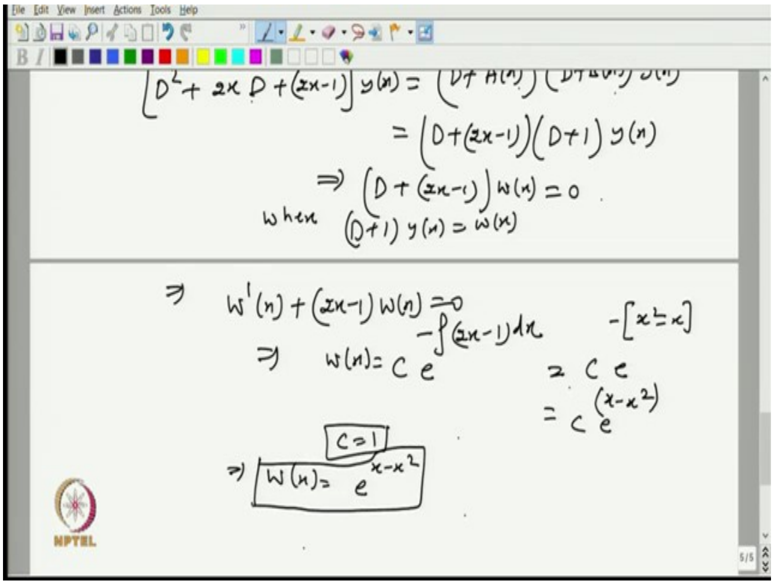

So, from here, now what I need to do is that, so this is equal to 0, because this was equal to 0. So, now we have to solve this equation. So, from here, first I will solve this one. So, this equation becomes W x + 2x – 1 W x = 0, and from here I will get my Wx is C e raised to power – 2 x – 1 dx.

Because in this case we need only one solution, so I will take C = 1. So, this can be written as C e and –. So, from here I can write it as x square because x squared 2 will cancel out, – x, so this will become. So, from here I will get C e raised to power x – x square, okay? And I choose C = 1 because in this case we need only one solution, so I can choose my C = 1. So, from here what I will get is my W x is e raised to power x – x square.

So, once I am able to find this one, now I want to solve another equation. So, that will be D + 1 y x = W x. So, W is known to me now. So, from here, I will get y dash + y = W x and from here I will get my y x = e raised to power, because P x is 1, so – D x integration, e raised D x, because 1 is there, into W x. So, W x is x – x square, dx + a constant of integration and then dx.

So, from here, my y x will be e raised to power – x and then I just integrate this one. So, it will be e x and x, so 2 x – x squared dx, this is the integration, + constant e raised to power – x. So, that is my solution for the corresponding equation whatever we want to solve. So, this is the equation 6 and that is the solution for the equation number 6.

So, with the help of the Riccati equation I am able to factorise the equation and then we are able to solve and find out the solution y x. So, this is a solution we are able to solve. Similarly, if you see that this is one of the solutions, so I can say that this is one of the solution. But we know when we were in lecture number 1 or 2 where we have discussed how to solve the Riccati equation.

So, in that case you know that if we know one solution of the Riccati equation, then we can find another solution of the Riccati equation. So, this is the solution corresponding to the solution when B x = 1. So, with the help of B x = 1 we can find another solution of the Riccati equation, and using that equation we are able to find another solution for the same problem and then this solution will be the linearly independent solution.

So, we are able to solve a homogeneous second order linear differential equation to find the solution that is y 1 x and y 2 x, because once we know that y 1 x and y 2 x then we are able to solve any differential equation with the function f x on the right hand side as a continuous function with the help of the variation of parameter.

