Introduction to Methods of Applied Mathematics
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Module No # 07
Lecture No # 31
Daubechies Wavelet (Contd..)

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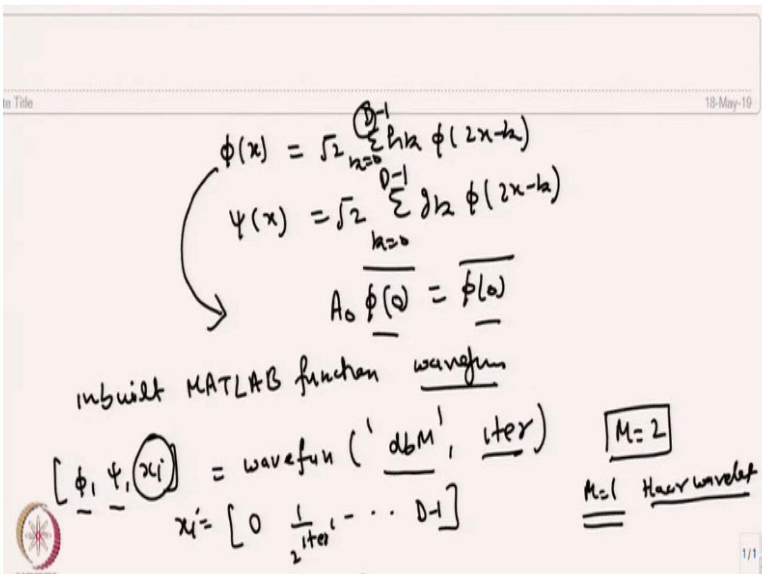

Welcome to all of you in the next class of this course so in the last lecture we were trying to compute Phi x using the following dilation equation or 2 scaled relations and if Phi x is known to us we can also compute Psi x using k wavelet equations. For k = 0 to D - 1 for k = 0 to D - 1 of course this time I am considering compactly supported wavelet that is why I am considering D here ok.

So in the last lecture we have seen two three methods cascade method, spectral method and then recursion method. If you recall the idea from the last lecture in recursion method basically we were computing Phi x with the eigenvalue problem or basically you could say I was trying to compute this way okay Where Phi 0 was the vector which contains the value of Phi at several integers okay.

So it is a finding Phi naught is the eigenvalue problem which is a problem of a linear algebra. So you can look at how we have boil down to the step of computing a eigenvector with this way ok.

So this is that was the idea behind recursion method. Now as I was saying in the last lecture there is also inbuilt MATLAB function fun okay. There is a inbuilt MATLAB function wave fun okay wave fun. So you can say the wave is coming from wavelet and fun is stands for a function.

So what it does for us the syntax of a this wave function in the MATLAB is this idea okay. So dbM is the name of the wavelet okay db stand for Daubechies and M is the vanishing moment of the Daubechies wavelet as I said in my previous lecture also it gives you the value of Phi which is a scaling function and value of Psi which is a wavelet function at grid xi okay. So what is grid xi? xi is basically this grid if you look at okay.

Of course Phi and Psi are compactly supported wavelet of order D so it is support will be from 0 to D – 1 that is why I am taking this and this iteration decides the resolution of the grid that is why the resolution is 0 then the next point will be 1 by 2 to the power iter. This is just a parameter so you could give this parameter value you could give 7, 8, 9 etc. So if m is 1 it will corresponds to Haar wavelet if m is 2 it will corresponds to Daubechies is wavelet of order 4 you could say m is 2 Daubechies wavelet of order 2 or vanishing moment sorry Daubechies is wavelet of order 4 or vanishing moment 2 okay.

So now I can also show you how it looks because you already know how it looks in case of a m=1 because m=1 corresponds to a first member of a Daubechies family which is a haar wavelet which we have already seen in our previous lectures. So corresponds to m=2 how it look let me show you in the figure yes so here we go.

