Introduction to Methods of Applied Mathematics
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Lecture - 03
Introduction to Second Order Linear Differential Equations

Hello viewers. So welcome back to the course. This is a lecture number 3. So today we will discuss about the second order linear differential equation because in the last lectures, we have dealt with the theory to solve first order linear and the nonlinear differential equation. So in this lecture, we are going to start moving further to the second order linear differential equation. So lecture 3.

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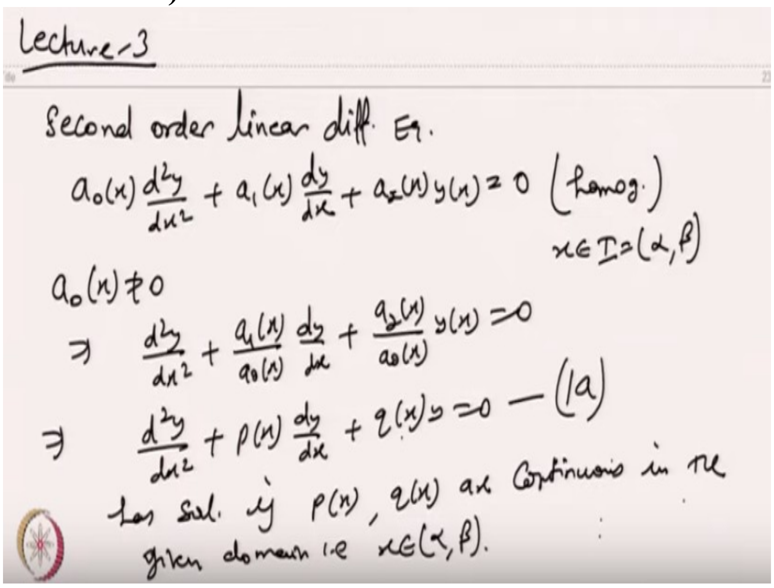

So today we will discuss a second order linear differential equation. So I write the linear differential equation as d square y over d x square plus a 1 (x) dy/dx + a 2 (x) y(x) = 0. So here I am talking about homogeneous forms. So this is a homogeneous. Now I am defining this one in where my x belongs to some interval. So this interval I just take as an open interval between alpha and beta.

So I assume that a x is not equal to zero. So I divide by a naught zero. So this equation is reduced to a 1 (x)/a 0 (x) dy/dx + a 2 (x)/a 0 (x) y(x) = 0. So I just make it give a new name. So this equation I can write as d square y over d x square plus so I give the new name to this one. So that is P x dy/dx + q(x) y = 0. So this is my equation, I call it 1a. So in this case I assume that so this equation has solution if my P x and q x are continuous in the given domain. That is my x belongs to open interval alpha, beta.

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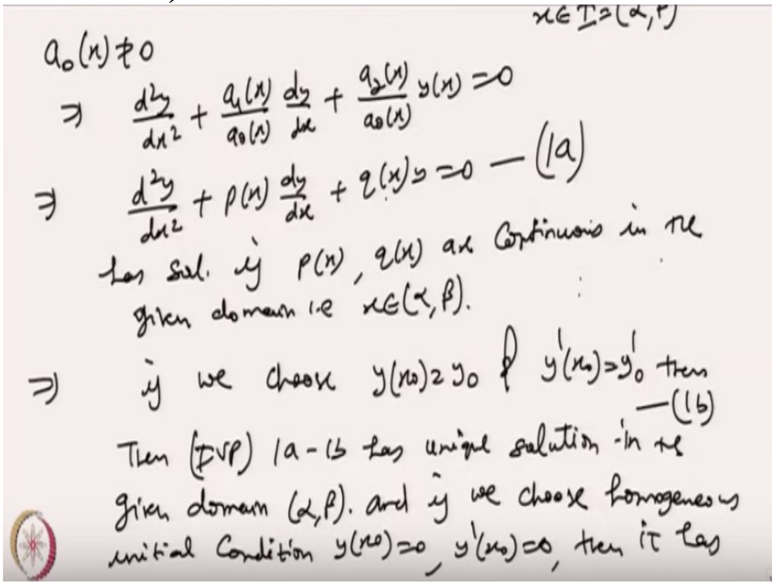

So now, so this equation if we choose some initial condition y (x 0) = y 0 and y dash at x 0 = y 0 dash then so I call it 1b, then the initial value problem that is 1a and 1b has unique solution in the given domain that is between alpha and beta and if we choose homogeneous initial condition, that is y(x 0) = 0 and y dash x 0 = 0.

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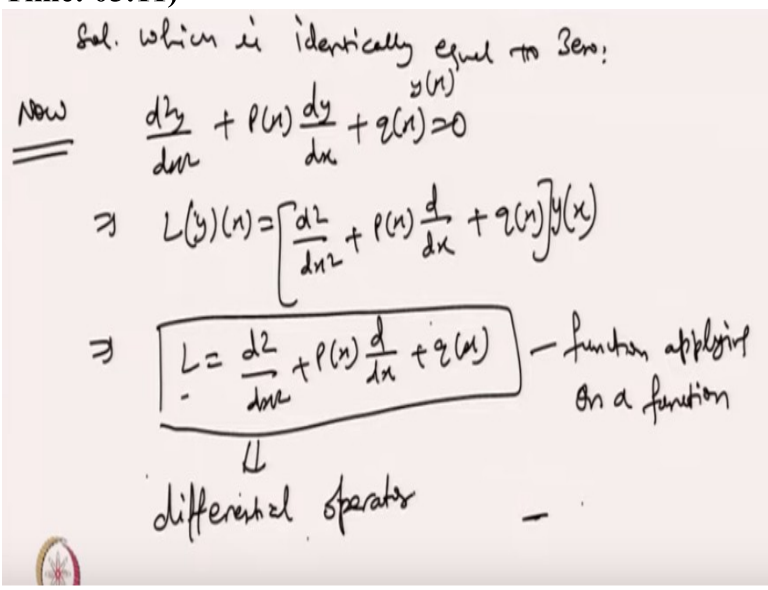

Then it has, then this equation has solution which is identically equal to 0 means that the solution y(x) = 0 is the only solution in that case. Now, so the equation 1a, now I introduce a operator, differential operator. Now the equation 1 which we have written like this one dx square + P x dy/dx + qx yx = 0. So this one I can write as I define the operator L is equal to y operating on x.

So I put it like this one, d square over dx square plus P x d by dx plus Qx and then put this one on x. So in this case, we define that L is equal to d square over dx square plus Px d over dx plus qx. So what is this? It is a function which is applying on a function, because here my I am applying on a yx, so this is yx. So this one I am applying on the yx and the solution, whatever the function I am getting, that is also a function.

So in this case, I can say that this is a function applying on a function. So applying on a function. So this is called differential operator. So differential operator actually operator is the extension of the functions, because the function we apply on a variable x in this case we are applying this operator on a function and the result is also a function. So this operator is extension of the function. So we define this L.