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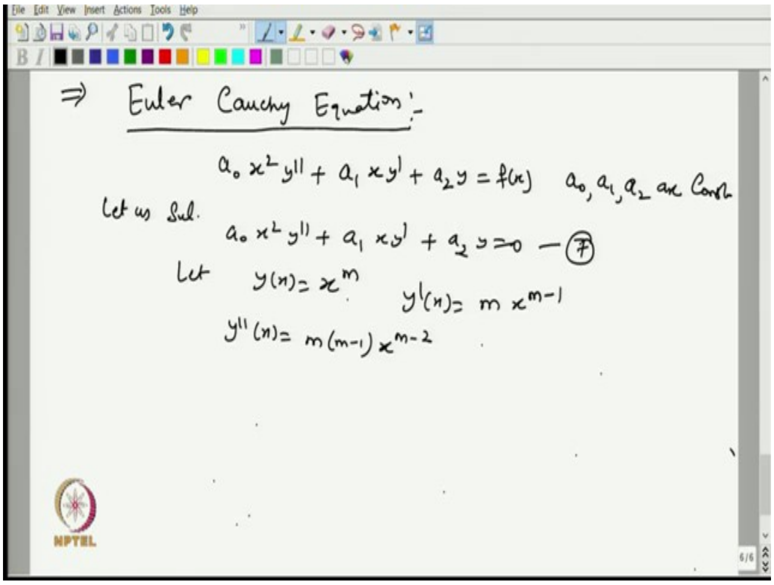

So, we move further and then I will discuss another form of the differential equation and that is called the Euler Cauchy Equation. Now, in the last example we have the function a 0 x, a function of x, a 1 x a 2 x, but now in the form of the Euler Cauchy equation, if we have some particular form of the coefficient of the differential equation. For example, suppose I have a differential equation of this type a 0 is a constant, x square y double dash + a 1 is a constant x y dash + a 2 is a constant y is equal to some function f x where a 0, a 1, and a 2 are constants.

So, in this case, if you see that the second derivative is multiplied by x square, the first derivative multiplied by x and this is the function y = f x. So, this is the linear non-homogeneous second order differential equation. So, in this case, suppose I want to solve this differential equation, so let us solve a 0 x squared y double dash + a 1 x y dash + a 2 y = 0.

So, let us first solve the homogeneous part of this equation. Now, so, in this case, I want a function y so there is a solution of this differential equation, which is being multiplied by x + also multiplied by x square and the summation is equal to 0. It means the function y should be a type of polynomial because if I take the derivative of that polynomial, again, I multiply by x and I take the second derivative of polynomial, again I multiply by x square.

And then the same degree of the polynomial should be there and then it combines and makes it equal to 0. So, from here, I just choose that, let my y x that is the solution is some x raised to power m, because in the form of polynomial I know that my y dash x will be one degree lesser than this one and y double dash x will be m m – 1 x raised to power m – 2.

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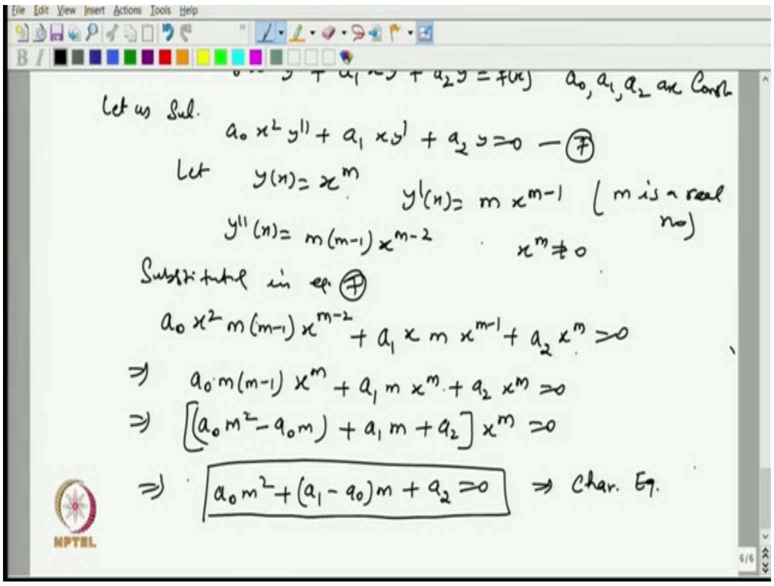

Now substituting in this equation, so I call it equation number 7, so substituting in equation number 7 I will get x square and this will be m m – 1 x raised to power m – 2 + a 1 x, this is m, x raised to power m – 1. I am considering here that m is a real number, + a 1 x raised to power m, right? So, it should be equal to 0.

