Introduction to Methods of Applied Mathematics
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Module No # 06
Lecture No # 29
Construction of Scaling Functions and Wavelets Using Multiresolution Analysis

Welcome to all of you in the next lecture of this course let us summarize what we have done in the last lecture. In the last lecture we were defining we were trying to define wavelet mathematically because it is not like that if I give you a function and you satisfy those 2 property which I have stated in my last lecture that and based on that you call it as wavelet or vice versa.

You give me a some function I try to satisfy those 2 property and I call that as a wavelet it is not that way because there are some other properties also which are desirable. So based on that how we could construct wavelet mathematically that was the idea behind last lecture or not exactly last lecture part of it. And this idea was given by 2 scientists is Stephane Mallat and Yves Meyer which I have already stated in my previous lecture.

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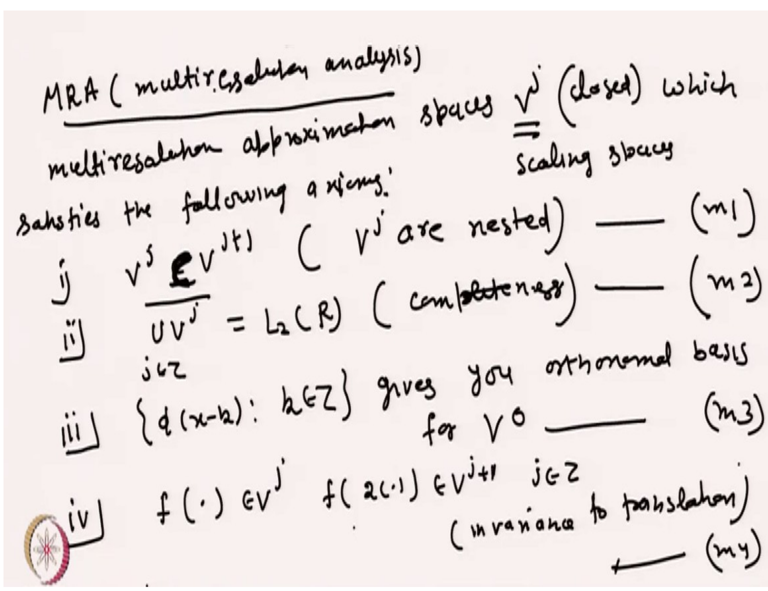

And they are the scientist who discovered the theory and that is the heart of wavelet theory and that is called multi resolution analysis or in a short form it is called MRA multi resolution analysis ok. So as I said that multi resolution analysis is explained with the help of 2 space Vj and Wj. So first we are concentrating on a space Vj so for that reason I am saying multi resolution approximation spaces Vj they are closed subspaces of L2 R which satisfy the following axioms ok.

In fact we are calling it as a multi resolution Vj these are also called scaling spaces this is also called scaling spaces ok. Why it is called scaling spaces that I will define it later. So now let us whatever we have done in the last lecture let us summarize all that into the form of axioms. So the first axioms are Vj's subspaces of Vj + 1 ok. Not this in inequality sign is not there this is just this subspace.

So it means this is called that spaces Vj's are nested ok. This is the first axiom of the MRA you can also call it as a m1 in literature many times it is called m1. Another second axiom of MRA is union of Vj where j belongs to Z = L2 R. This is called completeness property, and this is corresponds to m2 ok. The third axiom is Phi x – k, k belongs to Z gives you a orthonormal basis for V naught ok so this corresponds to m3 ok.

And the fourth and last axiom of the MRA is which I have already stated if function belongs to Vj its dilated element will belongs to space Vj + 1 for j belongs to Z this is called invariance to translation. So if you look at any book of wavelet this is the heart of the wavelet theory and any lecture or any book on wavelet is incomplete without these four axioms. So the last one is called m4 ok.

m1 and m2 these are the 4 axioms which are satisfied by the space Vj the first of it first of these are Vj are nested completeness and Phi x - k gives you an orthonormal basis. So the concept which I have given you initially translation they are here that if Phi belongs to Vj it translated elements will also be in the same space invariance to translation properties invoked here.

The next is this invariance to dilation I have already stated so I do not need to again say. So now you should have a clear idea how what are this Vj spaces will be ok. So of course, the one of the assumption which I have stated in my intersection of Vj's is a null set that you could conclude from these 4 axioms. So that is why I am not including that as a separate axiom that intersection of Vj is a null set ok. So and this Phi which we are writing here Phi x - k gives you an orthonormal basis for V naught.

