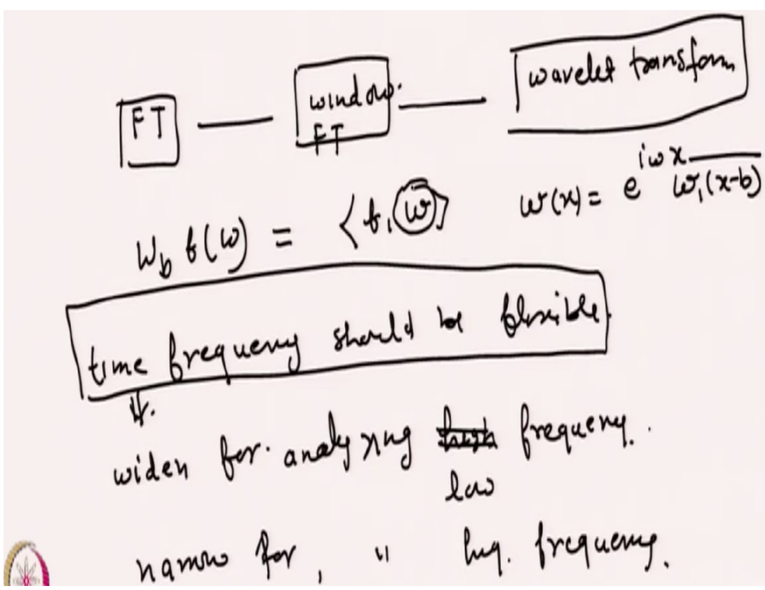
Introduction to Methods of Applied Mathematics
Prof. Mani Mehra and Prof. Vivek Aggarwal
Department of Mathematics
Indian Institute of Technology – Delhi

Module No # 07
Lecture No # 32
Wavelet transform and Shannon wavelet

Welcome to all of you again in my next and last class. So in the previous lecture what we have appreciated the wavelet series over Fourier series which was the motivation behind this part of the course and in this lecture we will look the advantage of wavelet transform over Fourier transform. So these 2 motivations we have set initially that I will show you the advantage of wavelet series over Fourier series which we succeeded.

And I will also show you the advantage of wavelet transform over Fourier transform or window Fourier transform. Because if you looked at the evolution of you could say wavelet transform initially people looked at the disadvantage of Fourier transform.

(Refer Slide Time 01:24)

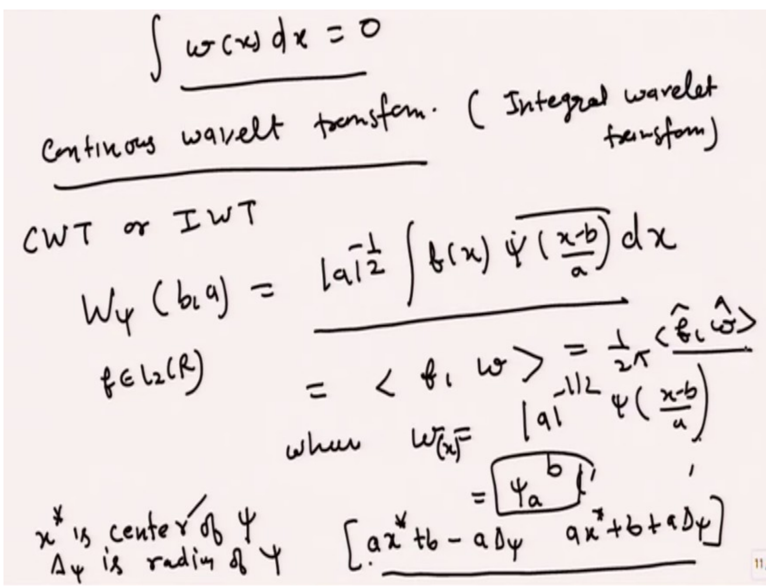


Okay then to remove those disadvantage of the Fourier transform people used window Fourier transform. Window Fourier transform or in short time it is also called short time Fourier transform and Gabor transform was a special case of it ok. And then so we have seen then we will see wavelet transform ok. So if you looked at the short form of window Fourier transform which we have defined earlier so it was like that where wx was this function ok clear.