So, from here, if you see, I will get a 0 m m – 1 and this will be x to power m + a 1 m x raised to power m + a 2 x raised to power m = 0. And now we know that we are looking for the non trivial solution. Then x raised to power m is never equal to 0 because we are looking for the non trivial solution. So, from here I can write this equation as a 0 m square – a 0 m + a 1 m + a 2 x raised to power m = 0.

So, here we have taken x raised to power m common. Now, x raised to power m is not equal to 0 which implies that a 0 m square +, this is my a 1 – a 0 m, so from here I can write this as + a 2 = 0. And this is again the quadratic in m, so I can call it as characteristic equation. So, this is a quadratic in m. So, from here, if I solve this one, I will get 2 roots.

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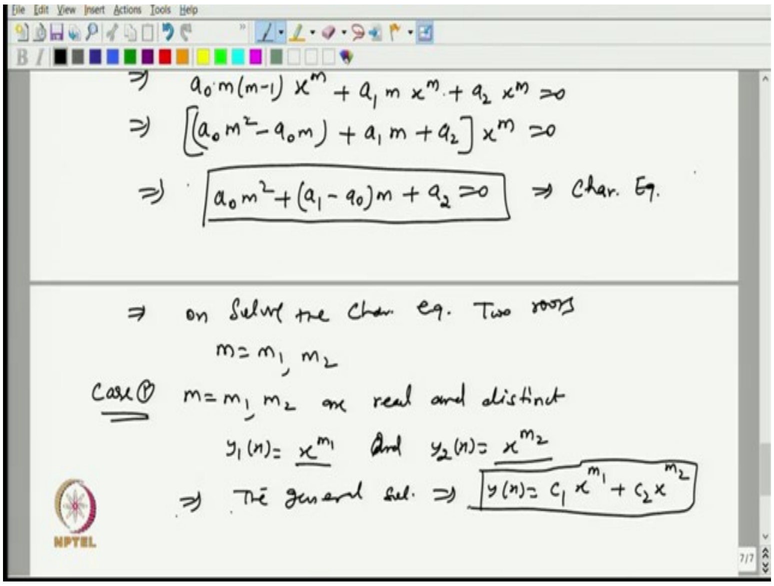

So, on solving the characteristic equation we will get 2 roots. So, root I call it as m 1 and m 2. So, it is up to whether this roots m 1 and m 2, so I just take the different different cases, the case 1 is that m = m 1 and m 2 are real and distinct. So, in this case, I will get my solution y 1 x will be x raised to power m 1 and y 2 x will be x raised to power m.

So, from here, if I want to write the general solution, then the general solution will be y x = c 1 x raised to power m 1 + c 2 x raised to power m 2 where my solution x to the power m 1 and x raised to power m 2 they are linearly independent. So, this is my general solution of the equation.

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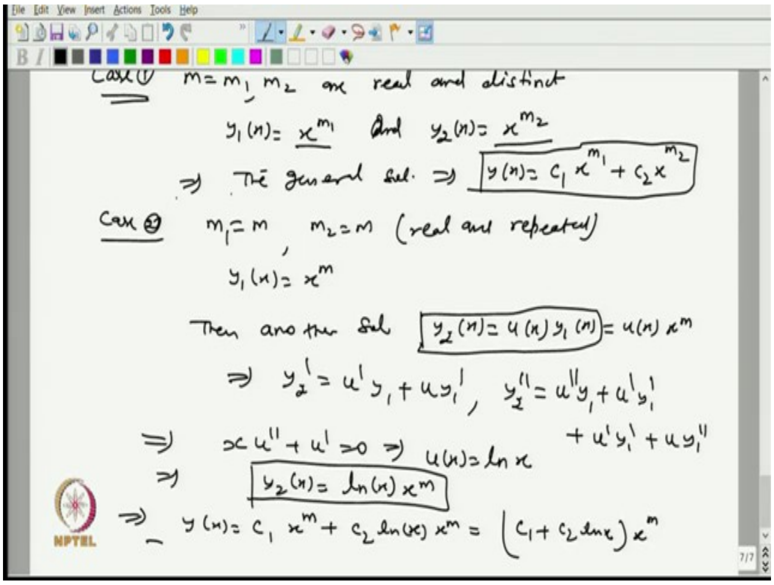