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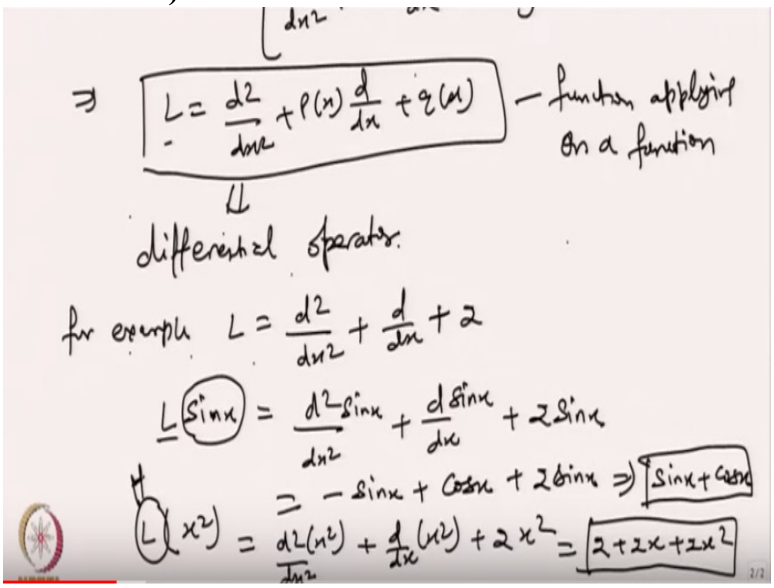

So for example if I apply L on so suppose I take L = d square over d x square plus d by dx plus 2 qx because I take this function this operator and I apply on sine x. So this will be again I am applying on a function. So this is my operator and I am applying on this function. So what I will get? To find out the value I will put this one here on the sin x plus d by dx on sin x plus 2 qx on sin x.

So in this case okay so what I do is, so this qx I have taken 2. So this qx will vanish from here and this will be to 2 sin x. So if I take the derivative, this the derivative sign is cos and again it is minus, so it will be - sin x + cos x + 2 sin x and which gives me that sin x + cos x. So this is the result by applying this operator on the function sin x.

And the result is coming sin x + cos x. So this is the operator operating on a function and the resultant is also a function.

Similarly, I can apply L on suppose I apply on x square. So putting the same one, I will get d square over dx square on the x square plus dy dx on x square plus 2 and that is x square. So after doing the calculation and taking the derivative of x + 2x and taking the derivative again. So it will be 2 + 2x + 2x square. So that is again the function. So in this case, that L whatever we are defining here, it is called the differential operator.

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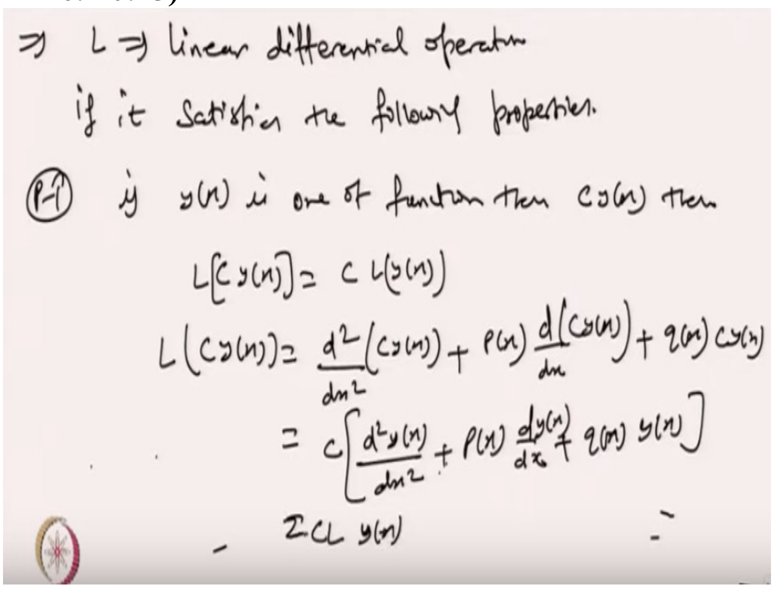

We define little bit move further and we will define that the operator I have defined that is L is a linear differential operator. So what do you mean by the linear differential operator? So linear differential operator is that, that if it satisfy, so it is called if it satisfy the following properties. So what is that property?

So P 1 the first property I am defining, that is called that if y x is one of the function we are defining by y then if I define a cy x then L[c y(x)] will be equal to c L y x. So this is the function, I have defined my L here and now I do the scalar multiplication of yx and operating the same function, then I should get this one.

So this is very easily we can verify that L of cy x = d square/dx square over c y x + P x. So I am taking the same differential operator, which we have defined on the equation 1a. So d/dx it is c y x + qx. And this will become c y x. Now it is a constant.

So I can take this constant common from here and then it becomes d square yx + d x square = P x d/dx and qx yx. So this is again the same as L C of L of yx. So the first property is satisfied. Then we define the second property.

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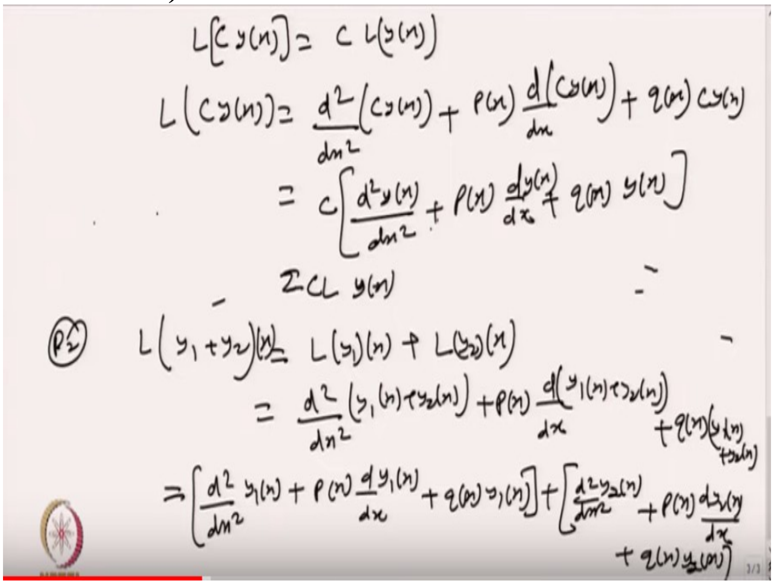

So second property is that property 2 is that L(y 1 + y 2). So in this case I have two function y 1 and y 2 and operating the L on this summation. So in that case it should be equal to L(y 1) that is operating on x plus L(y 2) that is operating on x. So this one I can also prove very easily. So we have operator d/dx square. So this is y 1 x + y 2 x + P x d/dx. It is y 1 x + y 2 x + q x and putting here y 1 x + y 2 x.

So this one we are writing. Now this is the summation of the functions. So from here I know that the derivative is a linear operator. So it can be taken as a separate form. So it can be written as d square over dx square y 1 x plus P x d/dx y 1 x + qx y 1 x. So I can separate this one and can write like this one and the another one is d square over dx square y 2 x + P x d y 2 x over d x + q x and y 2 x.