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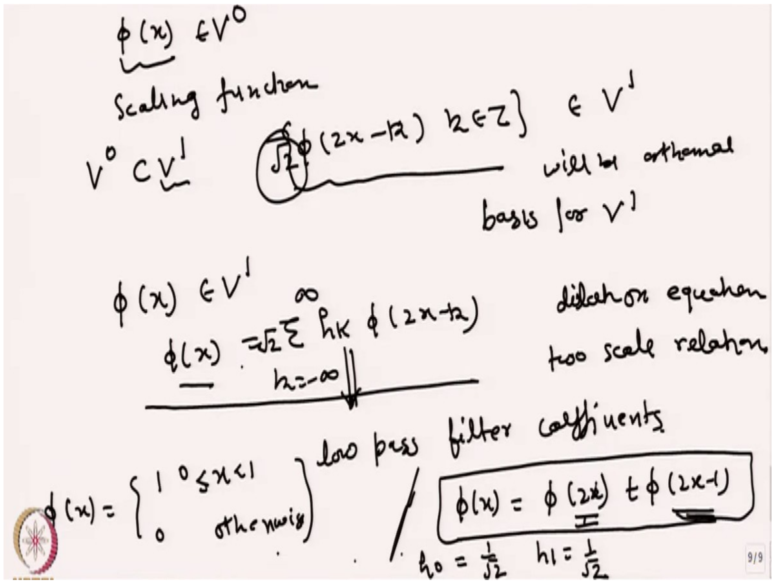

So this Phi x is called Phi x will also belong to space V 0 ok and this is called scaling function clear. So and then Phi x – k, k belongs to Z will give you an orthonormal basis for a spaces V naught. So with the help of MRA we are constructing one of the thing scaling function initially. So if now I use the nestedness property V naught is subspace of V1. So if I the Phi 2x – k, k belong to Z these elements you will have in V1.

If you look at invariance to dilation property and here I am also multiplying this with under root 2 so that I can define an orthonormal basis. So basically Phi x - k give you an orthonormal basis for V naught as we have seen in the m3 axioms and if I dilate that with 2 of course this I am multiplying so that I can give you an orthonormal basis it belongs to V1. So this will be an orthonormal basis for a space V1 will be orthonormal basis for V1 ok.

So if that is the case I can write down Phi x will also be V1 from this property. So I can write down Phi x is summation of hk Phi under root 2 Phi 2x - k ok. Where k is minus infinity to infinity ok. This is called dilation equation or what you can call it as a because here you say that we are connecting w different levels V naught to V1. This is this was belonging to V naught and this was belonging to V1 and with the nestedness property we have connected this. So this is also called 2 scale relation.

Dilation equation 2 scale relation because it connects a Phi x to different labels. So now you could yourself appreciate what are the beauty of MRA axioms ok clear to everyone. And this hk's are called low pass filter coefficient ok. Now if I am giving a one of the example that ok Phi x is given by this 1 0 if x is otherwise it is 0. So how that translation and dilations will behave with respect to this function that I am showing you with the help of figures as well.

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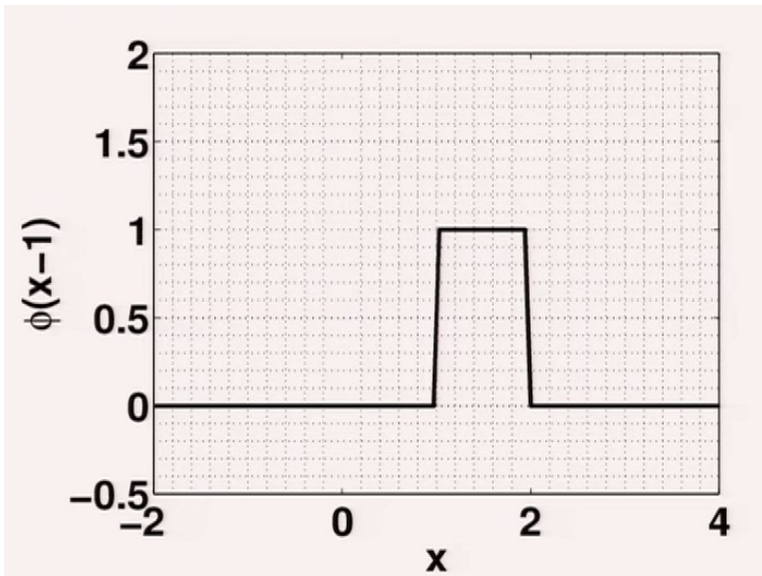

So this is Phi x ok 0 to 1 it is 1 rest it is 0 how the figure will look like of Phi this is a very simple box kind of a function. So how the translation will affect that I am showing you now. So Phi so this is Phi x - 1 it is translated by 1 that is why it is from 1 to 2 translation by 1. It has shifted from here to here that is how the translation will behave. Again now I am also showing you the effect of dilation Phi 2x.