So now we have to couple of disadvantage of Fourier transform was already removed it window Fourier transform that you do not need the whole signal information if you analyzing the signal in time and frequency domain that was one of the drawback which was removed in window Fourier transform. But now we I am saying that what is my expectation from one of the integral transform that when I am looking the high frequency phenomena my window by which I am analyzing it should be narrow and for low frequency phenomena it should be wider ok.

So it means I need time frequency window which is flexible ok time frequency window which is flexible and flexible also it means it is automatically widens. So time frequency window should be flexible it should be widened for analyzing high frequency sorry low frequency signal and it becomes narrow for analyzing high frequency phenomena ok. So our motivation is to look at flexible time frequency window flexible time frequency time this that is my motivation ok.

(Refer Slide Time 05:10)



So with that reason one of the property which is not posed by this window function w is this. wx dx is not 0. This is not because if you look at the window function in case of a Gabor transform we used gaussian. So gaussian does not satisfy this property ok gaussian does not satisfy this property. And now we want to introduce this property because with this property we will be able to say that our time frequency window is flexible ok.

Whatever was our motivations ok because so that time frequency window becomes widen for observing low frequency phenomena and narrow for observing high frequency phenomena. So with that motivation I am defining another part of the wavelet which is called continuous wavelet transform. It is also called integral transform integral wavelet transform which is a one kind of a integral transform. It is also called Integral wavelet transform or in a short form it is called CWT or IWT.

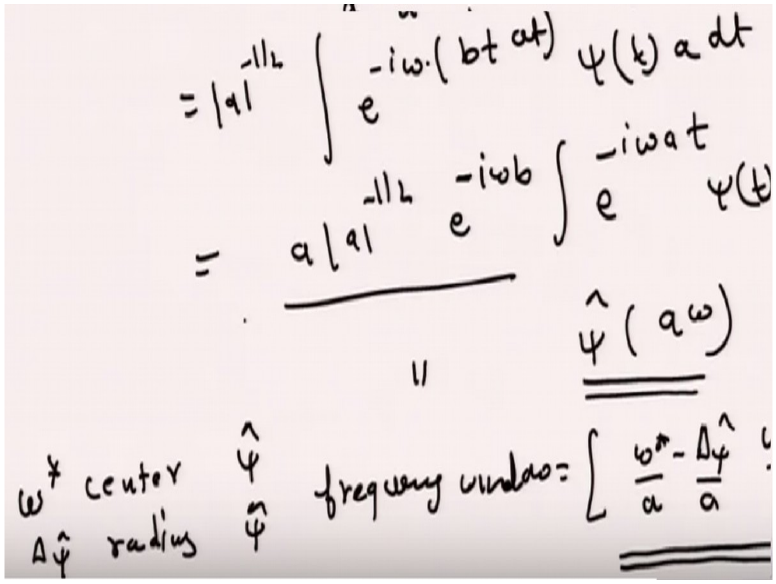
What is the mathematical formula of this? Mathematical formula of this is W Psi b of a f of x Psi x - b / a into dx ok. This is a wavelet transform of a function which where f belongs to L2 R with respect to wavelet Psi ok. This is the mathematical formula. Where b is transition parameter and a is called dilation parameter. The meaning of this dilation and translation we have already been seen when we are defining a multi resolution analysis.

So with that I can write this into the following way ok. Where w is Mod a Psi x - b / a or in a short form I will use this as a this symbol Psi a the more this is the function of x this will also be so symbol wise I will use this ok. So basically I am using Psi as a window function Psi the first of all Psi is a window function yes what was the quality of the window function? That function should be L2 R as well as if it is multi x w should also been L2 R.

So that we make sure that it decay very fast ok. So this is the window function and which window we should use in time frequency analysis where w as well as w hat should also be a window function ok. So first of all so if I am using this Psi which is a wavelet function as a window function. So now can you try to calculate what will be the time window for this what will be the time window for that reason.