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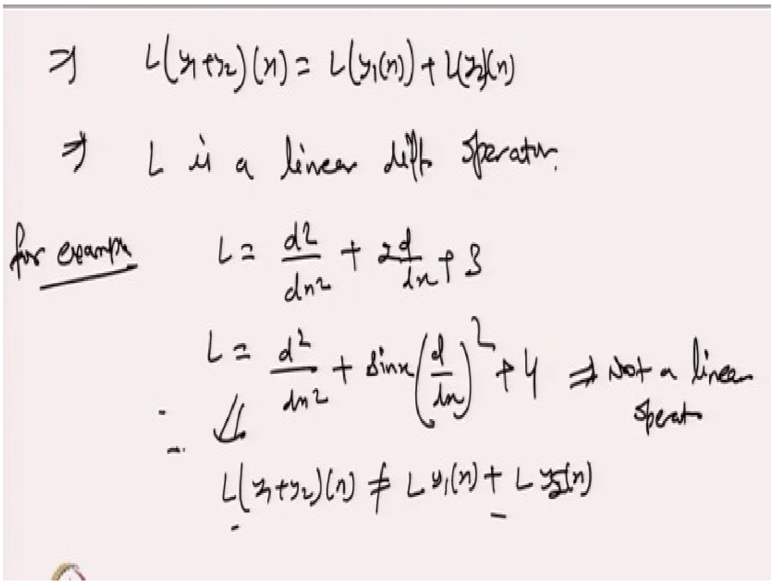

So this is also I can separate and from here, so after doing this one from here I can say that my L (y 1 + y 2) applying on this one they will be the same as L(y 1 x) + l(y 2 x). This is applying. So after doing this one we can say that if this is satisfied then I can say that L is a linear differential operator. Okay, so this is a linear differential (operator).

So the example I have taken, for example I have taken my L = to d square/dx square + 2d/dx + 3 that is a linear differential equation, because if I apply on this one, it is a linear. Another one I take d square/dx square + sin x d/dx whole square + 4 if I take this one. So in this case I have a square root of the derivative. So I can prove that this is not a linear operator.

So not a linear operator because from here I can show you that L(y 1 + y 2) x is not equal to L(y 1 x) + L(y 2 x). So it can be proved very easily. So that is you can do as a homework.

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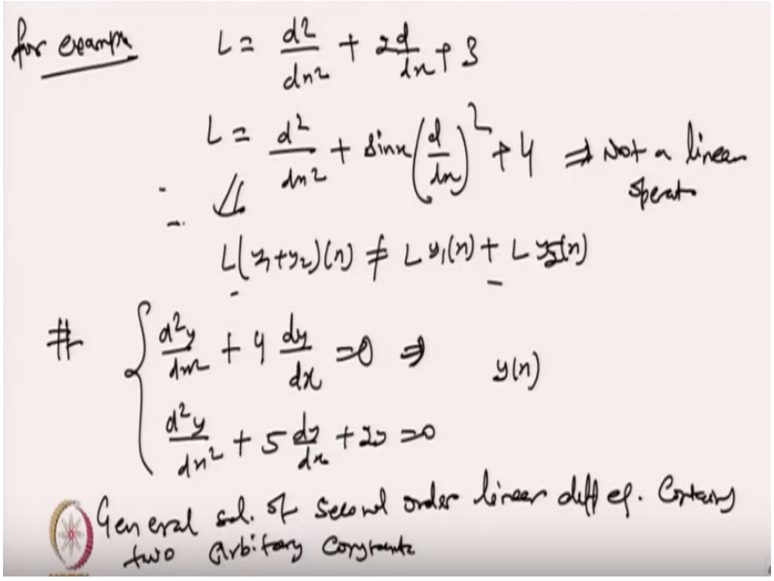

Then, so in this now after doing this one, so we know that this is a linear operator. Now, so the question is that that why we are doing this one, we are defining the linear operator. Because our main purpose is to solve the differential equation. So to develop the solutions, we need some theory that how the operator works on a operate on a function and which function we can say which operator we can say that is a linear operator or nonlinear operator.

So now so I know that this is my second order differential equation. So like so let us take one example. I have very simple differential equation dy/dx square plus some constant I want to take. So I just take 4. dy/dx = 0. So this is the differential equation I have taken in which the coefficients are constant. It can be a function of x, but here I am taking this one. So this equation I want to solve.

Or I want to solve d square y/dx square minus in this case I take 5, dy/dx + 2y = 0. So this one I can take. Now if I want to solve this equation, I need a solution that is the yx and we also know that the general solution of second order linear differential equation contains two arbitrary constants. So my main purpose is to find the general solution for this equation.

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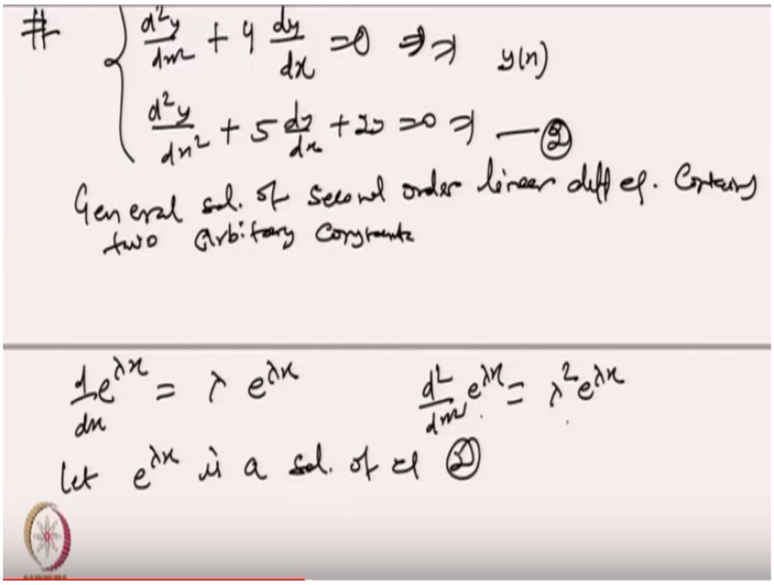

Now like this one, this is the my one of the equation, this is another equation, and I want to solve this equation. So if you see clearly from the equation, I need a function yx such that I do a scalar multiplication with that function, take the derivative as in the first equation, then I should get the value 0.

In the second one I should have a function such that this function is multiply by 2 and then taking the derivative of that function multiply by 5 and then again taking the derivative of that function and adding all these together, it should make zero. So in this case, I should look for the functions which has the derivative of similarity. So let us go for the functions which have the same derivatives like I can take a sin function.

So sin function, the derivative of sin function is cos and then taking another derivative it is a - sin x. So in this case sine x, but we have to take the two derivative which has the same function. Then so it means the sin x is not the function we are looking for, then I can go for the polynomials. But in the polynomials we know that whenever you take the derivative, its degree will reduce by one.

