Introduction to Methods of Applied Mathematics

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Lecture - 05
Second Order Linear Differential Equations with Constant Coefficients (contd…)

Welcome back viewers. So now start with the lecture 5. So this is in the continuation of the previous lecture.

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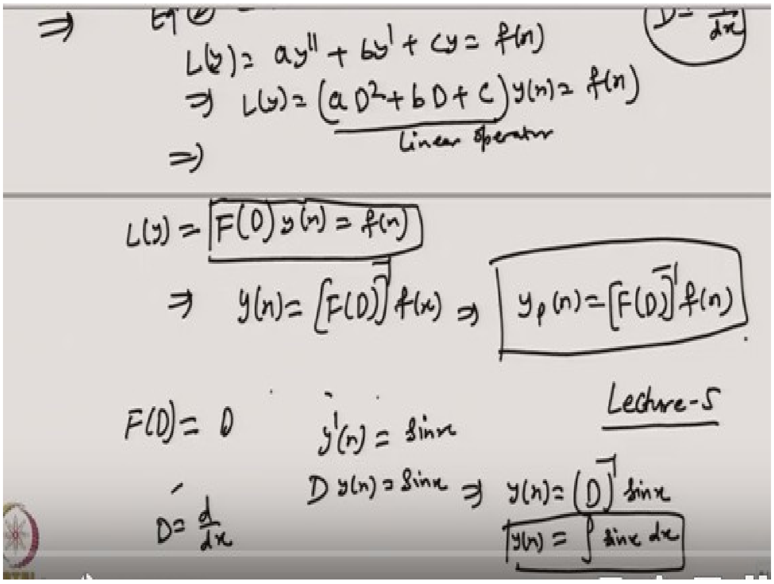

Now in the previous lecture I would define what is the linear differential operator. So F D I have defined. Now suppose my F D is simply a D, suppose I have a differential equation like y dash x is equal to sin x. So in this case I can write this differential equation as D of y x is equal to sin x and if I want to solve this one from here, I can define my y x as D inverse into sin x.

Now D is a differential operator and that I know that my D is dy/dx. So what is the inverse of a differential operator and that is the integration I know. So from here I will get D inverse will be integration of sin x with respect to x and that my y x will be the solution of this differential equation. So similarly, my F D now I have defined my F D.

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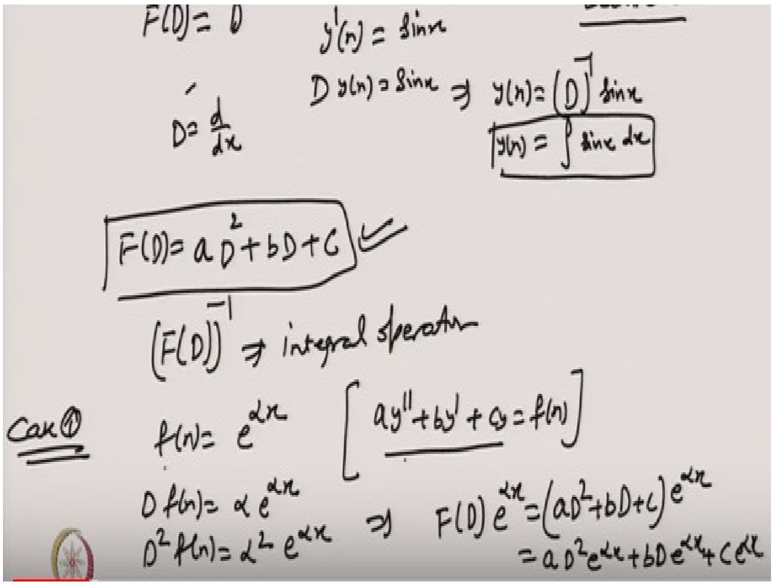

So if you see that if my F D is a D square plus b D plus c. Now this is my differential operator. Just I will differential operator. Now what will happen if I want to define the inverse of this one? So this is the differential operator and this is we generally call is integral of it. Because we know that the integration is also sometimes called the anti-derivative.

So if the differential operator we are taking is this one then the inverse of the differential equation we call it the integral operator. Now the question comes that how to find the F D inverse for any differential operator. If it is a simple one then we have defined that this is a inverse but what will happen if I have my F D of this type. So let us do this one for case to case that how to find this one.

So I will start with the case 1, number 1 when my function on the right hand side f x is some exponential function e alpha x. So this is my exponential function because if you see that in the previous lecture, the equation number two, I have defined this equation a y double dash plus b y dash plus c y is equal to FX and I want to solve this differential equation.

Then we have defined that this has a solution y c x + y p x; y c x is the solution corresponding to the homogeneous part putting f x equal to zero and then we define the particular solution and have to find a particular solution I will define that what is the differential operator and then we are going to deal with the operator methods to find out the particular solution.

So in this case, I am defining the case number 1 when my f x is equal to e raised to power alpha x. Now in this case what we will do is that I just want to find out that what will happen if I take D of f x. So D of f x if I do it will be alpha e raised to power alpha x. Now I want to define what is the D square f x in this case. So it will be alpha square e raised to power alpha x.

So from here what I do is that I take F D, so this is my differential operator into e raised to power alpha x. So from here my F D is already known that this is equal to b D plus c e raised to power alpha x because this is already known to me. Now I put this operate this operator on the exponential function. So I will get a D square e raised to power alpha x plus b D raised to power alpha x plus c e raised to power alpha x.

I can operate because this is a linear operator I can operate this function.

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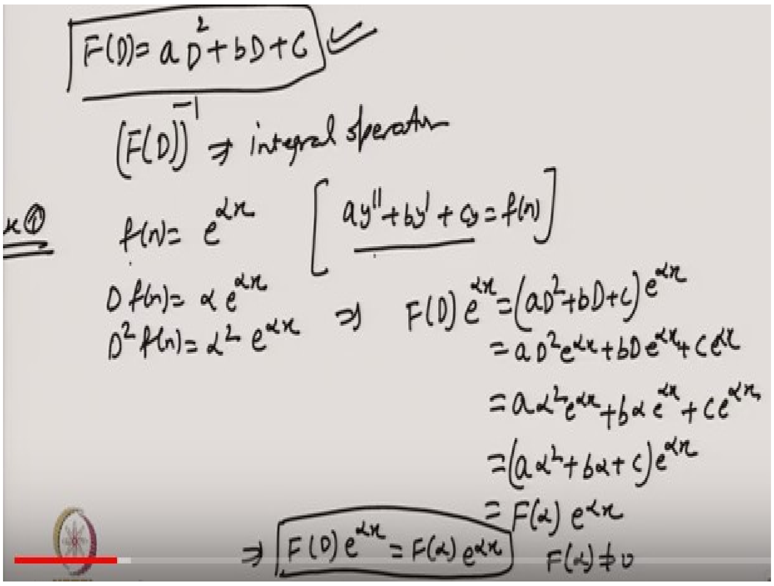

So after doing this one I will get a and d square e raised to power alpha x is this one. So it becomes alpha square e raised to alpha x plus b this becomes alpha e raised to power alpha x b alpha e raised to power alpha x plus c e raised to power alpha x. So from here I can take e raised to power alpha x common from this. So I get a plus alpha square b alpha plus c e raised to power alpha x.