Let say if x star is center of Psi and Del Psi is radius of Psi ok x star is the center of Psi and Del Psi is the radius of the Psi. So in that case if you look at the formula for defining center and radius of a window function this will become ax star + b - a Del Psi ax star + b - + a Del Psi this will become the time window which you could consider it as a exercise also and ok ax star + b how it is calculated to it to be a time window ok.

(Refer Slide Time 10:52)

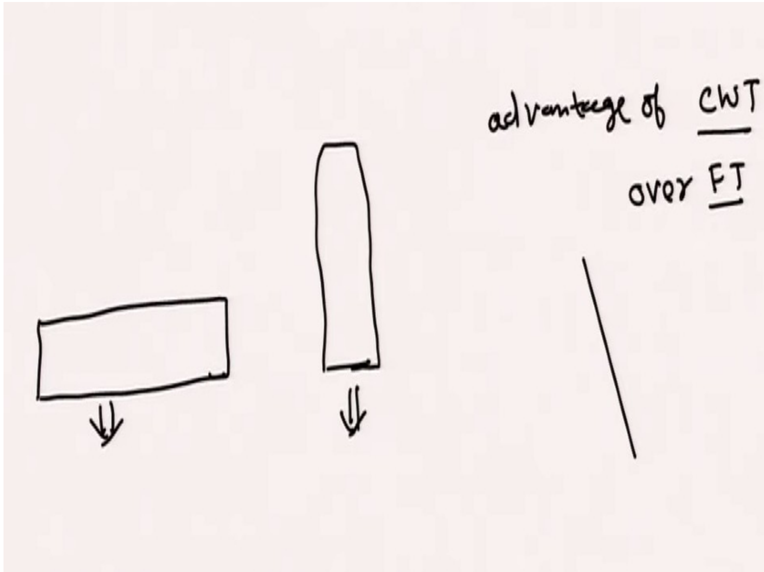


Now what will be my frequency window so for a frequency window first of all I have to observe what is this w hat? w hat is basically so how we just a minute so we call it as 1 over 2 Pi f into w hat we are using Parseval relation again here. So we have to basically look at this function w hat because h will become 1 over 2 Pi w hat. So, this will be e to the power - i Omega ok. So now if I am saying x - b / a is t this will be become e to the power - b + at Psi t dx is a dt. So this will become a e to the power -i Omega b I can take common a t Psi t dt.

So this will be this thing I am taking common a this will become Psi hat a Omega ok so clear to everyone? Now what I am doing so now this will be assignment question that how you will calculate the center and radius of this function whole function. So I am saying that let us say Omega star and Del Psi hat this is the center of function Psi hat and this is the radius of function Psi hat then ok. Then the center and then frequency window will become this ok.

This will be your assignment question also how exactly this will be ok. So now this is a frequency window and in the first page we have seen this is a time window ok. So you could look at that both the window is depending on the parameter a ok for the parameter a and if you calculate the area of a time frequency window that will be a constant that you could calculate and looked at that will be a constant.

(Refer Slide Time 14:26)