So that is also not going to fit for this type of equation. Then we come across the exponential function. So like I take the exponential function like e lambda x. So in this case I know that the exponential function has the derivative of the same form. So if I take the derivative of exponential function, it will be lambda e lambda x again, where lambda (is any) scalar I am taking.

Again I take the second derivative of e raised to power lambda x and from here I get lambda square e lambda x. So this function has the same derivative and whenever we are taking the derivative, this function is multiplied by the value of lambda. It means this function can be the solution of this equation. So we take that let us I solve this equation I call it second one. So let e raised to power lambda x is a solution of equation number 2.

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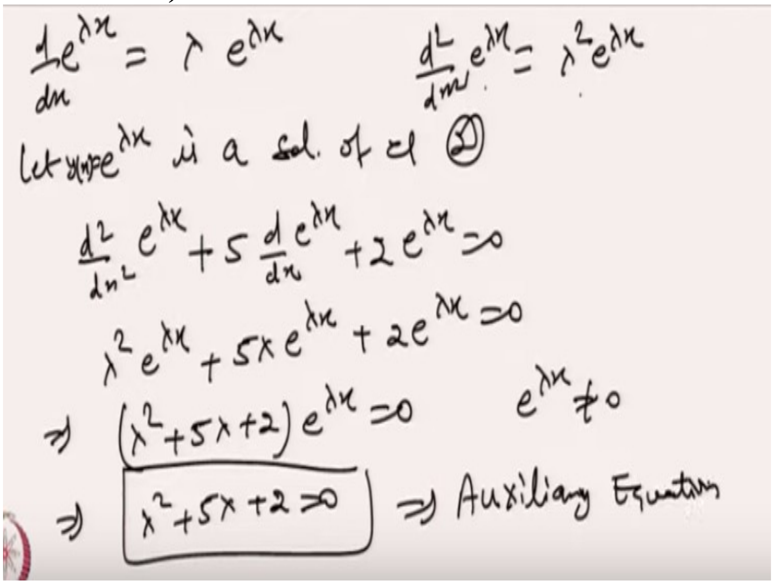

So if it is the solution of the equation number 2, it should satisfy that equation. So in that case, so this is my yx I am taking, yx is equal to this. So I should have d square/dx square e lambda x + 5 d/dx e lambda x + 2 e lambda x = 0. So in this case, I take the derivative two times, so I am getting lambda square. I take the derivative two times. So it will be lambda square e lambda x + 5 lambda e lambda x + 2 e lambda x = 0.

Now, so from here I can take the e lambda x common and I will get lambda square + 5 lambda + 2. So in that case, this is the remaining part. Now from here I know that e lambda x is never zero. So it is the factors two factors multiplied together and giving zero. So either one of them should be zero. So in that case I will say that the lambda square + 5 lambda + 2 = 0.

So whatever we are getting is called so this type of thing is called a auxiliary equation associated with the differential equation. So now if you see clearly that this is the quadratic equation and the quadratic equation we can solve.

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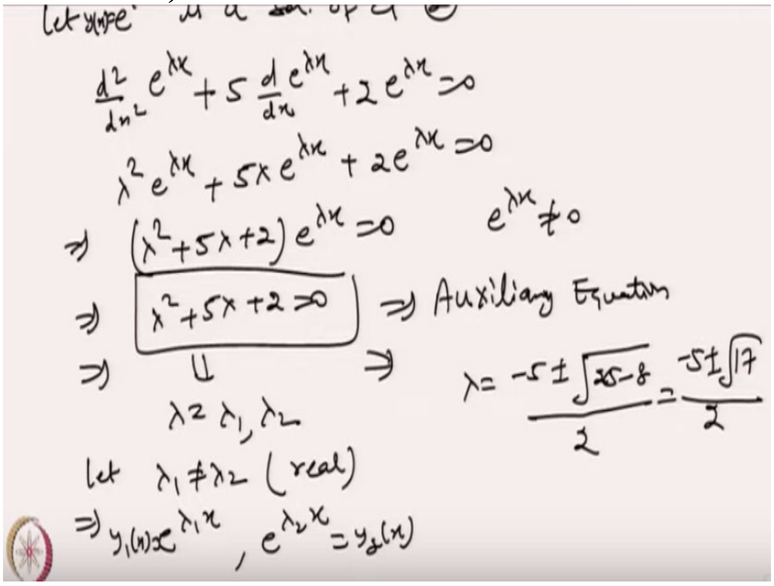

So if I solve this one I will get the value of lambda 1 and lambda 2. So the lambda 1 and lambda 2 are the two roots of this equation. So I can go further. In that case, so let I assume that lambda 1 is not equal to lambda 2. So it means and they are real. So if I say that the lambda 1 and lambda 2 are not equal and they are real because we can solve this one very easily.

So this will be lambda is equal to - 5 plus minus b square - 4 ac. So this will be 8 divided by 2. So whatever I am getting here is - 5 plus minus. So it will be 17 under root 2. So these are the fractions we are getting. So these are the real and lambda 1 and lambda 2 are not the same. So in that case, I will get two solution that is called lambda 1 x and lambda 2 x. So these are the two solution I am getting.

So I will call it this is the first one y 1 x and this I will call it as a y 2 x. So these are the solution, two solution I will get whenever I will come across such type of equation which has a quadratic as a auxiliary equation.

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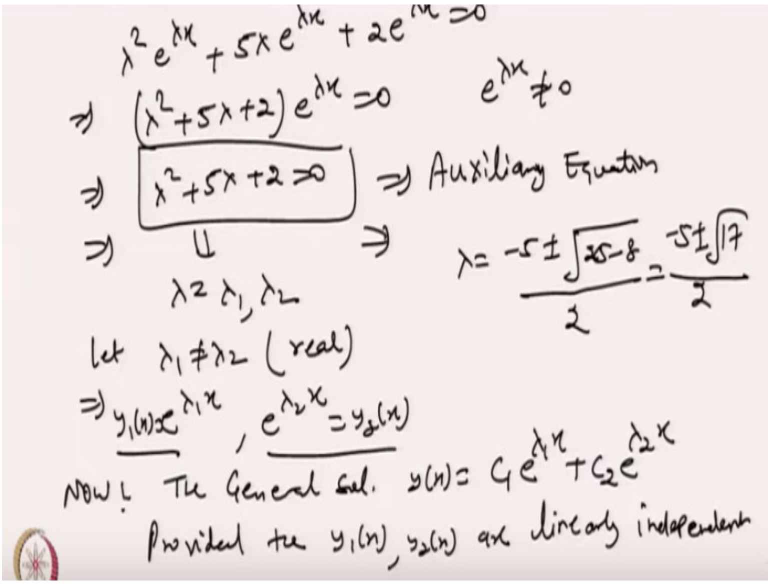

So now I have a two solution. But here this differential equation I have I want a one solution or maybe two solution. So in this case I have two solution. Now, I want to write the general solution of this equation. So the general solution yx will be equal c 1 e lambda 1 x plus c 2 e lambda 2 x. Because I told you that this is a second order linear differential equation and in this differential equation, the two arbitrary constraint will appear.