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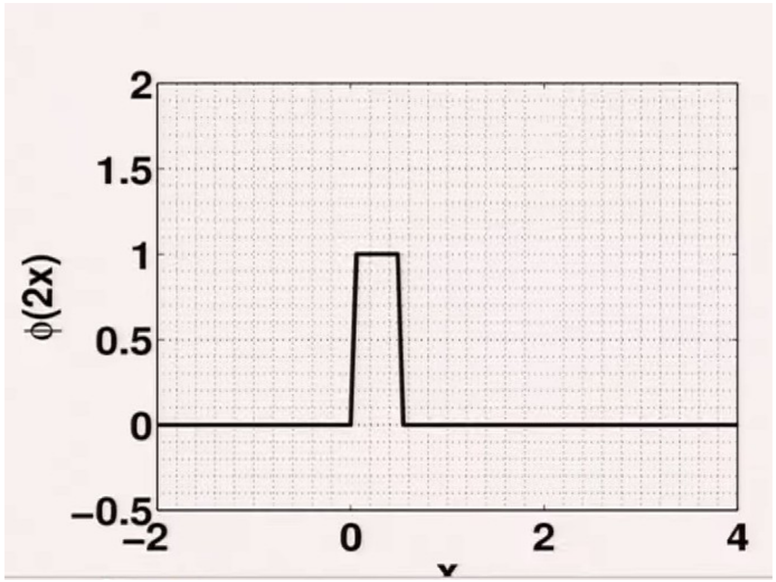

So If I am dilating the width will be half so this is 0 to half ok. So that is the function let me again reopen Phi 2x ok. Next thing how when I will translate this Phi 2x how it will look like Phi that will be Phi 2x - 1. So that also let me show you graphical Phi 2x - 1. So with the help of a one example I have shown you what will be the effect of dilation and translation. Now you have to observe this figures more carefully to conclude how I can write down Phi x in the form of Phi 2x and Phi 2x- 1 because if you look at Phi 2x - 1 was shifted from half to 1 this was 0 to half and Phi x was 0 to 1.

So if you look at all these four figures carefully which I have shown you here I can write down this relation Phi x = Phi 2x + Phi 2x - 1 ok. I can observe this graphically as well if I wanted to drive this mathematically, I can conclude from here ok. So what will be the conclusion so of course then h naught will be I can h naught will be because for k is equal to this corresponds to k = 0 this corresponds to k = 1. So k is equal to if k = 0 this will be h naught under root 2 is there. So I am taking 1 / under root 2 and h1 will also be 1 / under root 2 ok but ok.

So that is how we can construct, and this Phi x is called is one of the major disadvantage of this wavelet is I am going to say one the major disadvantages that this Phi every time you do not have an analytical form. Analytical form means closed form means I cannot tell you that Phi x will be like a Sin x function or the expression of Phi x will be this no I cannot say you. So that is one of the drawbacks of the wavelet you could call ok.

So that is why here how to construct a Phi itself is a non-trivial task and then how we will construct this Phi if we do not have a analytical expression that I will come to the later point because this was a very simple example I have written but how to construct this that is altogether different game ok. So these of course so in this case it is it will be called compactly supported scaling function because only for finitely value of k hk's are nonzero otherwise if they are 0 ok.

So later on I will based on this compactly supported scaling function I will also define compactly supported wavelet ok so that is the idea. Now in the same way if I wanted to generalize this scaling relations, I can connect any 2 scaling function at the different levels.

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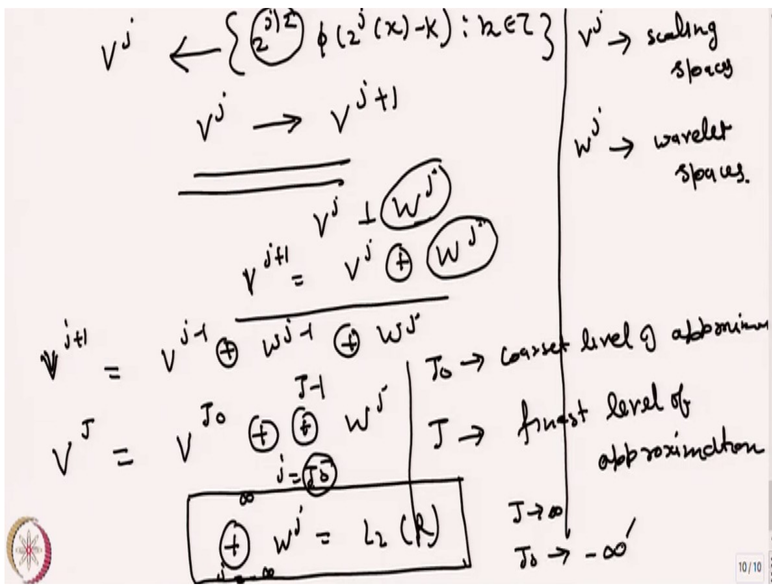

So now what will be the basis of space Vj the basis of this space will be again k belongs to Z. This will be an orthonormal basis for a space Vj. I have a started with Phi I am keep dilating Phi 2 Phi 2x will belong to V a V1. So Phi 2 to the power of jx will belongs to space Vj and I am multiplying this with this factor which I have already explained because so that it gives you an orthonormal basis.