Now what is this? It is similar to the my F D here. Only thing is that instead of D I have changed my alpha. So I can write this as a F of alpha e raised to power alpha x.

So from here I will get that my F D e raised to power alpha x becomes f of alpha e raised to power alpha x. Now in this case, what I assume is that my F of alpha is not equal to zero.

Because it may happen that when I put alpha instead of D, then this becomes zero. So I am assuming that my F alpha is not equal to zero. Now, so I want to solve my differential equation. So this is my differential equation.

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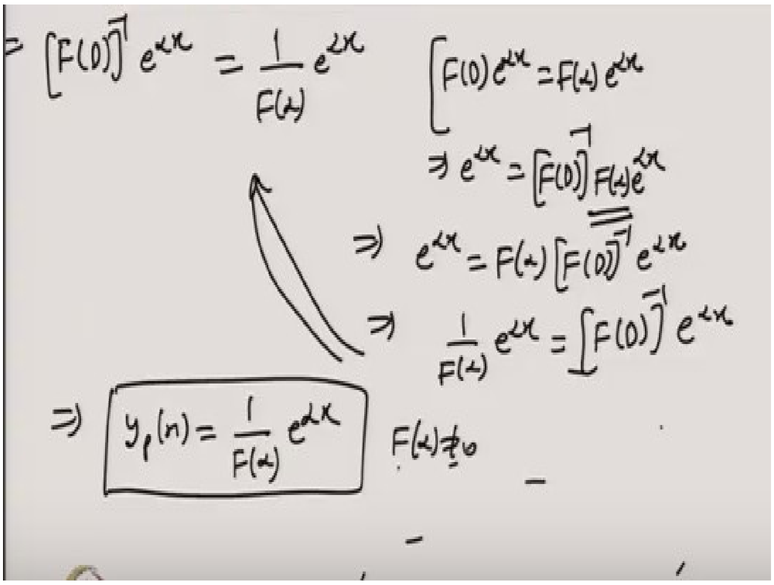

So this is my differential equation and I want to find my particular solution, as I told you, that the particular solution in this case will be F of D inverse e raised to power alpha x where alpha f x is used for alpha x. Now from the previous slides, I know that I get e of F D e raised to power alpha x is equal to F of alpha e raised to power alpha x.

So from here I can define that this e alpha x becomes, now this F alpha is a constant. It is just a scalar value, is a constant value. So if I am operating a linear operator on a constant, I can take this constant outside. So it becomes f of alpha F of D inverse e of alpha x. So from here, I can write that 1 of F alpha e raised to power alpha x can be written as F D inverse e raised to power alpha x.

So from here now I am going back to here. So from here I can write that F D inverse e raised to power alpha x can be written as 1 over F alpha e raised to power alpha x. So from here now I can say that my y p x in this case will be 1 over F of alpha e raised to power alpha x.

So what do we do in this case now that whatever the power e raised to power F x is there and put the value of alpha in the linear operator and we are assuming that F alpha is not equal to zero. So this is my condition that F alpha is not equal to zero, then this will be my particular solution.

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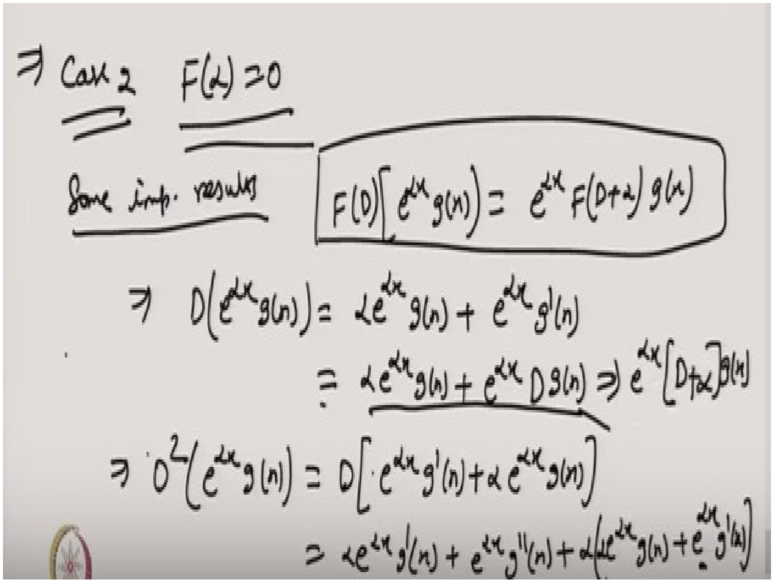

So for example, so let us take one example. Instead of example, I will take both the cases. I will take the another case 2 together that F of alpha is equal to zero. What will happen because in this case I have assumed that F alpha is not equal to zero but now I have F alpha equal to zero. So in that case, I cannot divide by F alpha because this is already zero.

So to deal with this one, before dealing with this one what I wanted to find is that some important results I wanted to find. So some important results. So what is that? I want to define that my F of D whatever the differential operator we have defined, if I operate this up on some function e alpha x into some function g x then this can be written as e raised to power alpha x F of D plus alpha into g x.

So this is my results and suppose I want to verify this one. So I want to verify. Now for this one what I do with that? Let us take the derivative of e raised to power alpha x g x. So if I take this one I apply the product rule for the derivatives, so I can write the derivative of the first function. So alpha e raised to power alpha x and then into g x plus e raised to power alpha x and g dash x.

This can be written again as I can write this as alpha, e raised to power alpha x g x plus e raised to power alpha x and this is a D g x. Which can further be written as now what I do is that I take this D here. That is a derivative operating on the function g x. So I can take from here e alpha x I can take common from the left side. So inside I will get D this D plus this alpha.