Next case is that, I take the case 2, when I will get, so m 1 = m and m 2 = m. The roots are real and repeated. So, in that case I have one solution. So, one solution will be y 1 x will be x raised to power m. So, this is one of the solutions. And we know from the previous form that another solution y 2 x can be found using the help of, let us take u x y 1 x.

So, it will be u x x raised to power m, and then I substitute this y 2 x in the equation and then with the help of the reduction of the order I know that if I solve this further then my y 2 to dash I can write as, now this is my y 2, so if I take the derivative y 2 dash, it will be u dash y 1 + u y one dash and y 2 double dash will be u double dash y 1 + u dash y 1 dash, okay, + u dash y 1 dash + u y 1 double dash.

And then if I substitute this one in the given differential equation and on substitution we will get the differential equation of this form equal to zero. So, this is the first order equation in the form of u dash dash and from here if I solve this one further my u x will be logarithmic x, l n x. From here I can find another solution y 2 x. This will be l n x into x raised to power m.

So, from here my general solution y x will be C 1 x raised to power m + C 2 l n x x raised to power m, and from here I can write as C 1 C 2 l n x x raised to power m. And if you remember that whenever we were solving the differential equation with the constant coefficient, and in that case, the corresponding characteristic equation of the repeated root in that case we put e raised to power m x + x times e raised to power m x. So, in this case, instead of putting x we are putting l n x.

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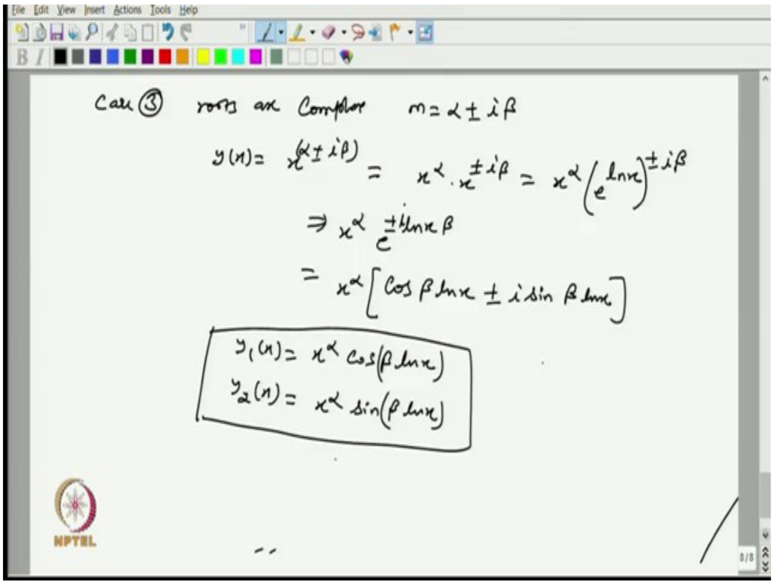

Now we will take the next case, case number 3. So, in that case, I am considering that the roots are complex. So, my m will be, I just call it alpha + i beta. So, in that case my solution x raised to power alpha + + – i beta. So, that will be my solution. So, if we solve it further, then I can write as x alpha and x + – i beta.

If we further solve this one, I can either it as x alpha and then from here I can as e l n x + – i beta because I know that e raised to power l n x is x, which can further be solved. So, it becomes e alpha and then it becomes e + – l n x into beta. So, from here, I can write my x alpha and this one I can write as cos beta l n x + – i sin beta l n x.

So, from here, I can find my two linearly independent solutions as, so I can further solve and I will get two linearly independent solutions that will be x raised to power alpha cos beta l n x and another solution y 2 x I will get x raised to power alpha sin beta l n x. This is the argument. And from here, I can find my general solution and then we can solve the differential equation. So, this is the form we have done right now.