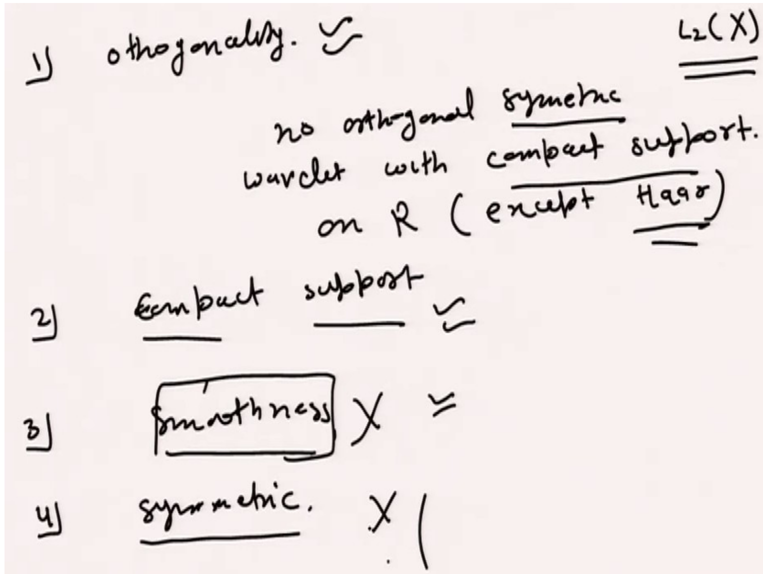


So basically, this will be like that sometime my window will be like this or sometime my window will be like this. For observing high frequency phenomena window will be narrow for observing low frequency phenomena window will be widened ok time frequency with area will remain constant. So with this thing I have shown you what is the advantage of CWT over FT. Advantage of wavelet transform over Fourier transform. It gives me flexible time frequency window to analyze the functions.

So if some function has a varying frequency ok low frequency phenomena is also there high frequency phenomena is also there. So we should analyze those type of a signals using wavelet transform ok. So this way we should appreciate wavelet transform over Fourier transform. Now at least as I said that some wavelet has some property some wavelet has another property or I could say in this manner that wavelets basic property I have said that integral of Psi x dx is 0 and Psi and Psi hat should be localized and other properties are desirable.

So this thing anyway we have discussed that how wavelet transform is can is doing better than Fourier transform. So with this words you should able to appreciate these advantages but now I am discussing that what are the other properties of wavelet. If you just ignore those basic not ignore if you means have some additional property ok. So basic anyway we have said that these are basic property to justify this is a wavelet function but what other properties.

(Refer Slide Time 17:13)



So in one of the other desirable property in the wavelet is orthogonality. Why orthogonality is a desirable property? Orthogonality is a desirable property because it helps you in the computation. If you remember we started this course with defining these basis orthonormal basis and frames. So you must have already noticed that what was the advantage of orthogonal basis. Because in that orthogonal basis we were when we were boiling down to the step of matrix or linear algebra problem if matrixes orthogonal, we can compute the inverse very easily.

So orthogonality for numerical computation you could say orthogonality leads to the fast computation. However, every time it is not possible to it is not means not possible I should not say I should say that means difficult to achieve difficult to orthogonality's are difficult to achieve. Here one point also I should mention that we were talking everything in term of a real line. But later on for a real line application you have to define a wavelet on L2 X. Where X can be interval X can be any complex manifold also ok.

So and it is very difficult to construct orthogonal wavelet on if X is a complex manifold or in fact there is a no orthogonal wavelet which is symmetric on the real line except Haar. Haar is a symmetric orthogonal wavelet ok which has a compact support also. So in a natural what I am saying that there is no orthogonal symmetric wavelet with compact support on real line on Haar except R ok.

So this way you can say in fact on a Haar we do not have orthogonal wavelet with other desirable property symmetricity and compact support. So if this is the what is the other desirable property compact support? Compact support is also a desirable thing and fortunately that is why Daubechies wavelet became so popular. Because Daubechies was the constructed a compactly supported wavelet using MRI ok.

So this is of course it also leads to a fast computation if you look at the computation this compactness ok. Other desirable properties is smoothness of course we need a smooth functions to approx. I should say we need a smooth basis to approximate a smooth function ok. So desireness of a smoothness should be clear to everyone. So here Haar is you could say because this is in fact not a continuous wavelet if you keep increasing the order of d you will get more smoothness. As we have already seen that the smoothness is a function of M.