So this will be alpha into g x. Now if we expand this one what we will get? Now this is a derivative plus some constant. So if it is operating on the g x, so it will be first derivative of g x. So it will be e raised to power alpha x g dash x plus e raised to power alpha x alpha g x. So you will get this value. So this can be written as this one. Similarly, I can define this one for what will happen if I take D square of e raised to power alpha x g x.

So in this case, I will operate on this one. So I will get this value this way I am writing here g dash x plus alpha e alpha x g x. This is a one derivative I have applied. Now I am applying the derivative again. So again e alpha x alpha g dash x plus e raised to power alpha x g double dash x plus. So now operating on this one, so it will be alpha, alpha e alpha x g x plus e raised to power alpha x g dash x.

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$$\Rightarrow D^2\left(e^{\alpha x} g(x)\right) = e^{\alpha x} g''(x) + \alpha^2 e^{\alpha x} g(x) + 2\alpha e^{\alpha x} g'(x)$$

$$= e^{\alpha x}\left[D^2 + \alpha^2 + 2\alpha D\right] g(x) = e^{\alpha x}\left[D+\alpha\right]^2 g(x)$$

$$D^2\left(e^{\alpha x} g(x)\right) = e^{\alpha x}(D+\alpha)^2 g(x)$$

$$\Rightarrow F(D)\left[e^{\alpha x} g(x)\right] = \left(aD^2 + bD + c\right)\left(e^{\alpha x} g(x)\right)$$

$$= aD^2\left(e^{\alpha x} g(x)\right) + bD\left(e^{\alpha x} g(x)\right) + c\, e^{\alpha x} g(x)$$

$$= a\, e^{\alpha x}(D+\alpha)^2 g(x) + b\, e^{\alpha x}(D+\alpha) g(x) + c\, e^{\alpha x} g(x)$$

$$= e^{\alpha x}\left[a(D+\alpha)^2 + b(D+\alpha) + c\right] g(x)$$

$$= e^{\alpha x} F(D+\alpha)\, g(x)$$

Which can further be written as. So this one can be written as D square e alpha x g x can be written as, now from here I collect the terms. So this can be written as e alpha x g dash x plus from here alpha square e raised to power alpha x g x. This can be taken here and this and this term together. So it is can be written as two times alpha e raised to power alpha x g dash x. So this can be written as like this.

Now from here what I do is that I take e of x common and from inside what I do I take D square from here and then from here I take alpha square from here plus two times alpha D. I can write like this one and I operate this one on g x because if you take this one operating this function on g x it will be second derivativefrom here alpha square g x is same and 2 alpha D x.

So from here I will get e alpha x D plus alpha whole square operating on the function g x. So from here, what I see is that from I can write this function as my D square. So d square e alpha x g x can be written as e alpha x D plus alpha square g x. Now like this one if I keep going like this one, then you will see that now I have my F D. So I want to apply the F D on the e raised to power alpha x g x.

So I want to see that what will happen. My F D is a D square plus b D plus c applying on e alpha x g x. Now this is operator I am applying on the equation, applying on this function. So this can be written as a D square e alpha x g x plus b D I am applying on e alpha x g x plus c e raised to power alpha x g x. So this one and from here, I know that it can be written D square I just written that.

So it can be written as e alpha x D plus alpha whole square g x plus b e raised to power alpha x. So this I have taken and this can be written as D plus alpha g x plus c times e alpha x g x. This further I can write this as alpha. So what I do is now e raised to power alpha x I take the common. So from here e raised to power alpha x I just take the common.

So inside I will get a D plus alpha whole square plus b D plus alpha plus c operating on the function g x. So this is what this is can be written as e raised to power alpha x. And this if you see, instead of F D I have put D plus alpha. So it will be d plus alpha operating on the function g x.

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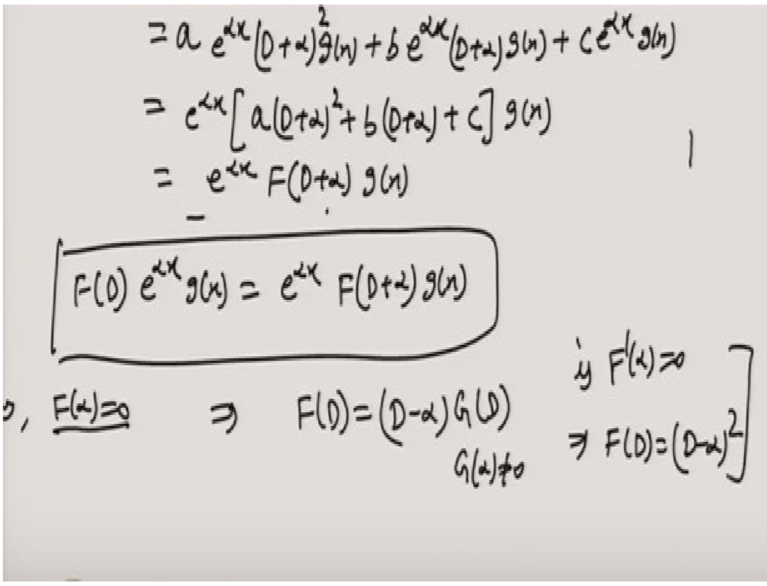

So from here what I will get is that F D e alpha x g x can be written as e alpha x F of D plus F g x. So this terminology we can keep in our mind. Now, so I will use the this one to study. Now we want to study the case when f of alpha is equal to zero in the, because when the F of x equal to zero I can apply it in my case whatever the case we have just studied.

So now see that F of alpha is equal to zero which implies that my F D would be of the form d minus alpha into some function I call it G D because in this case we are dealing with this quadratic or it may happen that F dash alpha is also zero. So if F dash alpha is also zero so in that case you will see my F D will be d minus alpha whole square.

So if my F alpha is zero then definitely by the knowledge of the polynomials, we know that d minus alpha be one of the factor of the F D and in this case my G of alpha is not equal to zero because only F alpha is zero. But what happens that F alpha is zero and then I take the derivative and F dash alpha is also zero. So in that case this is the possibility.