So based on this invariance to dilation nestedness property I can conclude that this property will be an orthonormal basis for a space Vj ok. So now the next issue comes what should I do to move from Vj to Vj + 1 what should I do? Ok so of course I will be adding some information in a space Vj to achieve this space Vj + 1 ok. So for that and for that reason I am defining another space Wj in the following manner ok such that this directs some other things that I have already explained so I do not need to re explain again.

So this is the first time I am bringing this space Wj ok. As I said Vj are called scaling spaces and Wj are called wavelet spaces. So if you look at more carefully you can call it that multi resolution is analysis explained with the help of 2 spaces Vj and Wj. Vj's are scaling spaces Wj's are called wavelet spaces ok. So now that is how we are defining Wj so if you can recursively use this Vj + 1 again Vj - 1 + Wj - 1 Wj ok.

So you can keep iterating this procedure so finally if I wanted to say this, this will be I can say VJ0 + j = J0 to J - 1 Wj ok clear. So this is Vj + 1 so here I have again used recursive Vj - 1 Wj - 1 + Wj so this is j to J0 - 1. So J0 is called coarsest level of approximation and J is called finest level of approximation. So that is J0 is coarsest level of approximation and J is finest level of approximation ok. So this is the idea we are using.

So that is how if there is any function that is how we can divide that information at different levels. So each Vj has some information about the function each Vj at different scales and if J tends to infinitive basically you could call it as a ok. You can so this the main as J also tends to infinitive and J0 tends to minus infinitive.

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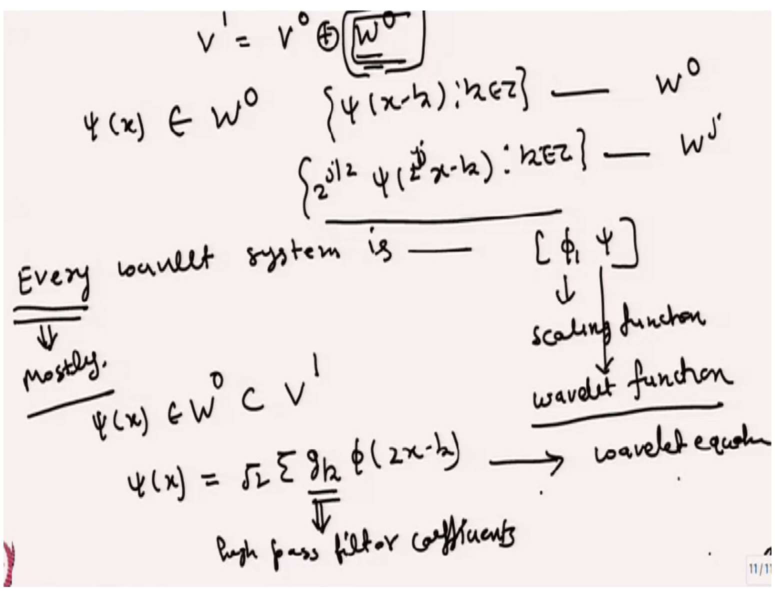

So now I will tell you basically V1 is V0 + W0 ok. So how this in when we are moving from a space V naught to V1 this will play a role. So that is the place where this wavelet spaces are coming into the picture. So for that I am going to explain with the help of figure ok. So that is you could see if - Phi x belongs to V naught and Psi x belongs to so how you could basically, I wanted to connect 3 different spaces V naught, V1 and W0 ok.