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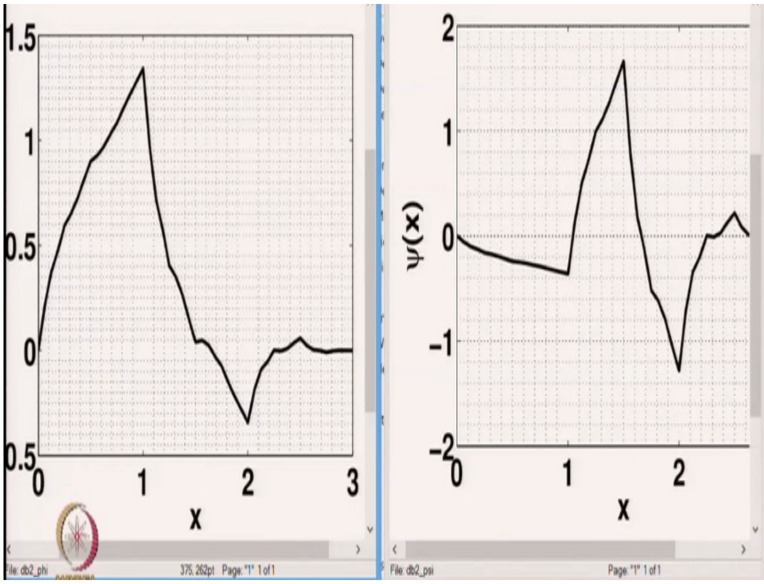

So this is our Phi x if you look at this is our Phi x versus x okay and this is how the scaling function corresponds to Daubechies wavelet of order 4 we will look you can observe this is this cannot be expressed as analytic function which we have seen in case of a haar wavelet and this is a Daubechies wavelet. This is Phi this is Psi so you could very well observe the basic property of the wavelet that integral of Psi x dx should be 0 of course what one more thing you could observe that this Psi x do you think it is symmetric the answer of my question should be no it is not a symmetric wavelet.

As well as it is not that a smooth also in fact in case of a haar wavelet it is that Psi x is a discontinuous function in this case also where it is non smooth it might be continuous but not derivative may not be continuous. So that one has to write precise a statement for the regularity of Phi and Psi that is anyway that is the topic I will discuss next to it. But here you could observe how the Phi and Psi will look like in case of a Daubechies wavelet of order 4 in case of a Daubechies wavelet of order 2 we do not need to use this wave fun function.

Because analytic expression is already there but if you can you if you want to use you can use it that is not a problem. Now I will also this is the first time I will also show you how the Fourier transform of this Phi and Psi will look like.

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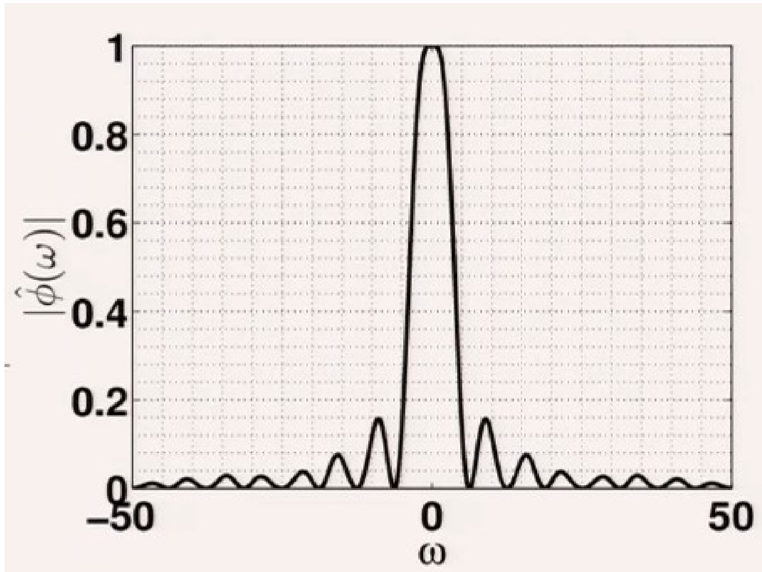

So you look at this is the magnitude of the Fourier transform versus Omega. So you can very well look at this is the how the magnitude of Fourier because this is a complex number that is why we are plotting with the magnitude how it should look like this is Phi hat.