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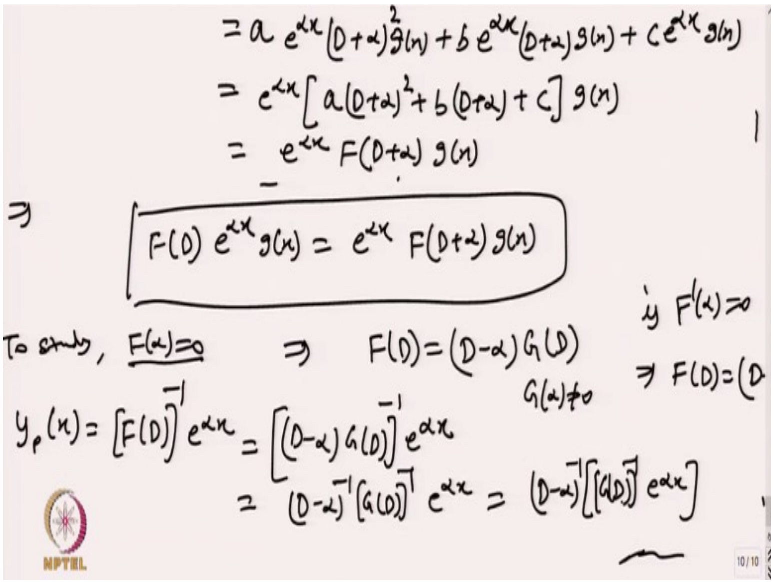

So now if this is happening, then suppose let F D is equal to this, then what is my purpose I want to find my particular solution and the particular solution I know that is Inverse of e raised to power alpha x. So that is my purpose. Now my F D will be this one suppose, I take this factor that F alpha is zero and F dash alpha is not equal to zero. So let this is the case.

So in this case I will take that D minus alpha G of D I am taking the inverse e raised to power alpha x. So this is the inverse. Now, so in this case also now this is the inverse. So I can separate e raised to power alpha x. From here I can write D minus alpha inverse and then this is the operator inverse e raised to power alpha x. Now e raised to power G of alpha is not equal to zero.

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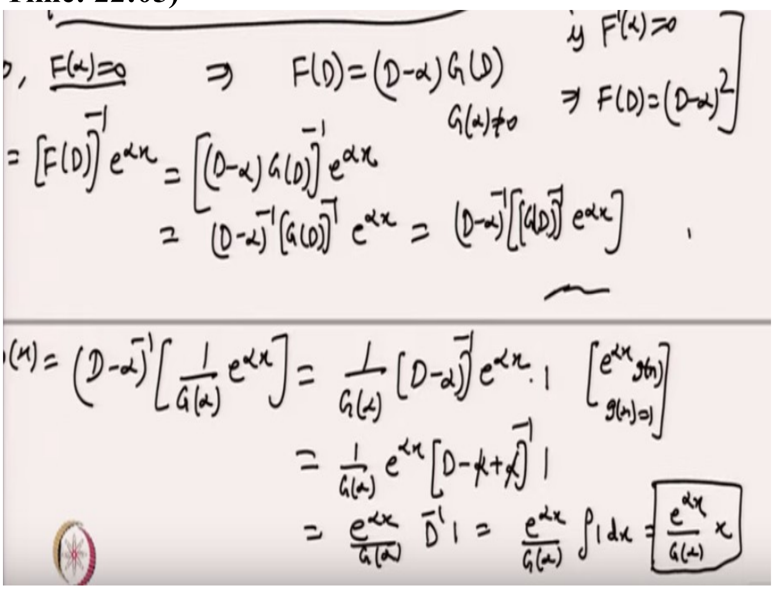

So the by the previous case I know that I can write this one as y I can write this as y p x is equal to D minus alpha inverse and then this one I can write from the previous one can be written as G of alpha e raised to power alpha x and now G of alpha is a scalar the constant term. So I can take as a G of alpha and then D minus alpha inverse e raised to power alpha x.

So this one is now just now we have from the previous one we have seen that this can be written as G of alpha will be here. Now I can take this function as multiply into 1 because now I am going to use that e of alpha x into g x and g x in this case I am taking 1. Now I can write this as, so I taking the e alpha x on the left hand side and then it becomes D minus alpha plus alpha inverse to 1, just by the previous value we have defined here, from here.

Now this can be written as, so this will cancel out. From here, I can define so it becomes e alpha x over G of alpha into D of inverse operating on 1. And what is this D inverse? It is just the integration. So from here e of alpha x over G of alpha integration of 1 with respect to x and this will be x. So it will be e alpha x over G alpha x. So that will be the particular solution in the case when F of alpha is zero.

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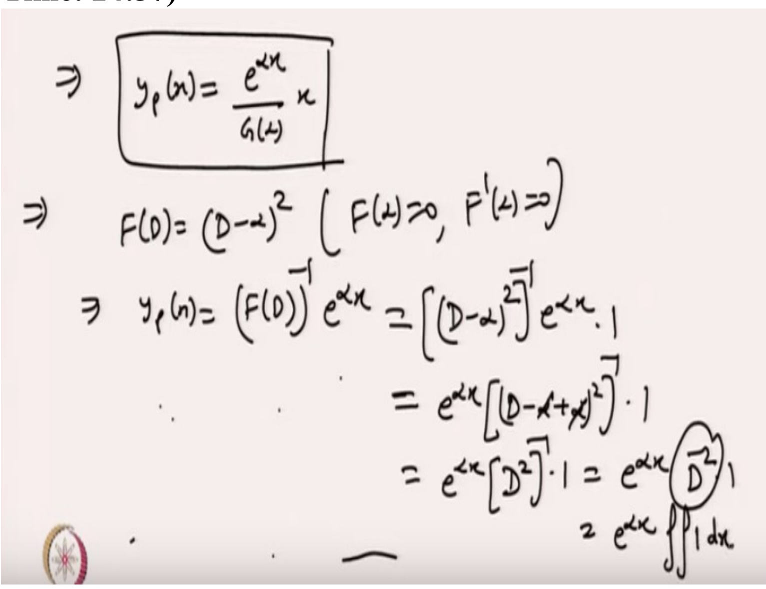

So from here I can say that my y p x will be e raised to power alpha x by G alpha x. So this is the case 1. Now what will happen if in that case suppose I have my F D is D minus alpha whole square. So in that case what is happening that F of alpha is zero and F dash alpha is also zero. So both the cases are happening. So in this case what will happen if I want to find my particular solution?

So this will be again my F(D) inverse and e raised to power alpha x. Now this is again D minus alpha whole square minus e raised to power alpha x. Now I will apply the shifting operator and into 1. So this one is there. So this one I can write as e alpha x. Now I am using the previous form. So it becomes D minus alpha plus alpha square minus one into 1. So this is the operator operating on 1 only.

So it will be e alpha x and this will be D because this and this will cancel out operating on 1. So if you see, this becomes a D, D inverse of operating on 1. So e alpha x and what is this? It is the two time derivative or integration of one with respect to x.