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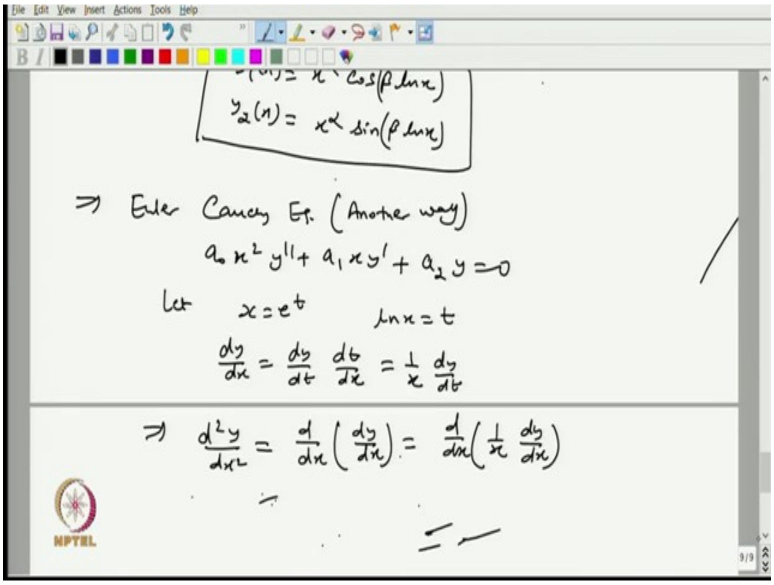

Now, sometimes the given differential equation, the same Euler Cauchy equation, the same equation just we have solved a 0 x square y double dash + a 1 x y dash + a 2 y = 0. Now, in this case what you do is that, this is another way of solving, I can write as another way. So, this way is in the form of the operator form.

So, what you do, we take a transformation, let x = e raised to power t or I call it l n x will be t. So, this is the transformation of the variable I am doing because x is my independent variable and I am putting x = e t. Now, in this case, if I want to find my dy by dx. Now, in this case, if you see that x is also dependent on it, so this will become, if I want to find the dy by dx, so it will be dy by dt and then dt by dx.

So, from here, I can find out that dt by dx is 1 by x and it becomes dy by dt because if I want to take the derivative here dx by dt 1 e t and that is x. So, from here I can find out this one. Further, if I want to find d square y over dx square, so this I want to write in the form of dy by dx and dy by dx is this one, so d by dx is 1 over x dy by dt.

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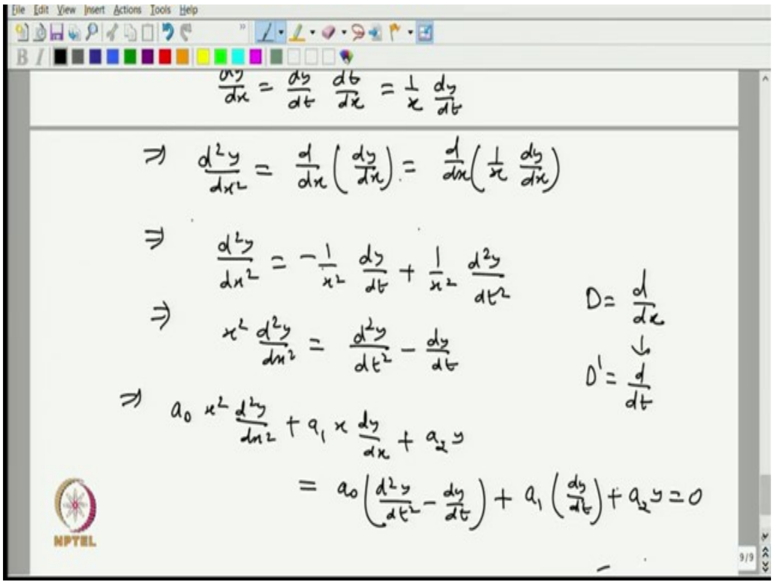

So, if I solve further my differential equation, so from here if you solve further, then I can find that my d square y over dx square, because further I can reduce this one, that becomes – 1 over x square dy by dt + 1 over x square d square y by dt square. So, from here, I can write x square d square y over dx square can be written as d square y over dt square – dy by dt.