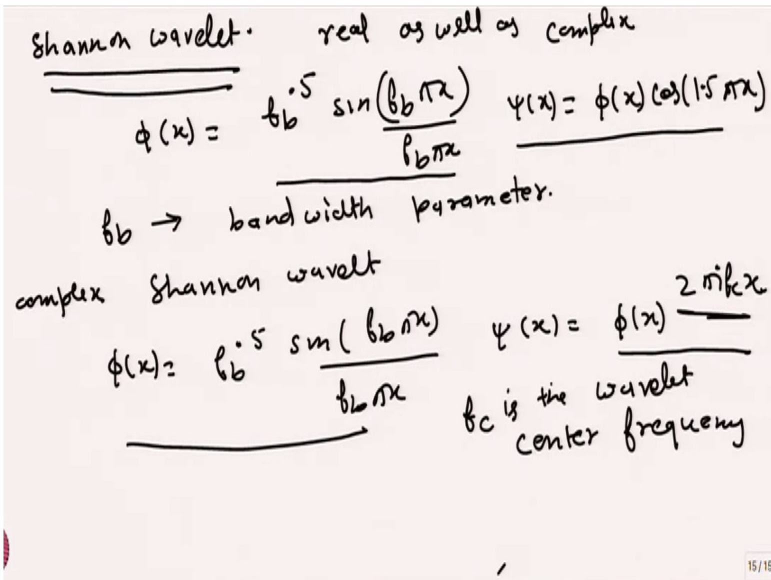
So if you keep increasing M you will get more and more smoother Daubechies wavelet. So and other desirable property in the wavelet is that it should be symmetric ok. Because when we you want wavelet and image compression and if image is symmetric you also want wavelet to be symmetric function. Again, this is a desirable function desirable property but that is an so depending on the application you have to choose wavelet. Wavelet may not satisfy all the properties ok.

At the same time symmetricity, compact support, orthogonality and smoothness all four you cannot have been in one wavelet that is what is said in this is statement. No orthogonal symmetric wavelet with compact support which is which is smooth also ok. Haar is orthogonal, Haar is as compact support, Haar is symmetric, but it does not possess the smoothness. What is the other kind of wavelet we have seen?

Daubechies wavelet which has a compact support which is orthogonal, but which has smoothness also if you work for higher d but it is not symmetric clear to everyone. So these properties are desirable. So it becomes very important for a wavelet community that which wavelet they should use**.** They should so they basically there are a plenty of wavelets available in the market as far as this course is concerned, I have introduced the very basic wavelet which is Daubechies family wavelet.

But in the market, there are plenty of you with this part of the course you should able to pick up the right wavelet depending on your application ok. So now at least I should be able to show you one another kind of a wavelet other than the Daubechies so what is that wavelet I am going to show you Shannon wavelet.

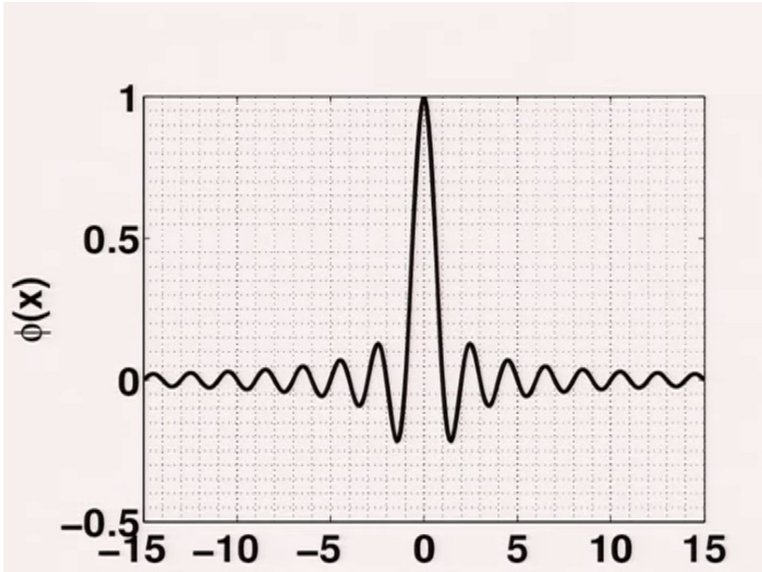
(Refer Slide Time 23:58)



Ok so this is the first time we are looking at wavelet which other than this Daubechies wavelets. So how what is the Shannon wavelet? Shannon is the name of the scientist who discovered it and these wavelets can be real as well as complex real ok. So in case of a real Shannon wavelet the Phi x is this ok and Psi x is ok. Where fb is some parameter and in case of a complex it is I could say fb is a bandwidth parameter and in case of a complex Shannon wavelet I am defining Phi x is fb 0.5 and Psi x is 2 Pi i fc x ok.