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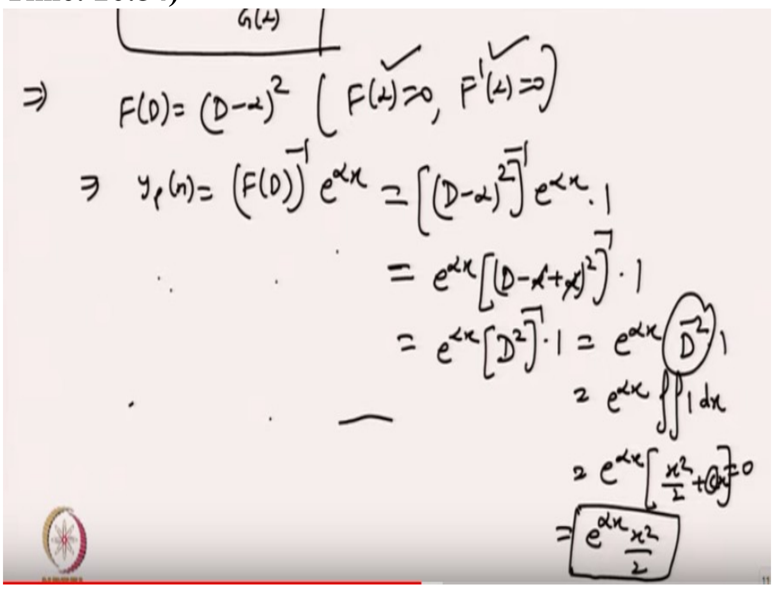

So this can be written as e alpha x and this is the two time integration I am taking so one is x square by two plus cx. This is taken because if I do the one time integration, it will be x plus some constant. So in that case I take that constraint is zero and then again. So in this case if you see this c I choose equal to zero. So it will get I will get only e alpha x square by two.

So that will be the solution particular solution when this is happening that the F alpha is zero and F dash alpha is zero. So you can see from here that when the F alpha is zero only then this will happen. And when F of alpha and its derivative is zero then this will happen. So this is a solution. Now let us do one example based on this one. So let us take one example.

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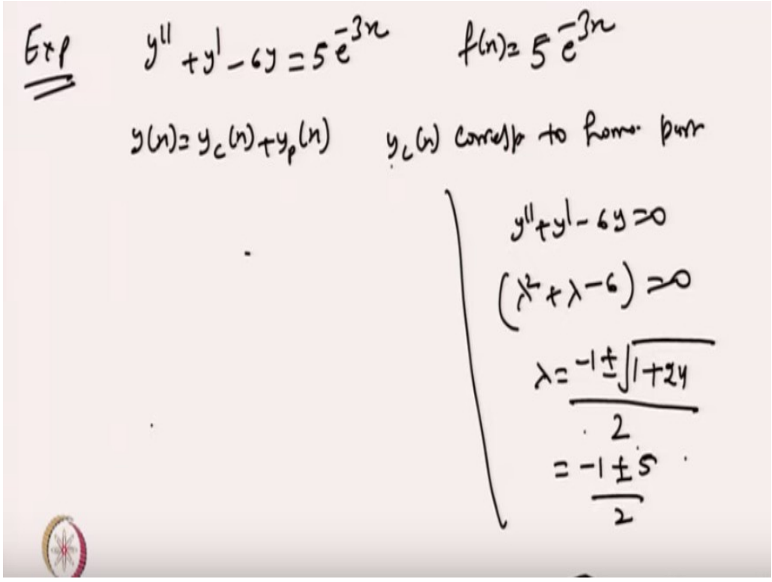

So example. I have a differential equation y double + y dash – 6y = 5 e raised to power - 3x. So this is the non-homogeneous linear second order differential equation with my function f x I am taking on the right hand side is e raised to power – 3x. And I know that this equation has general solution. So that will be y c x + y p x where y c x is a solution corresponding to homogeneous part.

So from that one I know that if I solve this one, so from here I can find out that when I solve 6y = 0 then I will get my solution so this is lambda square. The characteristic equation where lambda square + lambda - 6 that is equal to zero. And from here I can find out the solution. So this is the way we can find out the roots of this quadratic equation.

So from here I will get -1 b square, b square will be 1 – 4 ac. So 4 ac, so it is 24 by 2a. So this is a 2. So from here minus one plus minus it is 25. So it will be 5/2.

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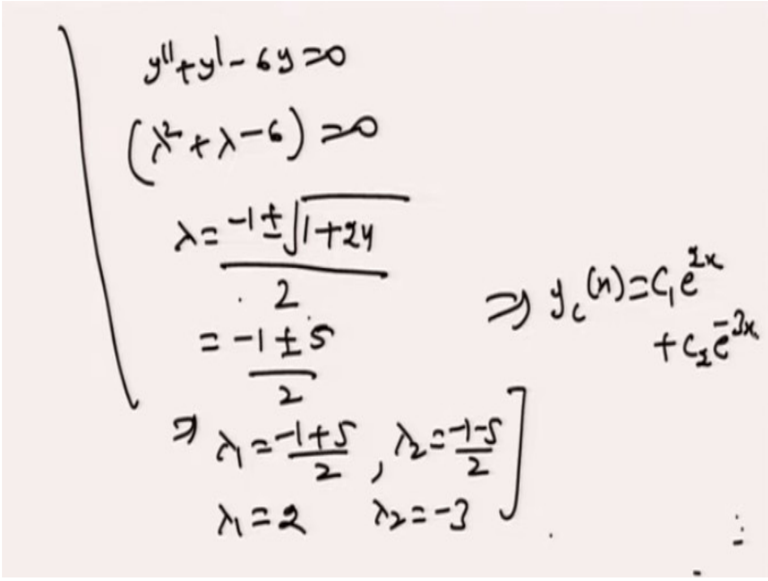

So from here I will get two roots are real and distinct. So from this one my lambda 1 will be – 1 + 5/2 and lambda 2 will be -1–5/2. So from here my lambda 1 is 2 and my lambda 2 is -3. So after solving this one, I can define my y c x from here and this will be linearly independent. So my y c x will be c 1 e raised to power 2x + c 2 e raised to power – 3x. So my complimentary solution is I am able to find.

Now the question is that how to find the particular solution. So particular solution is this one. Now I want to find my particular solution y x.