So c 1 and c 2 are the two arbitrary constants provided. So this is true provided the solution y 1 x and y 2 x are linearly independent. So what is the linearly in d (independent). So that condition is there that my solution y 1 x and y 2 x should be linearly independent. Linearly independent means that so let us introduce the concept of linearly independent.

So we must have seen this type of concept linearly dependence or linearly independence in the case of vectors where we have a vectors that contain the numerics. But here we are dealing with the function. So I want to discuss that how the functions can be told that whether they are linearly independent or dependent. So let us do this one.

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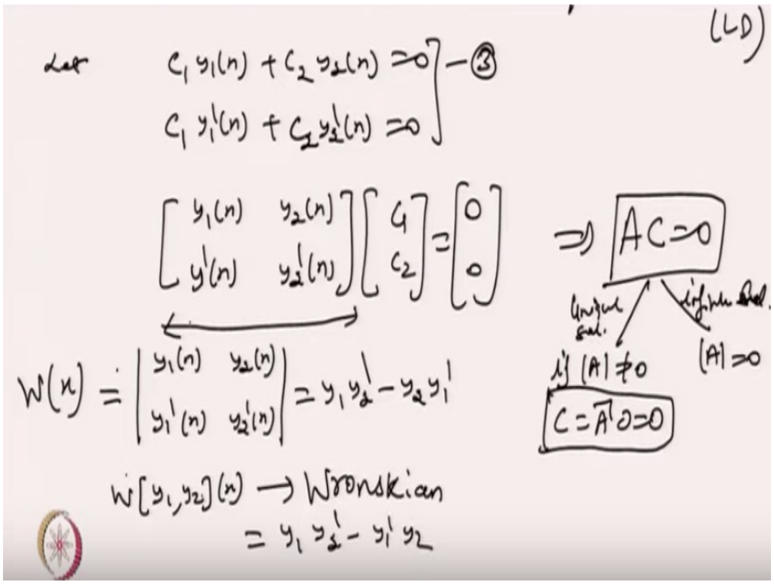

So I have two functions y 1 x and y 2 x and I want to check that whether these are linearly independent. So that is called LI or linearly dependent that is called LD. So what do you do? It is the same concept as we are doing for the vectors, take the linear combination. So let I take some constants I should take c I y 1 x + c 2 y 2 x = 0. So this is the linear combination I take.

That is the criteria to check that whether the functions or whether the vectors are linearly independent or no (not). We take the linear combination I put equal to 0. So this is the first equation. So I call it equation number 3. Now I have two c 1 and c 2. This one I want to find. So to find out this one I need one another equation also because we know that we have two variables to find out and then to find out the solution we need the two equations.

So what I do is that and I know that y 1 x and y 2 x is the solution of the linear differential equation. So these functions are differentiable function. So what I do is that I take the derivative of the equation 3 with respect to x. So when I take this one so I will get this one equal to zero. So this is my equation now. Now this is the system. So I can write that system as y 1 x y 2 x y 1 dash x and y 2 dash x and this one I can write as c 1 and c 2. This is equal to 0 0.

So if you see it clearly that this is a system two by two system and this I can write as AC = 0. So this is the homogeneous system of equations. And we know that this system has, so this system has two ways, unique solution and infinite many solutions. So unique solution will be in this case if A is nonsingular.

So if the A is nonsingular, I can take its inverse and in that case, my c will be A inverse 0 and that will be 0. So for the unique solution, we know that this is the only trivial solution, we call it trivial solution. And if A is equal to zero, then in that case we may have the infinite many solutions. So here also, so this is the case I am taking. Now to solve this one, so this is my matrix.

So I want to check that I defined this matrix by some so I take this matrix and want to find its determinant, y 1 dash x and y 2 dash x and take the determinant. So determinant will be y 1 y 2 dash minus y 2 y 1 dash. So this is the determinant. So we call this determinant as in with a new name that we represent by W and that is equal to x and also we call it sometime W. So this one is W.

So we call it W y 1 y 2 x and this W this is the determinant of the corresponding matrix is called the Wronskian of the, so this is called Wronskian. So this is called the Wronskian, this factor. So in this case, my Wronskian is equal to y 1 y 2 dash – y 1 dash y 2. So that is the determinant of this matrix made up of the functions.

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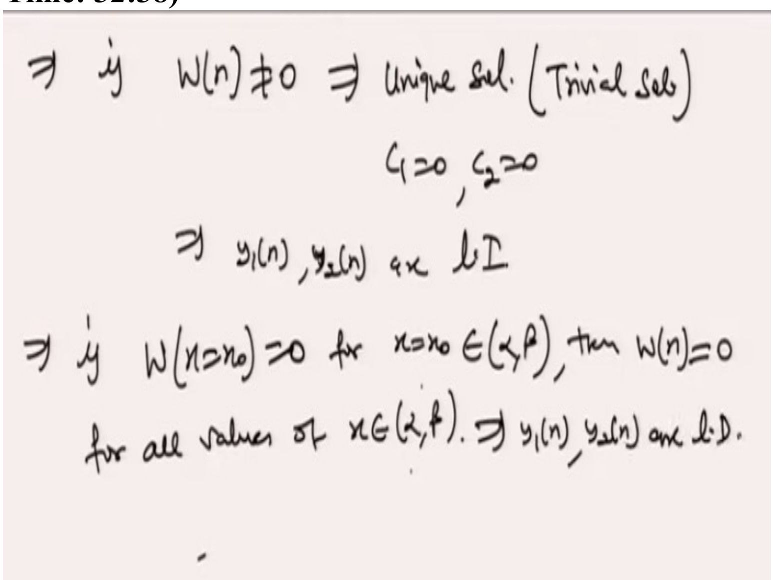

So now, so I will say that if W x the Wronskian is not equal to zero in that case I will have unique solution. So unique solution means trivial solution and in that case I will get c 1 = 0 and c 2 = 0. So in that case I can say that my function y 1 x and y 2 x are linearly independent. So they are linearly independent. Now there is another theory that if Wronskian is 0 at even a single point for any x is equal to x naught, that x naught belongs to the domain in which the differential equation is defined.

Then in that case then the Wronskian is also equal to zero for all values of x belongs to alpha beta. So in that case the Wronskian will be 0 for all value of x and probably in the given domain. And in that case, I will say that y 1 x and y 2 x are linearly dependent. So this is the related to the Wronskian.