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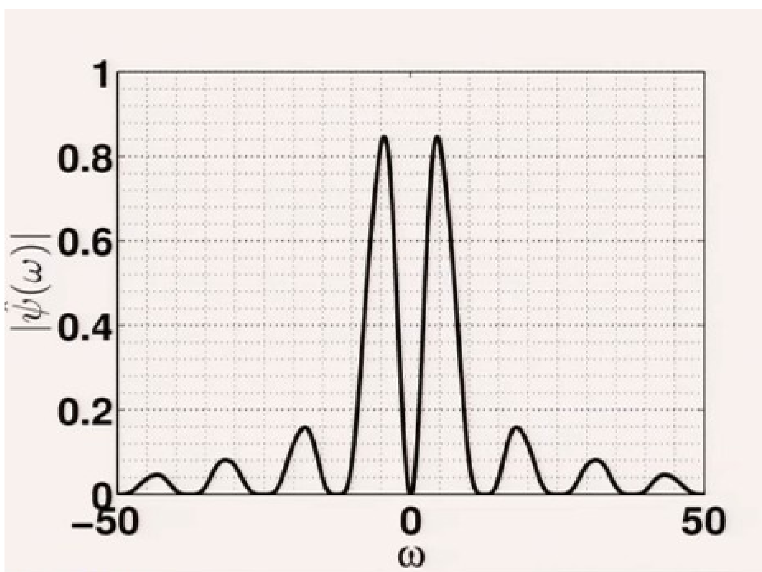

And then next figure I am showing you this Psi hat okay. So I said with the basic definition of the wavelet Psi hat both should be localized so of course in this case Psi compactly supported. So of course it is a localized and Psi also you could see it is not a compactly support either but it is a localized function. That you could observe from this figure and integral of Psi x D x = 0 which anyway you have already observed when I was showing you the figure of Psi. So in this way at least you have seen some members of the daubechies family similarly if you want to try this out later on you could change m 2, 4, 6, 8 onwards and you can keep playing with this how what is the effect of m on the appearance of the wavelet.

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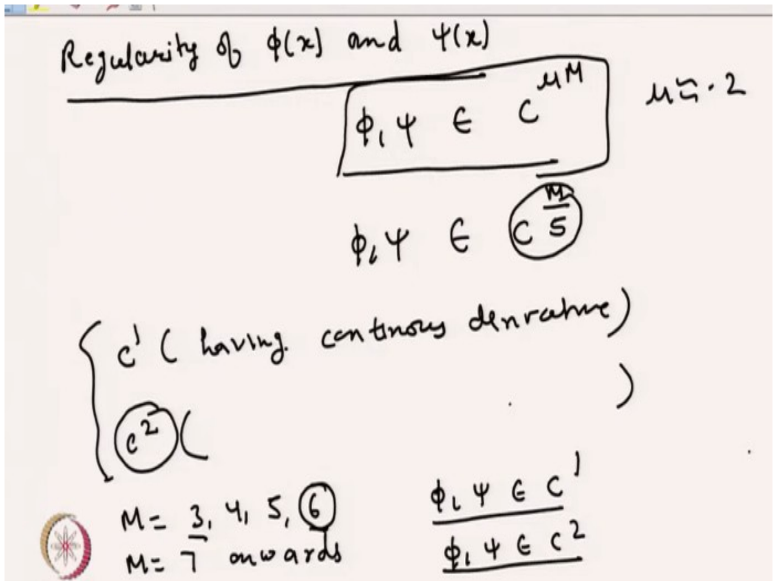

Now as I said now I will be discussing more mathematical caution behind any function which is what is the regularity? So regularity of Phi x and Psi x of course it is a very delicate matter to discuss the regularity of Phi and Psi and in fact whatever the regularity of Phi the same regularity will be for Psi. Because we are also writing Psi in terms of MRA of course whatever here I am saying with respect to MRA wavelet.

The wavelet which are constructed using MRA because in some cases wavelets are not generated with MRA they are called non MRA wavelet. So basically if you want classification of a wavelet in a broader sense MRA wavelets non MRA wavelets. So MRA case every time you can write down Psi in terms of a Phi I will contain the same regularity which Phi will have okay. So now I am writing a one statement that what is the regularity of this Psi Mu M where Mu is this okay.