So, if we substitute, because in this case if I put my d = d by dx, then we have changed this one into another d or I should write that as d dash. d dash is I am writing in the form of dt. So, from here, my differential equation a 0 x square d square y over dx square + a 1 x dy by dx + a 2 y, this will become a 0 and x square d square y by dx square is this one, so this will be d square y over dt square – dy by dt + a 1 and this I know that it becomes dy by dt + a 2 y. Now y is a function of t, so that becomes equal to 0.

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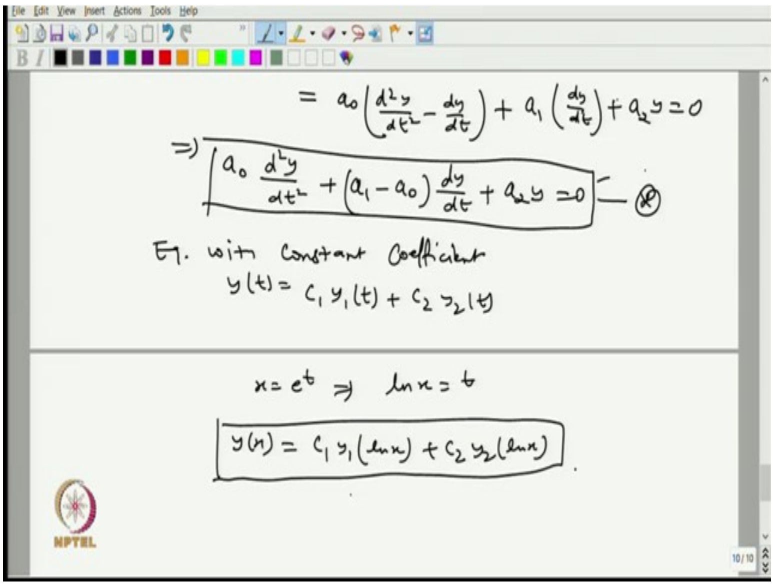

So, if we solve it further, then I will get my a 0 will be d square y by dt square, my a 1 + a 1 – a 0 dy by dt +, so this one and this one, a 2 y that is equal to 0. So, my equation has reduced to this equation and we know that my a 0, a 1, and a 2, they are constant values, so this equation that is reduced equation is equation with constant coefficients, and then we can solve this equation with the constant coefficient with the help of the characteristic equation.

And then after solving this one we will get the solution that would be y t = C 1 e raised to power, so I should write as y 1 t + C 2 y 2 t, and then I know that my x = e raised to power t, so from here my l n x will be t. So, instead of t I can put my l n x and the solution will be in the form of y x. So, this will be C 1 y 1 l n x + C 2 y 2 l n x. So, now in this case we are able to solve the Euler Cauchy equation in the two ways.

The first one was the just putting whatever we have done for solving the equation with constant coefficient putting y = x raised to power m and another one is taking the transformation and transforming the given equation, the Euler Cauchy equation, into the second order linear differential equation with constant coefficients.

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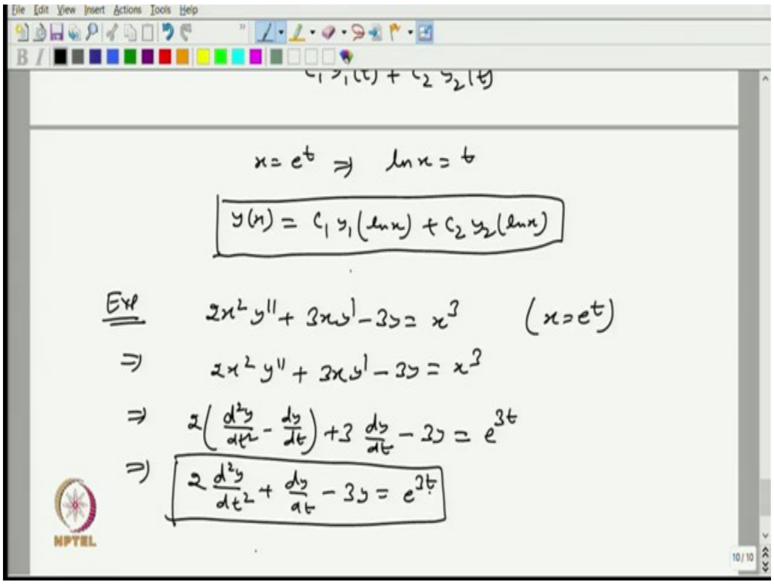