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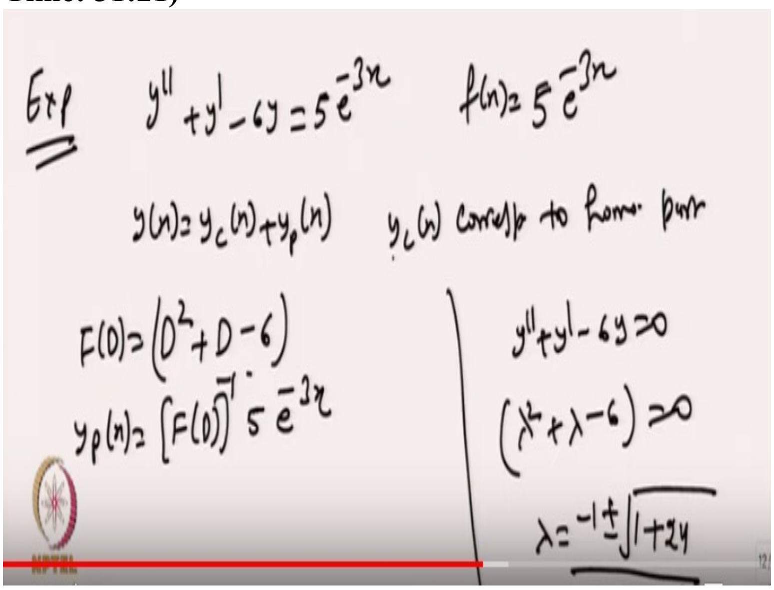

So in this case, what is my F D? So F D is my F D will be D square plus D minus 6, so this is my F D. Now if you want to find the particular solution y p x this will be inverse of this operator into 5 e raised to power – 3x. So this one in there.

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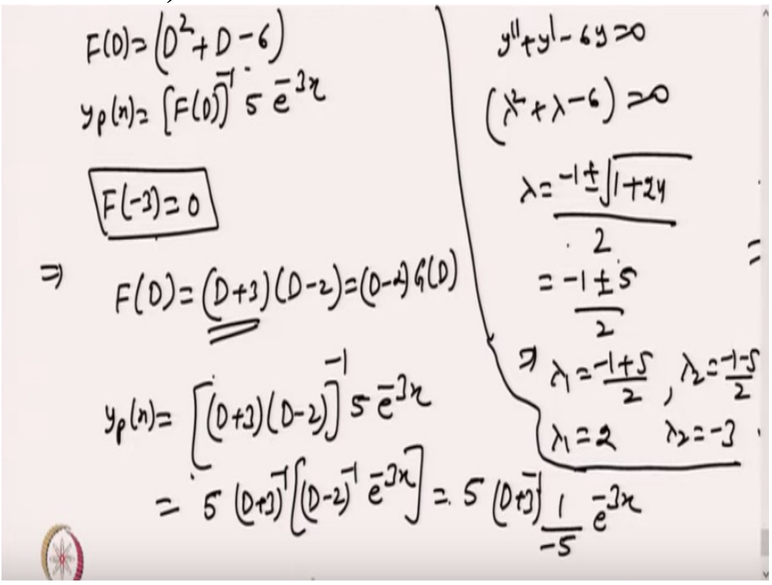

Now if you see that what will happen if I put F of -3 because this is -3 there. So and I know that from here that lambda 2 is equal to -3 is the root of this equation. So in this case this will be zero. So this is the case the case number 1 that my F of alpha is zero and F derivative is not zero because we have two distinct roots for this one. So in this case F dash alpha is not equal to zero.

So from here I will write my F D. So F D can be written as (D + 3) (D – 2). So I told you that if F alpha is the root of that one, then F D definitely can be written like this one. So this is the factor, which has become equal to zero when putting alpha is equal to -3 and this is my G x. So this is in the form D minus alpha into G(D). So G(D) that G we have defined is the capital letter, so G(D).

So my G(D) is D – 2. Now I apply the operator method to find a particular solution. So in this case my y p x will be, so I can either this as (D + 3) (D – 2) inverse into 5 e raised to power – 3x. So I can define this as 5 (D + 3) and (D – 2) – 1 e raised to power – 3x. So this one I want to apply. Now this I know that this G D inverse is not equal to zero when putting D = - 3.

So from here I can write as this value and then it will become one over so putting, instead of D I am putting – 3. So it becomes - 5 and e raised to power – 3x, okay.

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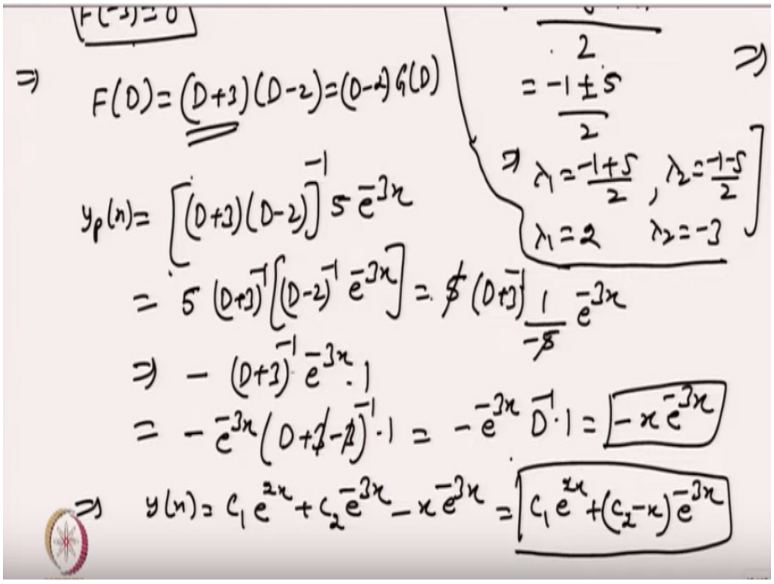

So now from here I will get minus of so this will cancel out. This is this one. So it will be minus and then (D + 3) inverse e raised to power – 3x into 1. Now I will use my operator methods just we have done. So I can take my e – 3x on the left hand side and this will reduce to – 3 into 1. So it will be minus e raised to power – 3x and this will cancel out.

So this will be D inverse into 1 and it can be written as minus so this is a integration. So it will be x e raised to power – 3x. So this is my y p x. So from here I can write my general solution as C 1 e raised to power 2x + C 2 e raised to power – 3x + y p x. So – x e raised to power – 3x. So it is e raised to power – 3x also coming here. So this can be reduced to this form plus c 2 – x e raised to power – 3x.