So this is the result given by Ingrid Daubechies okay for a compactly supported way with regularity of Phi and Psi will satisfy this asymptotically the main point to remember about this estimate is asymptotically. So mean it is true for large M otherwise it is almost true you could say okay. So Phi and Psi belongs to this space everyone understand what is the meaning of this see this space means all the C.

so I can say what is the meaning of c1 and c2 c1 means having continuous derivative function is continuous as well as it is derivatives are also having continuous derivative or in general this is derivative is continuous as well as the all derivatives up to order to our continues that you could that anyway this is a general spaces which notations which everyone should understand. So M by I could either I am so with this vanishing moments I can use same kind of regularity M/5.

So what I mean to say like if I take M = 3, 4, 5, 6 okay so Phi and Psi belongs to c1 and M = 7 onwards Phi and Psi belongs to c2. So if you look at this 3 and this 6 regularity of Phi and Psi are same but here we have wasted some of the vanishing moments okay. So of course this is not exactly true because this result is to asymptotically for a large and but if this is close to that closely if follows for a smaller values of M also so that is what I wanted to say about the regularity of Phi and Psi.

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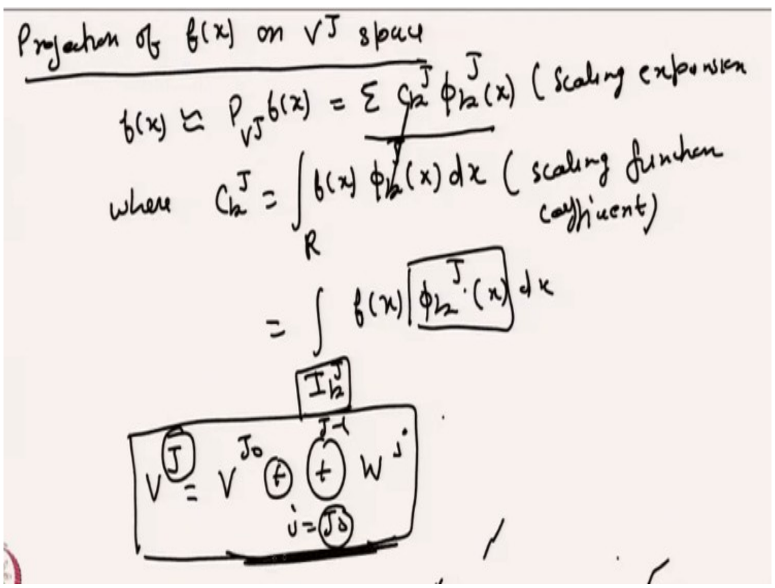

Now what I am saying projection of F on the VJ projection of function f x on VJ space ok so this will be this where ckJ are scaling function coefficient this is called scaling expansion called a scaling function coefficient okay. So this is over R which is also equal to you could say ikJ because the support of Phi kJ lies in Phi kJ that we have already seen earlier okay. So because of that reason instead of R I could write ikJ. Now if I wanted to observe this expansion at multiple levels okay which was the idea behind wavelet multi-resolution analysis I am decomposing my function at different levels.

So for that reason I have to use one of the exams of MRA nestedness property and if you remember with that exam of MRA we already saw what is the meaning of this okay clear. So we are J0 is the course of level of approximation and this J is the finest level of approximation. So using this property of multi-resolution approximation spaces VJ which we call it as a nestedness property also I am writing whatever I am approximating function on VJ space I am rewriting this scaling expansion because this was in VJ I will use now this right hand side okay.