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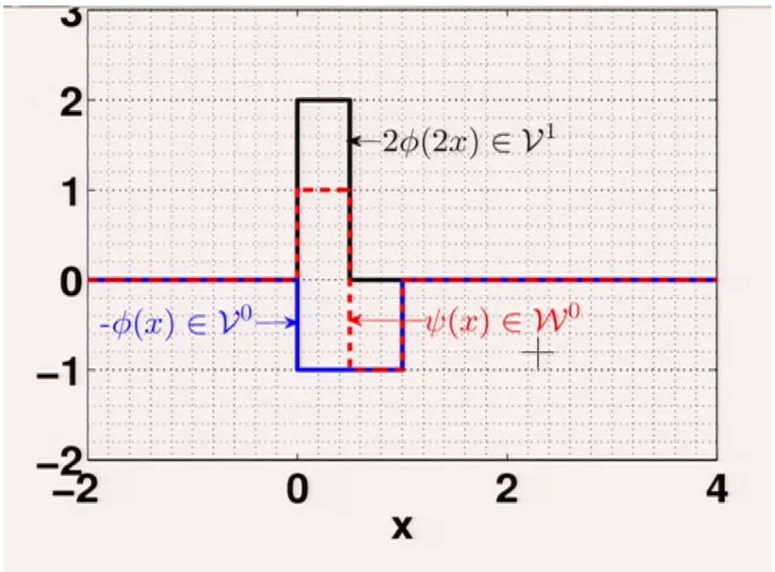

So basically, you could see I can write Psi x - Phi x = 2 Phi 2x belongs to V1. So with this relation Psi x - Phi x = 2 Phi 2x I am going I am moving into the space V1. So in a nutshell or in a short form how we define we define Psi x will be a function which will belong to space W naught such that Psi x - k, k belongs to Z will also be orthonormal basis for a space W naught and again if I use this dilation property 2 to the power j / 2 Psi 2j x – k, k belongs to Z will be an orthonormal basis for a space Wj ok.

So this will be an orthonormal basis for a space Wj. So if you every wavelet system is characterized by 2 functions Phi and Psi. The way I was saying that multi resolution in analysis is explained with the help of 2 spaces Vj and Wj same way every wavelet system is characterized by 2 things. I should not call it every wavelet system I should call it this every word should be replaced by mostly ok or every in that sense whatever wavelet we are constructing with the help of a MRA in that sense this every word will be there.

So every wavelet system is characterized by Phi and Psi if it is constructed using MRA ok. So now it should be clear to you. And this Phi is called a scaling function and Psi is called wavelet function ok. So basically, we are calling when we are defining wavelet using a MRA first we construct Phi then we construct Psi. And as I said Psi x belongs to W naught which is again a subset of V1. So this is with the this property if you look at so I can write Psi x as a linear combination of the basis element of V1.

What are the basis element of V1? We have already seen under root 2 Phi 2x – k, k belongs to Z will be an orthonormal basis for a space V1. So that is why with that concept I am writing this as a this ok. So where gk's are called high pass filter coefficient**.** And this is called wavelet equation or 2 scale relation for wavelet like previously we have seen if you look at more carefully.

This we have seen which was called dilation equation 2 scale relation for scaling function because Phi is scaling function. When we are writing Psi in terms of Phi then we are calling 2 scale relation for wavelet hk's are low pass gk's are high pass. So there is a very much similarity between the dilation equation and wavelet equations and corresponds to this we define this low pass and high pass filter coefficients ok. hk's and gk's are low pass and high pass filter coefficients.

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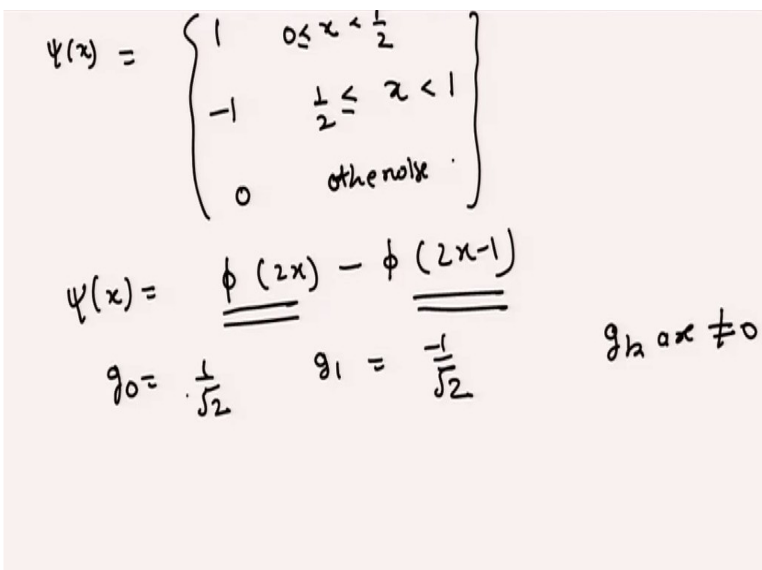

So I have already given you one example of Phi x box function. So with the help of that function can you construct wavelet Psi x? Yes so I am giving you one example of Psi x which will be 1 if

-1 if it is half 0 otherwise ok. So if this is the Psi x because if you look at this the kind of example I have given you initially also because it satisfy both the basic property it is a localized function as well as integral of Psi x dx will be 0.