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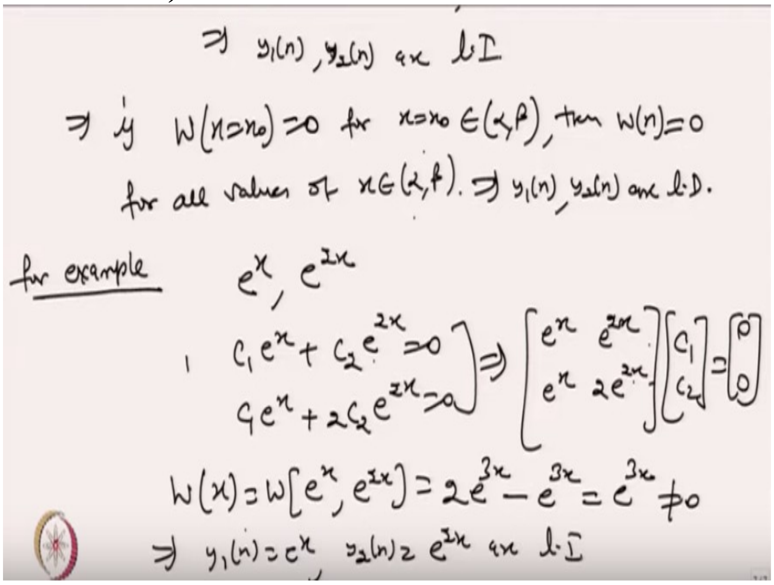

So for example, suppose I take two functions e x and e to power 2 x and just I want to check whether these two functions are linearly independent or dependent. So what I do is that I take the linear combination c 1 so this I will take c 1 e x plus c 2 e to power 2x equal to 0. And then I take the derivative because it is a differentiable function.

So I take c 1 e to the power x plus 2 times c 2 e to the power of 2x equal to 0. So this is the equation we get. So from here so this is the system of two equations. From here I get e x e 2x again e x and this is two times e 2x. So this is my system c 1 and c 2, c 1 and sorry it is c 1 that is equal to 0 0. So in that case my Wronskian will be that I can write as e of x and e of 2 x. This will be I am taking the determinant of this matrix.

So it will be two times of this. So it will be 3x minus this one e 3x. Because e x multiplied by e 2x so e 3x. So this will be e 3x. And I know that exponential is never equal to zero. So in that case, I can say that my function y 1 x e x and y 2 x e 2 x are linearly independent. So here, we are able to show that this two functions are linearly independent.

But sometime it happens because we are dealing with the second order differential equation. But when we go for the higher order, in that case we can have a 2, 3, 4 more than 4 functions as a solution of the linear equation. And to check that whether the functions are linearly independent, dependent, the matrix involved has a lot of functions. So in that case what we do is that sometimes it is very difficult to solve that matrix.

So we put a some value of x in that to make the matrix simpler. And in that case, we generally put x = 0 and then we see that whether the determinant of the matrix after putting x = 0 has a determinant zero or non zero. So in that case, if the value is zero, then from there we conclude that the given functions are linearly dependent.

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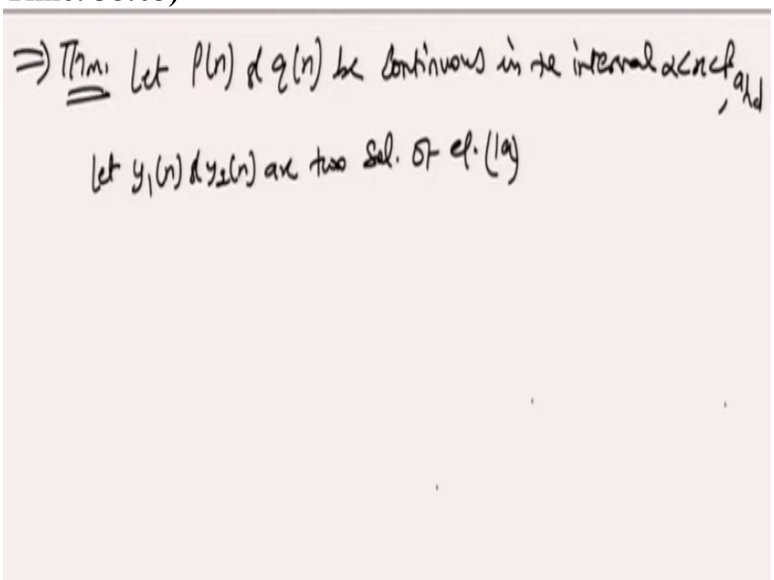

So now I want to take some another function. So for example, I take some other function. So before that I just define one theorem that let P x and q x be continuous in the interval that is given to me that is alpha x beta and let y 1 x and y 2 x are two solution of equation 1a, the equation 1a that we have started with so which contains the p x and q x, 1a.

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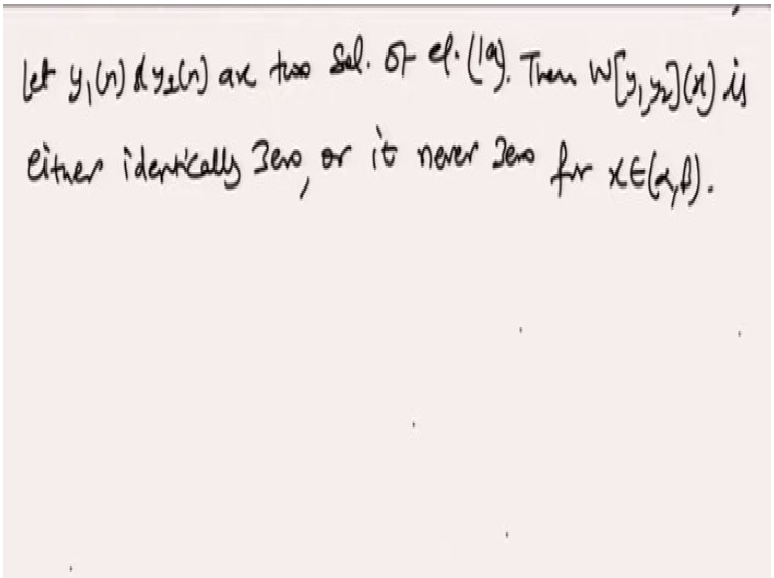

Then the Wronskian is either identically 0 or it never 0 for x belongs to the domain alpha beta. So that is a very strong results that even if we are able to check because it is very difficult to check that it is never 0 for any value of x. Because if I put a value of x and it comes zero, then I have to apply this formula for all the x and which is impossible. So this is a very strong statement regarding the.