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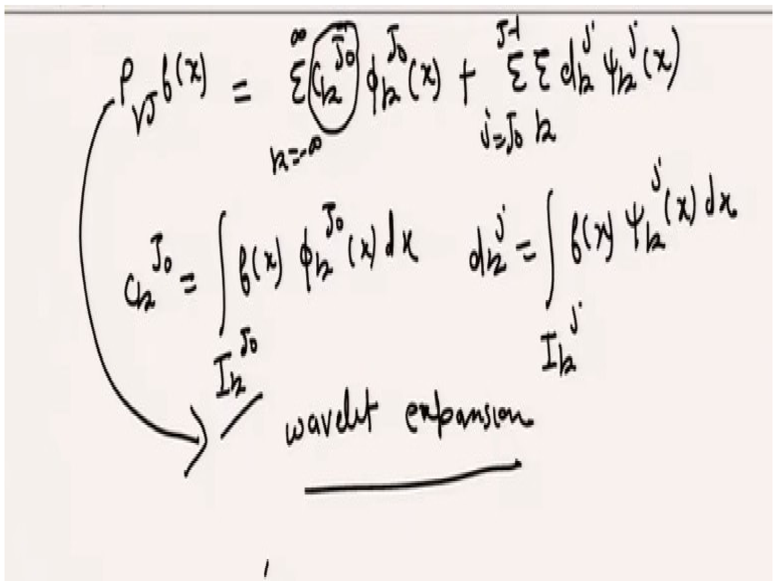

So with that this will be ck J0 Phi kJ0 x + j is j0 j -1 and this will be on dkJ Psi kJ x ok. Of course k you could say k varying for minus infinity here also k is varying from minus infinity to infinity. So ckJ's are again called a scaling function coefficient so again you could with the same logic because support of Phi kJ0 lies here instead of R I am writing this and dkJ are called wavelet coefficient okay.

And this is whole thing is called wavelet expansion or you could call it as a wavelet series. Because if you have to appreciate the beauty of wavelet series over Fourier series so with respect to that terminology better you call it as a wavelet series okay. So now I am going to write the key estimate for wavelet series with that estimate only you can appreciate that how wavelet series is doing better than Fourier series for some function for some specific class of a function.

And that was the aim of my this part of the course that I should be able to tell you the advantage of wavelet series over Fourier series and advantage of wavelet transform over a Fourier transform. So then in that case you appreciate whatever application of Fourier transform has also same set of applications and the same way you could say for a wavelet series with some additional advantages okay.

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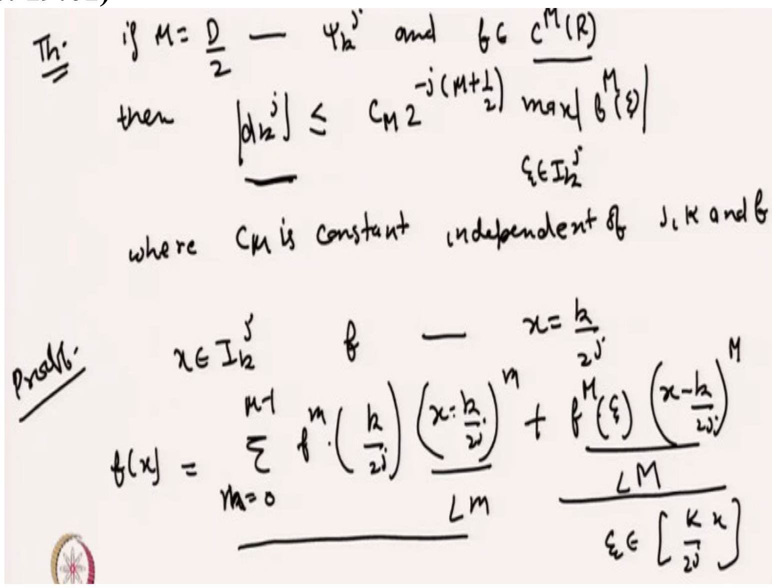

So what is that key estimate that key estimate I am going to write in the form of a theorem so that I will write on the new page what this theorem says if M this is the relation we always see M is the vanishing moment D is the order of the wavelet. So M is the vanishing moment of the wavelet Psi kJ and f belongs to CMR. I have already explained what the meaning of this space is so in that case dkJ will satisfy this estimate okay We CM is a constant independent of j K and f.

So as I already said that this is the key estimate of the wavelet so let me prove this how it is derived? So for that reason I am considering a point x which belongs to this then I am writing a Taylor series of a function f around point x = k / 2J ok. Of course because f belongs to CMR so for that reason I am writing a Taylor series m = 0 to m-1 fm k / 2 J x- k / 2 J to the power M by factorial m + f m zeta x – k / 2 J factorial M where zeta will belongs to k / 2 J into x.