So that is my general solution for this equation where c 1 and c 2 are the arbitrary constants. Now, so this is the case we have just now solved. Now I take the next one.

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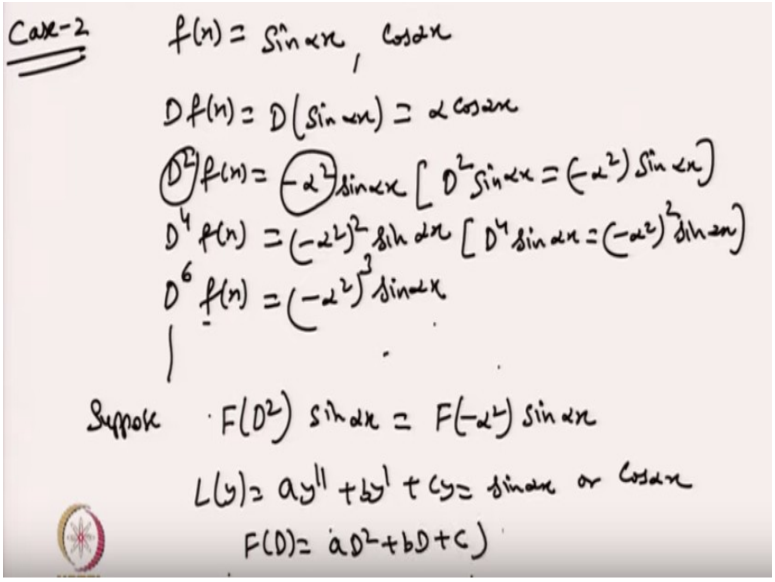

Case number 2 where my function f x on the right hand side is just a trigonometric function, some sin alpha x or cos alpha x like this. So what will happen in that case? Now my f x in the previous case it was exponential, but now we have a sin alpha x or cos alpha x. So in this case what I will do is what I will find what is the D f x? So if I find out the D f x, so D(sin alpha x) can be written as just taking the derivative.

So it will be alpha cos alpha x. Now I am taking D square f x. Then it becomes minus alpha square and then again sin alpha x. Now what about D 4 f x? So it will be minus alpha square then sin alpha x. So now you see that if I take the derivative of this one sin alpha x or the cos alpha x taking the first derivative we will get this value.

But taking the second derivative, I will get the same function back but multiply by minus alpha square and D raised to power 4 f x becomes minus alpha square raised to power 2 sin alpha x. So similarly I can define that in this case if I take the (even) power of f x so it will become minus alpha square power q sine alpha x. So like this one I can define all the even power of D to find out the derivative of that one.

So from here now suppose I have my differential operator that is defined in terms of F D square. So F D square sin alpha (x) will become what? So F D square sin alpha x become, if you see this one, if I take D square, then D square will be just minus alpha square sin alpha x. From here if you see, d square sin alpha x becomes minus alpha square sin alpha x.

So instead of D square I am putting minus alpha square. Similarly D 4 f x becomes minus alpha square then whole square sin alpha x. So F D square sin alpha x becoming F of minus alpha square and instead of D square I am putting minus alpha square sin of alpha x, okay. So now suppose I have my differential equation, my differential equation L y is this form plus c y is equal to sin alpha x.

Or similarly I can define for the cos alpha x. So in that case my F D will be a D square + b D + c. So this is my F D.

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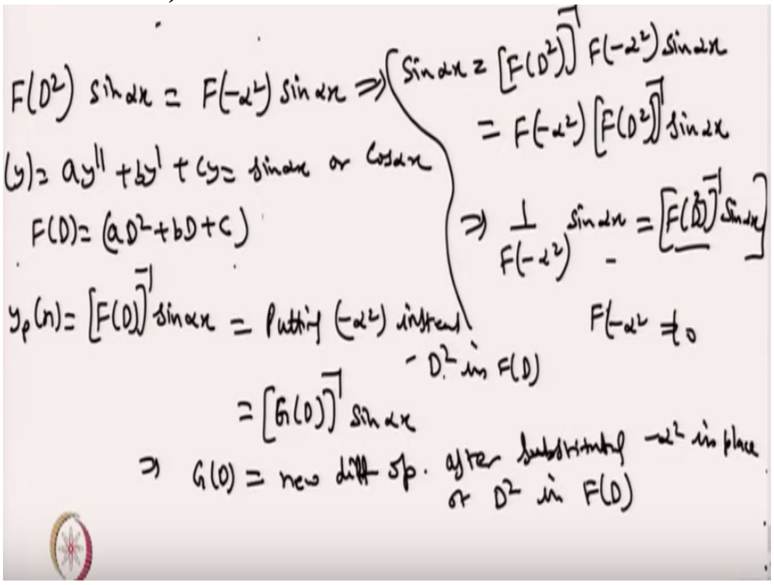

Now in the F D I have D square value only here not here. So what I will do that I want to find my particular solution. So particular solution will be F(D) inverse and into sin of alpha x. Now from here I can show that from here my sin of alpha x can be written as F of D square inverse F of minus alpha square sin of alpha x. So from here I can define, this is a just a constant value.

So I can take this one on the left hand side and I will get sin of alpha x. And from here, I can write that 1 of F minus alpha square sin of alpha x is equal to sin of alpha x provided my F of minus alpha square is not equal to zero. So that is the condition. So from here, so what I will do is that now here it is a D square, but here we have the all the value, D square is there D is there.

So what we do is that, instead of D square I will put the minus alpha square and so from here what I will do is, I can from here, this can be done as F of minus alpha square d square into sin alpha x or I can write here what I can write is that putting minus alpha square instead of D square in F D. So whatever I will get I will after that after putting this one I will get a function.

So that function will contain only new function I can say that the G D inverse e raised to sin alpha x. So G D is that I can define here that G D is getting a new operator. It is a new operator, new differential operator after substituting minus alpha square in place of in place of D square in F D, okay. And then we will solve this one. So let us do. Let us take one example based on that one. So let us do one example and it will be more clear that how we can solve this one.