But now the question comes up how I can write this Psi x in the form of Phi x ok. Which I have already stated as a box function. So you for that reason you could also observe a figure or you could write down Phi Psi x will be Phi 2x - Phi 2x - 1 ok. For that reason you could observe the figures which I have shown you for Phi 2x and Phi 2x - 1 then it will you will get this. So basically if you look at now this wavelet equation carefully g naught will be this and g1 will be this ok.

So again I am saying in this case also gk's are non-zero only for finitely many case means only for k = 0 and 1 that is why it is called compactly supported wavelet. It is not that every wavelet is compactly supported. I have never made any statement that wavelet will be a compactly supported functions ok. Wavelet can be compactly supported and if it is a compactly supported function it is a plus point. So to observe this dilation and translation with respect to this Psi function I am showing you the figures.

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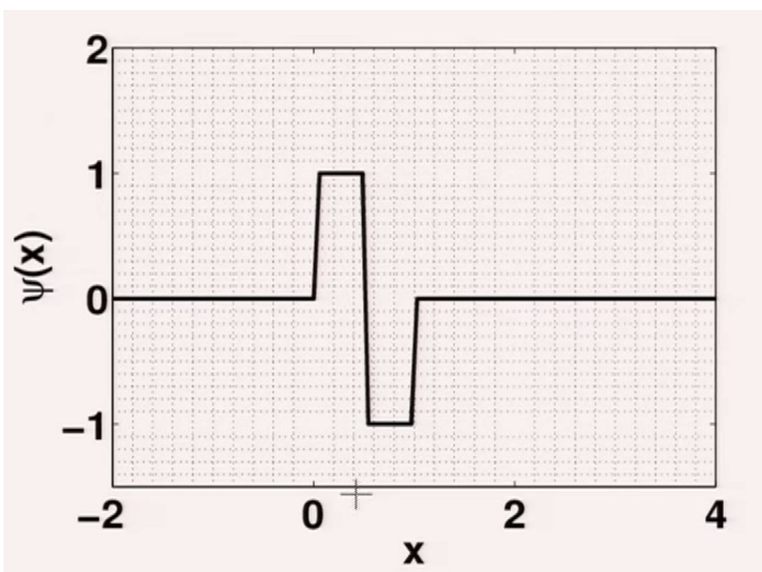

So you look at Psi x ok everyone can observe how the Psi x will look like so this is the 1 between 0 to half, half to 1 this is - 1 ok. This is the graphical representation of Psi x function which I have shown you. Now if this is the Psi x with the respect to translation how it will look like ok.

So this will be the Psi x - 1. So it is translated by 1 earlier it was supported in 0 to 1 now it is supported in 1 and 2 ok.

So this is the translation effect dilation effect you will see now. So this is dilation will support is this is squeezed. So this is you could see this from here earlier it was supported in 0 to 1 now it is supported in 0 to half ok this is Psi 2x. And again if I translate this side 2x effect will be this. So again it is shifted ok. So that is how one could observe the dilation translation effect of Phi and Psi with the help of different figures ok.

So now by this time you must have knowledge of how to construct Phi and Psi with basic properties ok with basic properties. Now I am going to define another important property of the wavelet which every wavelet should have.

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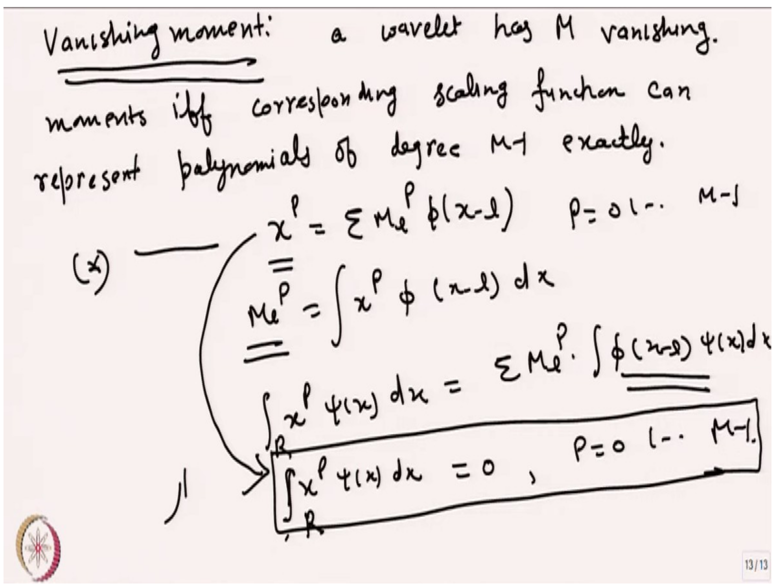

And why it is important that I will define anyway and that properties is called vanishing moment ok. So the basic idea of this property is as follows a wavelet has M vanishing moments if corresponding scaling function can represent polynomials of degree M - 1 exactly ok. So what is the mathematical way of writing this? A wavelet has M vanishing moment if and only corresponding scaling function can represent polynomial of degree M-.