This all of you how to write down a Taylor series this is a simple thing so we are expanding function f this is a remainder term of the Taylor series all of you know where zeta will belongs to this interval clear to everyone that here just based on this assumptions I am expanding the function in the Taylor series form around the point k / 2 J. So now I have to determine this estimate so for that reason I am writing what is the formula of this wavelet coefficients dkJ's are called wavelet coefficient CkJ’s are called a scaling function coefficients that anyway we have seen already.

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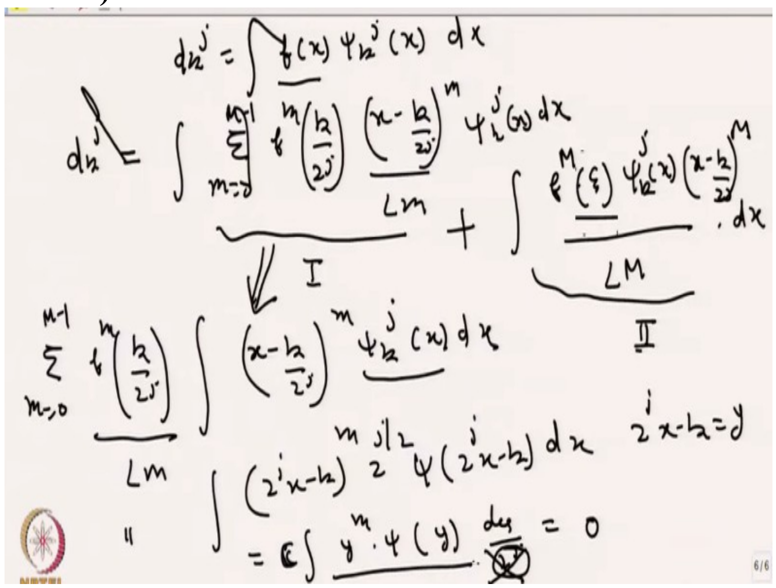

So and then so what is the formula for dkJ f of x Psi kJ x dx so now in this formula I am inserting the expansion of fx which I have written just now m = 0 to m – 1 fm k / 2 J x – k / 2 J by factorial m okay and then Psi kJ dx plus integral of fM Psi kJx x – 2J to the power M by factorial M into dx ok. So let me simplify this by considering this as a part 1 and this as a part 2. So first let me take this part 1 ok.

So how to take a part 1? So I have to simplify this term ok integral factorial Lm also I can take it out from the integral okay. So this is part 1 so if I now again this I can take it common 2jx-k m and if you look at what is the meaning of this symbol psi kJ this is 2 to the power Psi 2jx-k into d x. So if now I use the change of variable 2jx – k = y this will be y m Psi y dy / 2j of course then this 2j/2 will also in all of whatever coefficient value here it is coming I am denoting with this sum c this 2j can also be clubbed into this c.

So that is why I am doing that but what is this? And what is the value of m? m is from 0 to m - 1 so if you look at the vanishing moment property of the wavelet this is basically 0 this is 0 with the vanishing moment property this is 0 so part 1 is 0.

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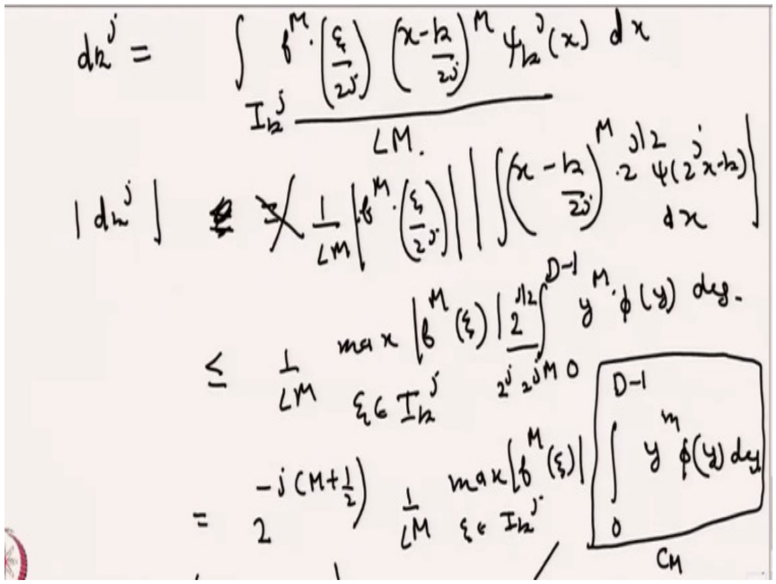