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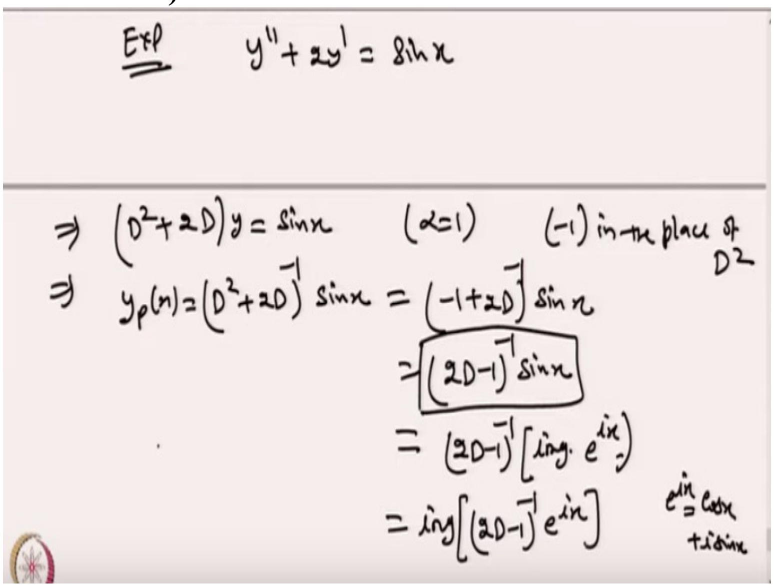

So let us take that I have a differential equation y double dash + 2 y dash = sin x. So let us do this. So in this case I want to find my solution. So solution can be written as, so this is my equation. Now I want to solve this one. So my differential operator can be written as (D square + 2D) y = sin x. From here what I will do is that now I know that how to find the complementary solution.

Complimentary solution I can find. But here my concern is to find a particular solution. So I will define my y p x. So y p x in this case will be it will be (D square + 2D) inverse sin x. Now what we have done here that now my alpha here is 1. So what I do? I will put minus alpha minus 1 instead of or in the place of D square. So what I will get? I will get here – 1 + 2D inverse sin x.

Now what will happen? This is instead of D square I am putting -1. So we get from here I can write (2D – 1) inverse sin x. Now how to solve this one? You can solve this (one) further. Then what we can do is that there is a trick to solve these. Generally what we will do is the sin x I can write as imaginary part of e raised to power i x, because I know that e i x can be written as cos x plus i sin x. So instead of sin x I can have this one. So this is the imaginary part of (2D – 1) – 1 e i x.

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$$\Rightarrow (D^2+2D)y = \sin x \quad (\alpha=1) \quad (-1) \text{ in the place of } D^2$$

$$\Rightarrow y_p(x) = (D^2+2D)^{-1} \sin x = (-1+2D)^{-1} \sin x$$

$$= \boxed{(2D-1)^{-1} \sin x}$$

$$= (2D-1)^{-1}\left[\text{img. } e^{ix}\right]$$

$$= \text{img}\left[(2D-1)^{-1} e^{ix}\right] \qquad e^{ix} = \cos x + i \sin x$$

$$= \text{img}\left[\frac{1}{(2i-1)} e^{ix}\right]$$

$$= \text{img}\left[\frac{2i+1}{(2i-1)(2i+1)} e^{ix}\right]$$

Now imaginary part. Now this is what? So this I already done in the previous that whenever the function f x on the right hand side is exponential, then what we do is that this becomes equal to one over and in the D I am putting the i. So it becomes (2 i – 1) e i x. So from here I get imaginary part of the 2 i - 1. So I simplify this one, 2 i – 1 multiply and divide by 2 i + 1) e i x.

(Refer Slide Time: 48:35)

$$= \text{img}\left[(2D-1)^{-1} e^{ix}\right] \qquad e^{ix} = \cos x + i \sin x$$

$$= \text{img}\left[\frac{1}{(2i-1)} e^{ix}\right]$$

$$= \text{img}\left[\frac{2i+1}{(2i-1)(2i+1)} e^{ix}\right]$$

$$= \text{img}\left[\frac{2i+1}{4i^2-1} e^{ix}\right]$$

$$= \text{img}\left[\frac{2i+1}{-5} e^{ix}\right] =$$

$$= \frac{1}{-5}\text{img}\left[(2i+1)(\cos x + i \sin x)\right]$$

So this can further be written as imaginary part of, so it becomes 2 i + 1 and from here you will get to 2 i square. So it will be 4 i square – 1 e raised to power i x, okay. Because 2 i 2 i is 4 i square minus 1 this one. So from here I can write as 2 i + 1. So i square is -1 so it becomes -1 e i x. So now the I want to take the imaginary part of this. So this will be, so this is my imaginary part.

Now imaginary part will be 2 i + 1. So I can take this as a common. This one multiplied by cos x + i sin x. So this will be. Now I can take little bit further.

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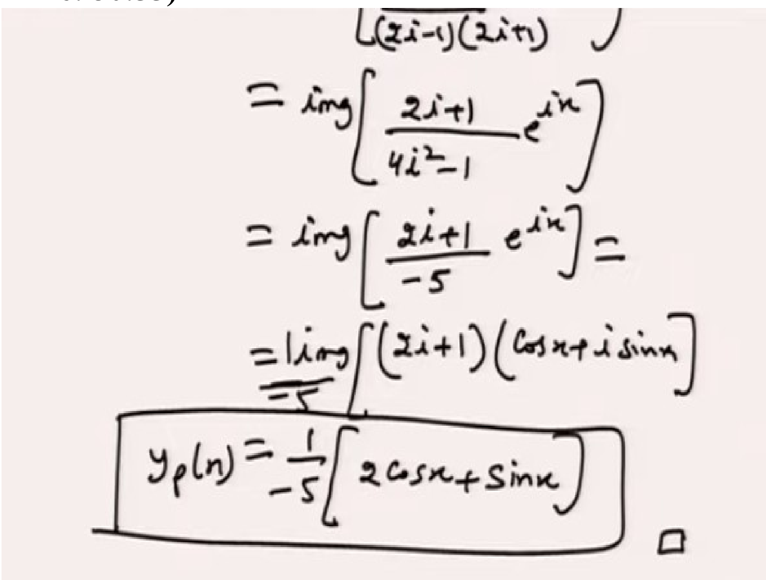

So this can be written as, so this is, so from here I will get 1/-5. Now 2 i will multiply by cos. So I will get imaginary part. So it will get 2 cos x. Now 2 i multiplied by this one, so that is real part. One is multiplied by cos that is the real part. One is multiplied by this one this is a sin x. So this will be into sin x. So this will be my y p x in this case. So this is the way we can find out the solution.

So in this lecture we have developed the methods to find out the particular solution for the differential equation when the function on the right hand side is either the exponential form or the trigonometric form that is the sin alpha x and the cos alpha x. So in the next lecture we will deal with that when the function on the right hand side is some a polynomial. So thanks for the watching this lecture, thanks.