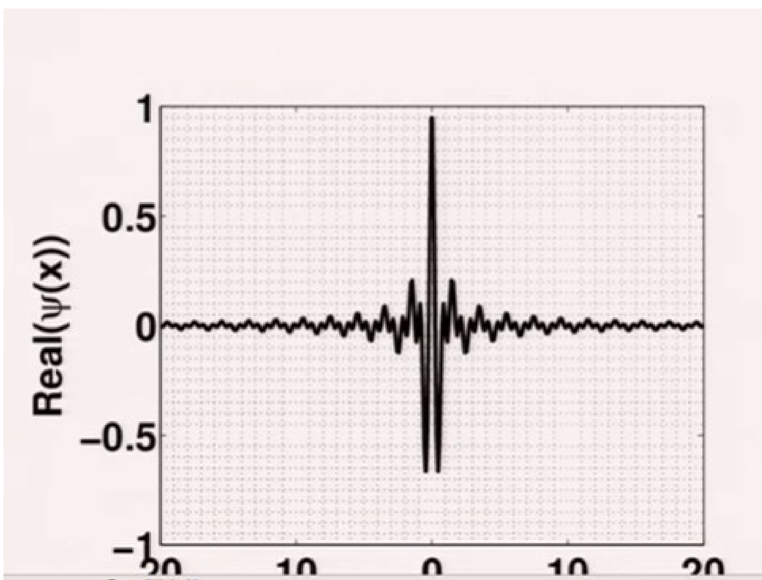
So where fc is the wavelet center frequency ok so, this is a scaling function corresponds to real Shannon wavelet this is a wavelet function corresponds to real Shannon wavelet. This is a scaling function which is same as in case of real. But here it is changed ok because of that that is a complex Shannon wavelet. So now let me show you the diagram how it looks.

(Refer Slide Time 26:48)



So this is a Shannon scaling function ok again, I will show you Phi and Psi side by side ok.

(Refer Slide Time: 27:14)



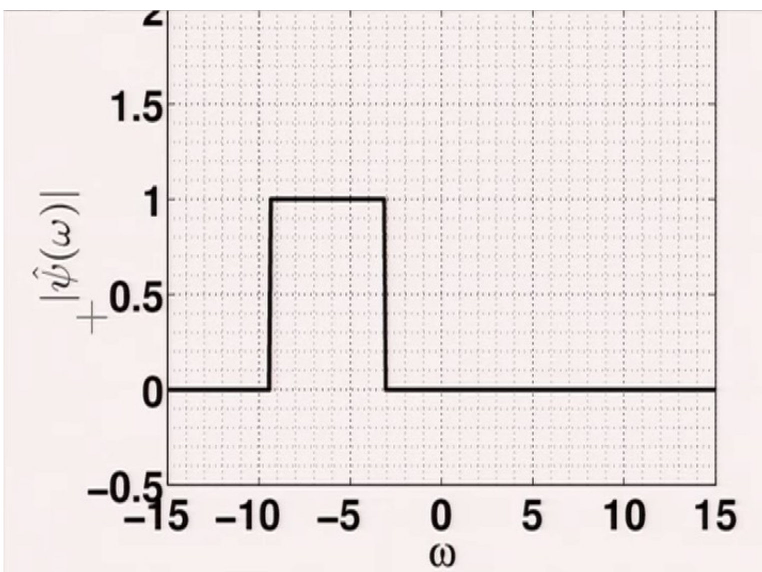
So Shannon Phi scaling function and this is a complex wavelet whose real part is plotted ok. So you could see and this is a imaginary part of Shannon wavelet is plotted ok. So you could observe this is the imaginary part of a Shannon wavelet this is a real part of the Shannon wavelet and this is how the scaling function. This is anyway in the form of a sinc function. If you look at the analytical expression of Phi this is in the form of a sinc function ok.