So then I am left out with that term this okay integral of f m x – k / 2 J m Psi kJ x dx by factorial M and okay because part 1 is already 0. So now if I am writing this way then this will what this will become right now okay in the next step I will do this in equality but right now I can take it in equality itself 1 by factorial n fM Psi / 2 J this will also be the mod integral of okay. So just I have taken the absolute value both the sides okay.

Of course one and now I can use the inequality by see 1 over factorial M and maximum of f m Psi were because this thing I was taking from the integral IkJ if you look at that is what we have done here this was the part 2 I am going to simplify okay. So fM Psi zeta in fact if that is the case I could I would taken the inequality here itself so zeta belongs to IkJ and the left term will be y to the power M Phi y dy what will be the coefficient 2 to the power J / 2, 2J and 2JM.

So the support of why is from 0 to D – 1 that is that is why that is the limit I am putting so this will become 2 to the power – j M + half okay 1 over factorial M maximum of f M zeta. Zeta belongs to IkJ in fact you can overlook this step and you can start thinking from this step then it will make more sense okay. 0 to d – 1 ym Phi y dy clear. So now this you can now what if you were trying to prove if you look at this was the case so we got this term and we got this term so now you could take CM as this constant CM is basically this constant which is independent of Jk and f .

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$$|d_k^j| \le C_M 2^{-j(M+\frac{1}{2})} \max_{\xi \in I_k^j} |f^M(\xi)|$$

Remark: if function ~~has some~~ discont is not smooth in some part of domain then d_k^j will not decay fast. and if function is smooth in some part of domain then wavelet coeff. will decay very fast.

So now if you look at the so we have finished the proof of this key estimate so again I am writing this estimate in the next place to understand more clearly okay. So now you have to observe this first of all do you think when you write a Fourier series such kind of estimate available for Fourier coefficients? The answer is no okay we do not have such kind of estimate for the Fourier coefficients.

But we have this kind of estimate for wavelet coefficient of a which is a part of the wavelet series that is okay. One thing that estimates is there but what is the beauty of this estimate that is another thing okay. So what is the beauty of this estimate? So I can show you if function is a polynomial of degree let us say M – 1 okay. Then in that case this will become 0 because if you are taking a derivative of degree Mth derivative of a polynomial of a degree M – 1 this term will be 0.

So it means wavelet coefficients are 0 so if wavelet coefficients are 0 then you can expand the whole function only with the fewer coefficients and that is the idea behind compression of the wavelet that is why wavelets are used for compression whether it is a signal it is a image or compressing the grid points on which we solve differential equations and when we compress or we reduce the grid points on which we solve the differential equation then it is called a we are solving a differential equations on the adaptive grid.

So that is the application of a wavelet in altogether different domain for solving a differential equations. But as far as engineering is concerned wavelets are used for compressing the signals image processing etc. But everywhere if you look at more mathematically the key thing key estimate lies here so that wavelets are used for compressions. So if you want to put this as a remark whatever I have said just now.

So let us put that in the form of remark first thing that these wavelet coefficients tell you the local behavior of the function because you know in this domain. So if function has some discontinuity in some of its derivative in some part of the domain okay. If function has some discontinuity or in some derivatives or you could say function is not smooth in some part of domain it means you can say piecewise smooth?

So a function is not smooth in some part of the domain then wavelet coefficients will not decay very fast then dkJ will not decay fast. But if function is smooth wavelet coefficients will decay very fast and if function is smooth in part of rest part of the domain then wavelet coefficient will decay very fast okay. So for a smooth function you do not need to store that many wavelet coefficients because some of the wavelet coefficients will already be 0 that is the idea behind compression.