We can take the example. So, I will just take 2x square y double dash + 3x y dash – 3y = x cube. So, this differential I want to solve. So, if you see this one, if I do not know anything about the Cauchy Euler equation, then I will be stuck that how we can solve this differential equation, because in this case we know that constant coefficient is a function of x.

But, once we know that the second derivative is multiplied by x square, first derivative multiplied by x 3 y, it means that I want to solve first this differential equation, 2x square y double dash + 3x y dash – 3y. So, this is the homogeneous equation I want to solve. And once I solve the homogenous equation then I can solve it by the variation of parameter to find the particular solution.

So, if I solve this one with the help of may be the more recent one we have done, then I know that this becomes 2 d square y by dt square – dy by dt + 3 dy by dt – 3y = 0, or maybe I can solve it directly because here now I am putting x = e raised to power t. So, first, I will change this one. So, I will write it as x cube and then it will be, where y is a function of t now.

So, from here, I will get 2 d square y over dt square + 3 – 2. So, it will be dy by dt because 2 is this one and 3. So, – 3 y = e 3 t. And this equation we already know how to solve. So, in this case I can solve this one. First I will solve the homogenous part putting equal to 0 with the constant coefficient methods and then I will use the case 1 in which the right hand side function was exponential function.

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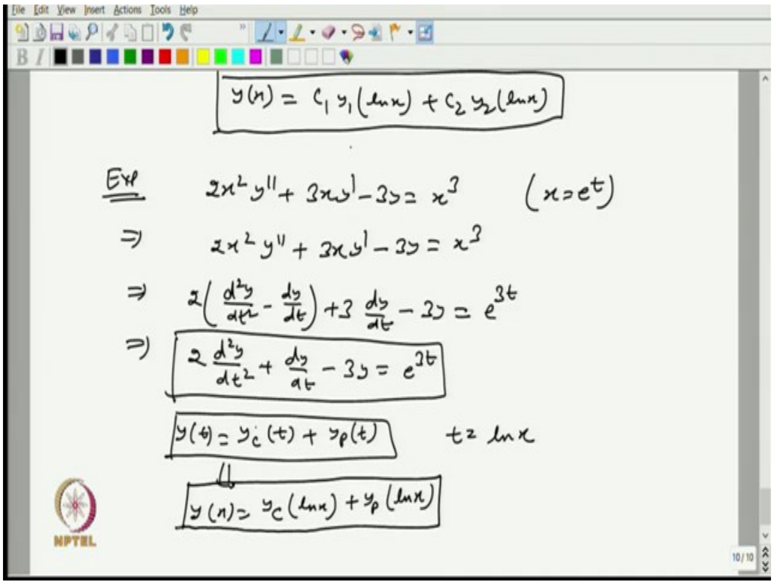

And then with the help of this one we can find my y c x + my y p x and that will be my general solution for this equation, because once I will get the solution here I will get in the terms of t. So, I will get function y 1 t and y p t, and then I will put my t = l n x and then I will get the solution in the form of t.

So, putting this value in the t, then from here I will get my solution in the form of y x because my equation is given in the form of y x. So, this I will get the solution y p l n x. So, this is how we are able to solve the second order linear differential equation in which either the equation has a constant coefficient or the equation is of the form of Euler Cauchy equation.

And, in this lecture also we have discussed how we can factorise the differential operator into the factorisation form if the corresponding Riccati equation we are able to solve, then the given differential equation can be solved with the help of Riccati equation or the factorisation form. So, thanks for viewing this lecture. Thanks.