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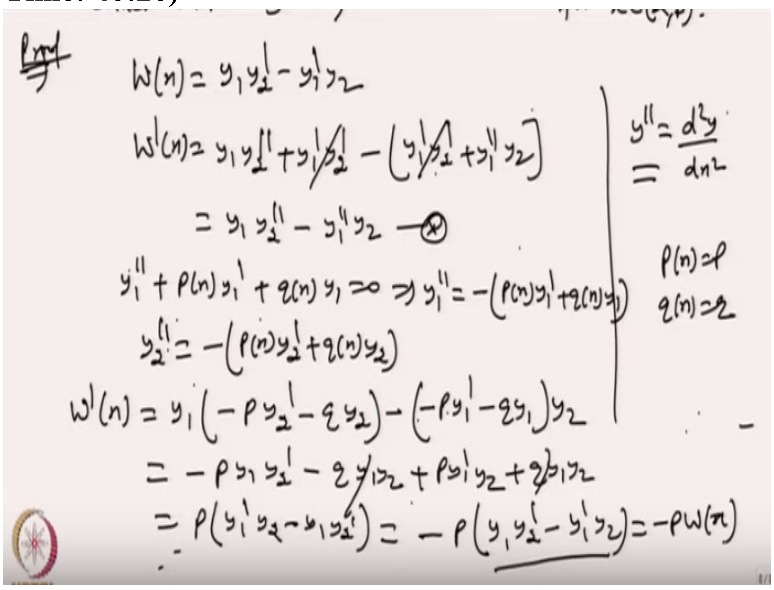

So this is a theorem. So before that theorem, I just check, so I just give you the proof because I know that I have two functions y 1 and y 2. So I know that my Wronskian will be y 1 y 2 dash – y 1 dash y 2. So I am satisfying this one I try to prove this one. So let my Wronskian is made up of y 1 y 2 dash – y 1 dash y 2. So this one I have taken. Now I take the derivative of my Wronskian.

So this will be y 1 y 2 dash two times + y 1 dash y 2 dash. So this is a derivative of this one minus the derivative of this one. So this will be y 1 dash y 2 dash + y 1 double dash y 2 dash. So this way. So if you see this one it will cancel out. So from here I will get y 1 y 2 dash – y 1 dash y (2). And I know that from my because this y 1 and y 2 are the solution of the equation.

So I know that y 1 double dash plus from here I just want to tell you that y dash sometime also can be written as dy dx this one. So to make it easier to write we also write like this one. So here I am writing y 1 dash, then my P x y 1 dash plus Q x y 1 equal to zero. Because y 1 is the solution of the equation this one. So from here, I can write my y 1 double dash will be minus of P x y 1 dash plus q x y.

Similarly, I have my y 2 because y 2 is also the solution. So y 2 can be written as minus of P x y 2 dash plus q x y 2 this one. So substituting this value in the above equation, so this equation I am substituting. So from here I can write my W x can be written as y 1 and this is a minus P. So now I am some time to make the things easier we can write P x as P and q x is q.

So I can I just as I am writing this one as minus P y 2 dash plus q, sorry this is a minus, so it will be minus sign also. So this is minus q y 2. This is minus y 1. So this is again it will be – P y 1 dash minus q y 1 into y. So if you this, see this clearly, from here I will get minus of P y 1 y 2 dash minus q y 1 y 2. It will be plus P y 1 dash y 2 and from here it will be plus q y 1 and y 2.

And this will cancel out and we left with this one. So I take the common factor P and then I can write as y 1 dash y 2 minus y 1 y 2 dash. Or I can write it as minus of P y 1 y 2 dash minus y 1 dash y 2. And if you see clearly then it becomes minus P and this is what again the Wronskian. So that is my Wronskian we have started with.

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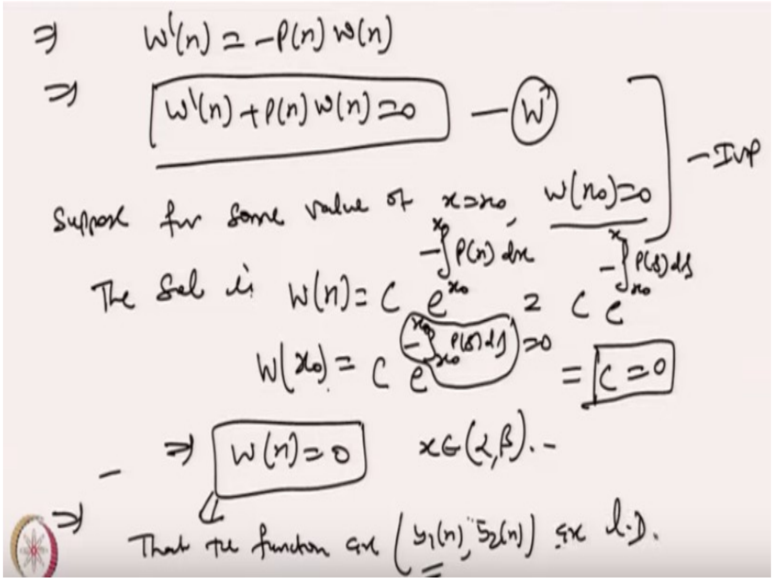

So now from here I can say that my Wronskian satisfy this differential equation and what it is? It is the first order differential equation. So I have this differential equation. So that is the first order differential equation, we have already solved that was linear. Now in this case, so what do we have? Now this is always true that the Wronskian always satisfy whenever we put the Wronskian in the differential equation, we come up with the first order linear differential equation satisfied by the Wronskian.

So this equation I call it W, Wronskian equation Now suppose for some value of x is equal to x naught, my Wronskian is zero because whenever we want to find out the determinant and suppose we are dealing with so many functions then the determinant whatever the determinant is coming that is a combination of the function and from there we are unable to say that whether that is going to be a zero or non zero for all value of x.

So what do we do that we just put the some value of x there. So I put the value of x naught and after putting this value of x naught, I found that my Wronskian is zero. So what will happen in that case? Can I say that this Wronskian will be zero forever? So let us this Wronskian is zero for some value of x. Then if you see this one the combination of these it will become the initial value problem and we know that the solution is Wronskian.

So that solution we have already know that how to find out. So my solution in this case will be c e raised to power. So I know that how to solve this differential equation. So this will be minus P x dx. So this equation we know we have solved this one starting from x naught to x. So in this case or sometime also we write like this x naught to x P(s) ds. Now I know that I substitute the initial condition.

So at x naught I have c and then exponential minus, so this is x naught to x naught my P(s) ds. So this integral has a limit starting from x naught to x naught. So this integral is zero. So from here I will get that c, this value is equal to c. But it is a given that the Wronskian is equal to zero. So that imply that c is zero. So if c is zero from here I can say that my Wronskian is always zero for all value of x belongs to the interval a, b.

So from here by just putting one condition, we are able to see that our Wronskian zero forever. So in that case I can say that Wronskian is zero for all value of x. So if this is happening in that case I will say that the functions are so for example, I take y 1 x and y 2 x are linearly dependent. So from here I can say that the functions y 1 and y 2 are linearly dependent.