And so wavelet coefficients most of the wavelet coefficient will be non 0 in that part of the domain where function is showing some non smoothness nature. So that is the idea and that is the beauty of this estimate and this kind of estimate is not available for Fourier series. That is why wavelet series are better than Fourier series for representing non smoothness non a smooth function for representing it or non smooth I should not say piecewise smooth function which is having some localized information in some part of the domain okay.

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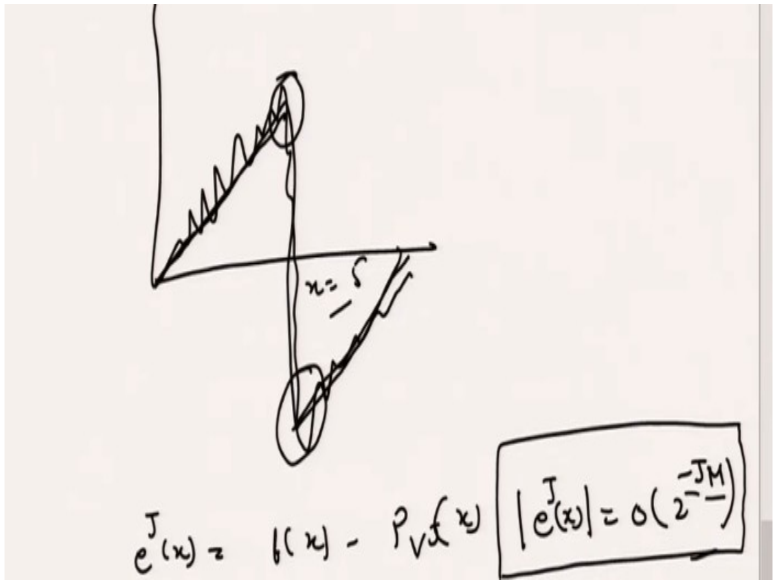

And with the help of if you remember when I was pointing out one of the drawback of the Fourier series what was the drawback if you remember we were it was a Gibbs phenomena that Fourier series approximations does not gives way good approximations near the discontinuity so if you remember that part I have shown you by taking the example of it this kind of a function of course x and then x – 1 of course this is just a diagram of that function.

And then when we looked at the Fourier approximations it will look like that so means because of this discontinuity which is let us say x = 0.5. This is a diagram that is why I have drawn a line but otherwise it is not a continuous at this point. These oscillations were spread throughout the domain that was the Gibbs phenomenon idea. But in a wavelet series you do not see that these oscillations are throughout the domain you see oscillations only in this part in fact those oscillations can also be removed with some help ok.

What is that some help? Because just now we have looked at the projection idea. So if we are approximating a wavelet series we are approximating the function on VJ space. So the idea behind if I define this error e J x I can define this error this in the following way ok. So here one of the estimate eJx will be order of 2 to the power – J M. This estimate I am writing without proof if anyone of you are interested in looking at the proof it is given in the reference book which I am using for this course wavelet theory and its application a first course by myself itself.

So this is the estimate and to prove this estimate we also need the key estimate for wavelet coefficient which I have proved just now okay. So means if you want to reduce the error you can play with 2 parameters M and J. M is the vanishing moment and J is the level of approximation on which I am approximating the function okay. If you are increasing J error is decreasing if you are increasing M error is decreasing.

So whatever little oscillations you see near the discontinuity or in the near the non smoothness of the function you can in fact you can remove those little oscillations with the help of by increasing J or increasing M. Increasing M means you are using higher order wavelet increasing j is you are approximating a function at higher level space V J. So it means whatever disadvantage we have seen in case of a Fourier series we can overcome with that this advantage with the help of wavelet series.

So that was the idea behind this that how I should be able to deliver you that what is the advantage of wavelet series over Fourier series? So now with these words I am closing this lecture and so in the next lecture which will be my last lecture I will be showing you the advantage of wavelet transform over Fourier transform okay. Thank you very much.