So we have seen when we were looking at the Fourier transform that Fourier transform of a box function is a sync function. And here also you will see that Fourier transform of this Phi x which is in the form of sync function is a box function. So but in either case both are localized this was the basic definition of the wavelet. So if you now if you compare the properties of a Shannon wavelet with Daubechies wavelet how can you compare.

What is the main basic difference you observe let say in scaling function itself. It is has a it is not compactly supported wavelet. So this is the first example which we are looking in this course where it is wavelet are not compactly supported. Because as I said at the beginning that all the wavelets are not having a compactly supported, does not have a compactly support. So this has a infinite support this is the first example Shannon wavelet other than that what is the property you observe here this is a symmetric wavelet.

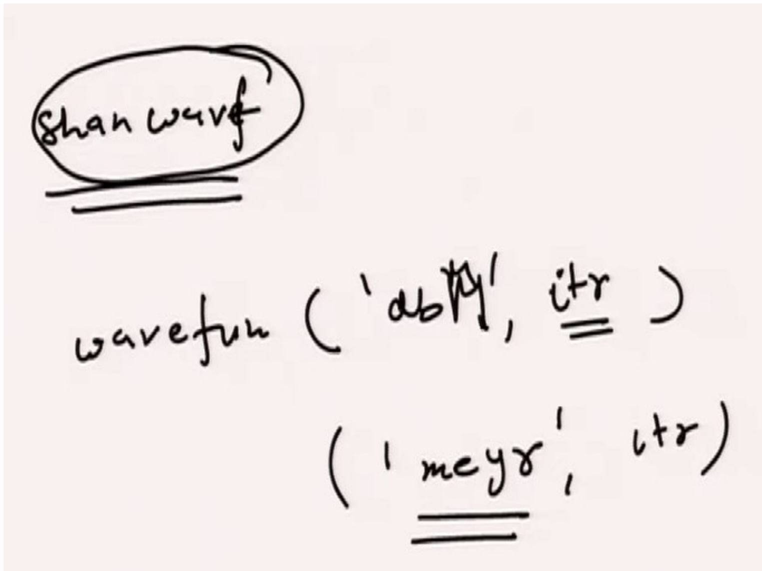
While Daubechies wavelet were not symmetric. As I said when I was discussing about different properties of the wavelet that symmetric see it one of the desirable property of the wavelet it is desirable depending on the application. So Shannon wavelets are have are symmetric they are not compactly supported what is the other property you could observe? Ok any other property which you could observe? Smoothness you could observer? Of course, it is that one has to observe more carefully to look at this.

(Refer Slide Time 30:20)



If you look at this function of course it is I can also show you the Fourier transform of a scaling function which is a box function which I already said, and this is Psi hat ok this will be Psi hat. So there are many other kinds of wavelet available in the market Meyer wavelet, Morlet wavelet, lifting wavelet, spectral graph wavelet, diffusion wavelet so but all the wavelets does not pose all the desirable properties ok. So if you wanted to this construct Shannon wavelet in the matlab there is a inbuilt function ok. And what is that inbuilt function?

(Refer Slide Time 31:19)



Inbuilt function is Shan wave f ok. Sometimes some of the wavelet in the Matlab can be constructed if you just use the wave fun function which I have by changing this dbm. This is the syntax I was using when I was constructing the Daubechies wavelet dbm and itr. But Shannon wavelet there is a separate inbuilt function which is called Shan wave f like there is also a Meyer wavelet which you can also if you want to know the detail of the wavelet you can looked at in the reference books.

That can be constructed if you change this dbm to Meyer ok and then coma itr. So Shane wave f is also inbuilt Matlab function and about wave fun I have already talked. So with this part or with this lecture particularly you should be able to choose a wavelet depending on the application. So what we have seen? We have seen that wavelet series has advantage over Fourier series. Wavelet transform also has a advantage over Fourier transform which this Fourier series and Fourier transform you have already looked at initial part of the course as well as I have also given some basics of it.

And now you should be also able to appreciate which property good depending on which applications. Ok with this word I am closing my last lecture and so good luck with this course. Thank you very much.