So P where P is polynomial degree M - 1 up to M - 1 exactly ok well up to M - 1 exactly. So this I am writing with this ok. So this is just a mathematical representation of the same thing which I have said. So here to write down the polynomials of degree M - 1 I do not need any wavelet term. And that is why that vanishing means is means wavelet coefficient are vanished. That is the meaning of vanishing term and Ml P is basically x P Phi x - l dx and this is called moments of the scaling function which will be subsequently used for some calculations ok.

But most of the time this property which we call it has a M wavelet has M vanishing moment property is given to you in an alternate fashion and what is that alternate fashion and driving that fashion. So if you integrate this expression star with Psi x ok Ml P integral of Phi x - l Psi x dx. So because Phi and Psi are orthogonal so this will be 0. So basically this is equal to 0 for P is 0 1 to M - 1 property.

So this is the property most of the time we call it as a M vanishing moment property which is basically constructed from here. That when scaling functions can represent exactly polynomial of degree up to M - 1 ok or alternate way of doing the same thing is oh the of course this integral is over R- infinity to infinitive. So if my question to you is whatever Psi x I have given you the example like this was 1 in 0 to half and -1 in half to 1.

So what will be the vanishing moment for that Psi x is because in that Psi x we are we can say integral of Psi x dx is 0 ok. So that will have a vanishing moment M because it has a M vanishing moment if it is satisfying up to M - 1. So if M is 1 P is 0 and integral of Psi x is 0 which we can observe ok. So the wavelet which I have shown you earlier has only 1 vanishing moment.

Now ok this is one of the property I have stated. But why I am stating that it has many advantages and because in I should not say it has many advantages because of this wavelet have many advantages I should say in that manner. Because how it helps the wavelet it this is the property which helps the wavelet in any domain either you could call it in the form of the compression for engineers or producing the adaptive grid if someone wants to take the application of wavelet to a differential equations or in any other way ok.

Compression signal compression image compressions helping detecting in the singularity in any way. This is the property because of that wavelet has plus point and wavelets are so powerful you could say ok. So it this is very important property of the wavelet and it is called vanishing moment. Later on I will explain also how it helps in detecting the singularity of a function what one could observe that because the very much that this property is not posed by Fourier basis function ok.

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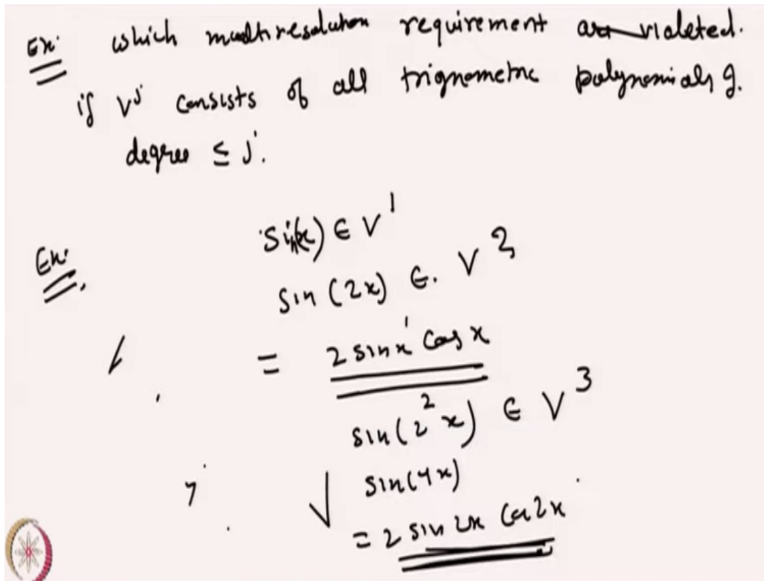

Now let me give a one because so far you have seen ok I have given you in any case I am giving you a one example because we have seen many property of the multi resolution approximation spaces. So I am giving a one example where you could see the which property is violated so that it does not form a multi resolution approximation spaces. Which multi resolution requirements are violated if Vj consist of all trigonometric polynomials of degree less than j.

Because what we have seen so far, we have seen MRA and then I have also stated one of the very important property of wavelet which is called vanishing moment. And now I am giving you one of the example where which multi resolution requirements are violated if Vj consists of all trigonometric polynomial of degree j. So can you think how to tackle this examples or which M because which of the multi resolution requirement is M1, M2, M3, M4 which one is not satisfied.