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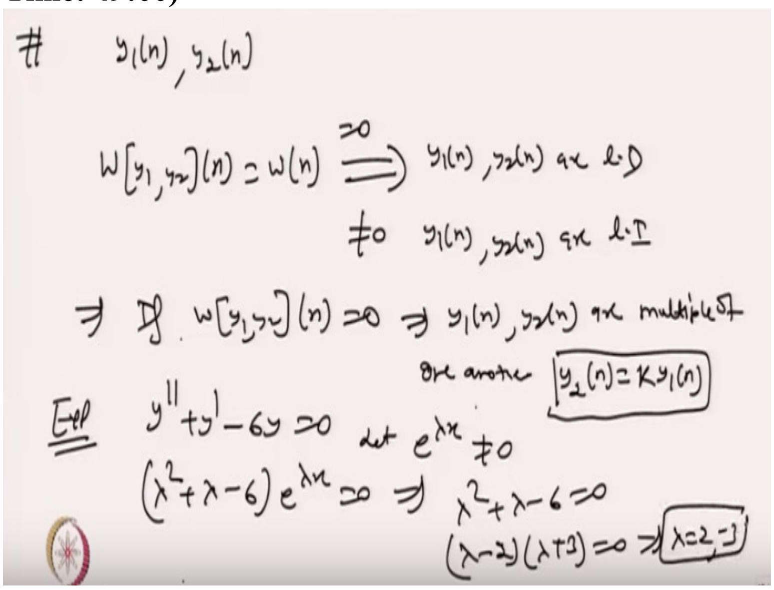

So from here we have developed a theory that if I have the solutions. So suppose I have solution two solution, y 1 x and y 2 x and from there I find out my Wronskian that is also written as this one and if this Wronskian is zero equal to zero, then my y 1 x and y 2 x are linearly dependent, okay are linearly dependent and if this is not equal to zero in that case y 1 x and y 2 x are linearly independent, okay.

One more observation I want to try that if my Wronskian y 1 and y 2 x is equal to zero in that case I will say that my function y 1 and y 2 x are multiple of one another. That means that in that case I can write one function say y 2 x can be written as some constant multiple K y 1 x. So y 2 x can be written as the constant multiple of y 1 x, okay.

So this case will happen whenever we have the roots of the characteristic equation the auxiliary equation are similar or the repeated roots. So for example, so let us take one example. I want to solve this differential equation. So y double dash + y dash – 6y = 0. So in this case I will that let e raised to power lambda x is the solution and substituting this one, it will give you the lambda square.

So I can write my its auxiliary equation very easily. So it will be lambda square + lambda – 6 e lambda x = 0. So from here, because e lambda x is never equal to zero, so this is never equal to be zero. So from here my auxiliary equation will be lambda square + lambda – 6 = 0.

So in that case if I find out the root of this equation, so I will get lambda – 2 lambda + 3 = 0 because lambda square – 2 lambda + 3 lambda and this. So from here, I can find the value of lambda is 2 and - 3. So the roots are real roots and distinct roots.

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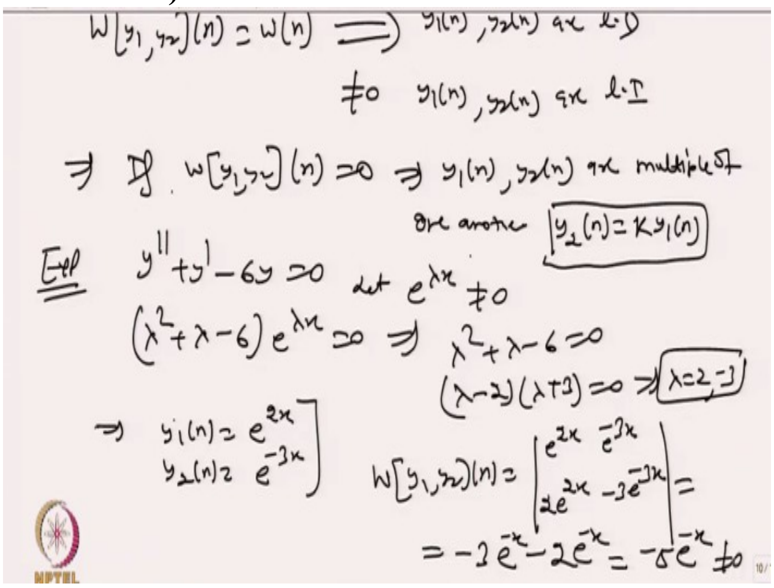

So from here I will get the solution y 1 x equals e raised to power 2 lambda and y 2 x e raised to power – 3 lambda. Now I want to check suppose I want to check whether this two solution makes the, weather these two solution are the linearly independent or not. So in that case if I want to find out Wronskian, so let us try to find out the Wronskian y 1 and y 2.

So this will be e 2 lambda e - 3 lambda then 2 e 2x lambda sorry, okay sorry. So in that case, I will write it e 2x e – 3x. So the same thing I will do here. So it will be e 2x e - 3x. So taking the derivative here and taking the derivative here and if I solve this one, so it will be if I am taking the determinant, so this will be minus of 3, e – 3x or 2x, so it will be – x minus this two times again e – x. So it will be – 5 e raised to power – x which is never equal to be zero.

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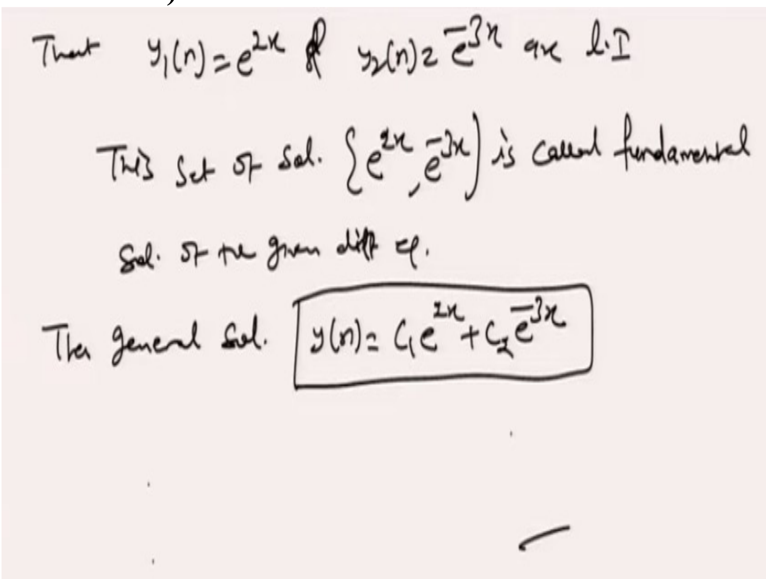

So in that case I will say that the y 1 x is equal to e 2x and y 2 x is equal to e – 3x are linearly independent. So if they are linearly independent, then this set, this set of solution, so this is a set I call it e 2x and e 3x is called fundamental solution of the given differential equation. So it is a fundamental solution of the given differential equation, fundamental solution that any other solution if available that will be the linear combination of only these two.

So in that case I can write the general solution will be y x = c 1 e 2x + c 2 e – 3x. So this is my general solution for the given differential equation and this is second order equation so it should contain two arbitrary constant, so c 1 and c 2. So basically this linearly independent solution that makes the space of functions, it is if you have some idea about the linear algebra, so they created a vector space of the function and these are the basis of that vector space.

So any solution is belongs to this category the solution will be the linear combination of these two solution. So that is my general solution. So today we will stop here and in the today lecture, we have developed little theory about that how we can solve a second order linear differential equation and then how we can define the fundamental solution so that we should be able to find the general solution of the second linear differential equation.

So in the next class we will move further and solve some different types of differential equation. Thank you very much.