Because think for a while otherwise let say V0 Sin x trigonometric, Sin x is a trigonometric or polynomial so Sin x will be P1 trigonometric polynomial of degree j ok. So if I dilated with 2 Sin 2x it should be in V2 ok. So if you look at this expression on Sin 2x this is 2 Sin x into Cosine x so it has a degree 2 that is ok up to V2 it was fine. But Sin 2 to the power 2x because that is what I am using the dilation property it should go into the space V3 ok.

But this is what is this? this is Sin 4x Sin 4x = 2 Sin 2x into Cos 2x. But here it will not be contained into V3 because it is a just a this is Sin x ok ok. So you but what is the degree of this polynomial? Degree of this polynomial is 4 so according to the MRA it should be V3 but it is not in V3 ok. So this is mean dilation properties violated means when I am dilating it is not going to the next space. So that is why this will not be V3.

So this is the property which is violated. So I have also given you one of the one example where you could see that if Vj's are spaces which contains trigonometric polynomial of degree less than equal to j it will not satisfy M3 axioms of the MRA that is why it does not form a MRA. So it is not that you can say everything is a scaling function wavelet function of Vj and Wj.

Vj are the approximation spaces this satisfy certain axioms only then corresponds to V naught if there is a one element which generates the whole space V naught by translation then Phi is called a scaling functions ok. So now the next questions comes this is example we have already done.

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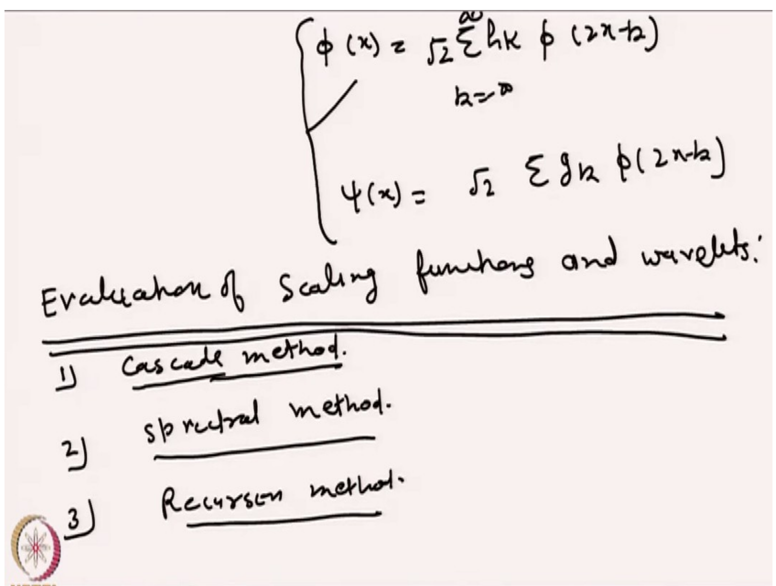

So now the next questions come if Phi x is given in the following form ok. This is what we have concluded from MRA 2 scale relation for Phi 2 scale relation for Psi dilation equation wavelet equations ok. How to construct this Phi and Psi if they are not given analytically ok. If they are not given analytically how to construct that is a my next question. And that is a non-trivial to answer and because of that there is a mathematical construction behind it that the construction of this boiled down to the problem of a linear algebra.

Because Phi is connected at 2 different space so I can initially I will compute Phi and then recursively I will compute so that is how it will be a problem of linear algebra ok. So now basically the next topic will be evaluation of scaling functions and wavelets. Why I am saying evolution of scaling functions and wavelets because they are not in the form a closed form expressions. We have to compute them so how to compute them itself is a non-trivial task ok.

And because of that wavelets are not like many engineers are not very comfortable to use it. But nowadays there are many inbuilt Matlab routines are also there how to compute Phi x and Psi x. There are 3 popular method for computing Phi x and Psi x and 3 popular methods are like that. Cascade method, spectral method and recursion method ok.

So I will explain each one of them in briefly in the next lecture but before that let me give you a idea cascade as all of you know what is the idea behind cascade that iteratively we are using the same thing. So that is will be the idea behind using cascade method. Spectral method will be based on the Fourier transform and recursion method will be like it will be boiled down to the linear algebra how we are recursively using it.

There is also inbuilt Matlab routine for computing the value of Phi x and Psi x and that inbuilt Matlab function uses which method that I will also tell in my next lecture. So before that you could think yourself little bit how if Psi x is unknown Phi x is unknown and that is available in the left hand and right hand side both. How we can compute them ok. The methods which are used in the literature for computing them are these 3 these methods I will explain in my next lecture so thank you very much.