Introduction to Methods of Applied Mathematics
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Module No # 06
Lecture No # 28
Window Fourier Transform and Multiresolution Analysis

Welcome to all of you in the next class of this course. So first let us summarize what we have done in the last lecture. In the last lecture we have seen couple of disadvantage of the Fourier transform and so remove some of the disadvantage of Fourier transform we defined Gabor transform.

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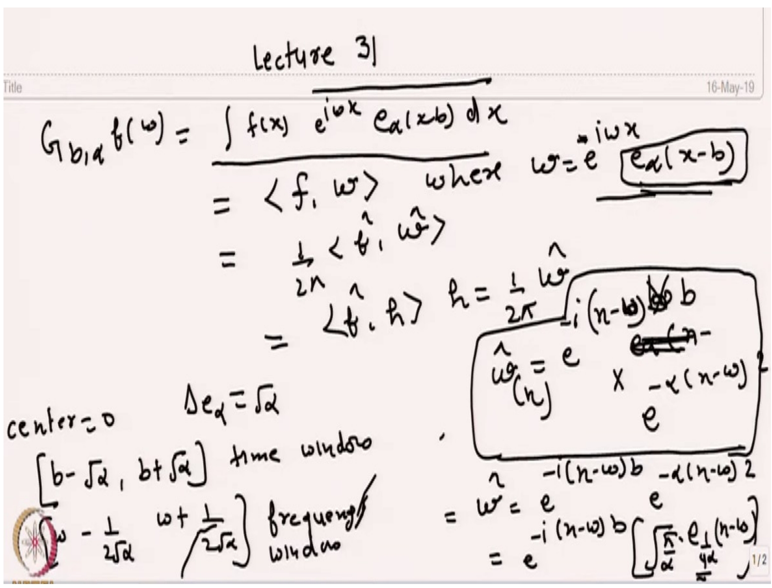

The definition of the Gabor transform which we looked at was this ok. This is the way we defined Gabor transform in the last lecture. So if we wanted to write this in the form of a inner product we can write f into w ok. So now where w will be function e to power i Omega x e Alpha x - b. This is a real function so we do not need to bother about the complex conjugates here. Ok so basically, we have seen that we are localizing Fourier transform.

If you look at this definition more carefully which we have written f inner product with function w. And w is a window function so it is localizing function f that is why when we want to analyze the function in the time domain this will be called time window. Now if I use Parseval relation which we have seen in the last lecture also then this will become this. So you can define this s AS= f hat into h where h will be 1 over 2 Pi w hat.

Now what is w hat? w hat you can look at from here that will be this function or let say what we are defining here e to the power minus Alpha into Omega square. This is the inner product of this so that is we are defining w hat ok. So what we see that Fourier transformer of Gaussian is Gaussian itself is different with what I have mentioned in the my last lecture itself. So we can also localize function in the frequency domain because h is also window function that is also in the form of Gaussian ok.

We can look at the signal in time window as well as in the frequency domain. And we need only localized information because we are localizing the signal with the help of this window function w and h that was the idea behind Gabor transform. So in this course you must have heard another kind of a transform as well like Laplace transform, Fourier transform in fact I have also discussed Gabor transform etc.

So Gabor transform is also one kind of a integral transform which is used in time frequency and analyses very often. To remove some of the disadvantages of Fourier transform. So this is the point where I left in the last lecture and I said that now I will define window Fourier transform. So in window Fourier transform the idea is instead of using this as Gaussian we are using some general window functions. So we are denoting with this w ok.

So the idea behind window transform is that is instead of using Gaussian here we are using any other general window function. So with that idea let me define what is window Fourier transform in the next page let see.

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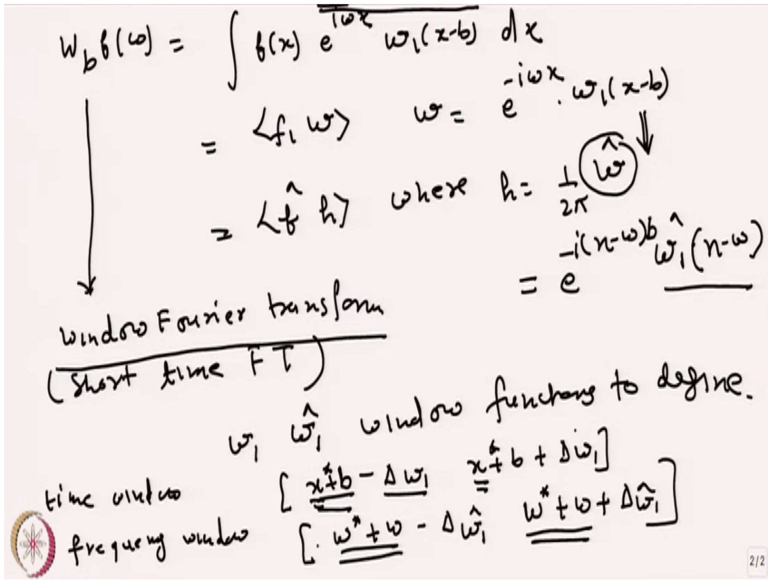

So again, f x e to the power of i Omega x w 1 x- b into dx so again, we can write down this as a this way. Where w is e to the power i Omega w 1 x - b ok so and similarly we can write down f hat into h where h is 1 over 2 Pi w hat ok. So what will be the w hat here that you could calculate from here ok. So again, this will be e to the power i Eta minus Omega, and this will become w1 hat Eta minus omega.

So that will be the idea behind this w1 hat if you look at the previous page here also it will be Eta minus Omega b ok just Eta minus Omega b. So this is the idea behind window Fourier transform ok it is also called short time Fourier transform ok. So here if you have analyze the concept what is the assumption I should make on w1 and w hat w1 hat. The assumption again I am using on w and w hat or w1 and w1 hat is that it should be a window function both should be a window function.

So in a nutshell w1 and w1 these are these should be window functions to define window Fourier transform ok. Because only then we can localize the function in time domain as well as in frequency domain. Which was the we were not able to localize this function in a Fourier transform and with the help of a Gabor transform we are able to localize that is the idea behind. So if you look at more carefully the Gabor transform is a special case of a window Fourier transform where w1 is the is equal to the Gaussian function itself ok.

So now what will be the time window if you look at in the previous page also what should be my time window? If you look at this the center of the Gaussian is 0 we all know. So what will be the center of this e alpha x - b it will be shifted Gaussian. So just for clarity I am again writing this w hat which can also be written in the following way ok. This is e1 / 4 Alpha that is what we are writing because it will help us into finding the area of the time frequency window.

That is why I have rewritten this in the following form. So what I was talking that ok the center of the Gaussian is 0 and radius let say we are denoting by this Alpha. Though this will be assignment question, that this will be equal to under root Alpha for Alpha greater that 0 but that we will anyway prove later. So if center is origin and radius is this what will be the time window? Time window will be b- this into b + under root Alpha ok and that was the reason I have written this w hat in the following sense so that you can directly calculate what will be the frequency window.

So this w hat of course this is the function of Eta that you could write down here. So this will be Omega -1 / 2 under root Alpha Omega + 1 / 2 under root Alpha ok. So this will be the frequency window and this will be the time window clear. How we are calculating the time window and frequency window? Ok and what will be the width of time window 2 under root Alpha? What will be the width of frequency window that you could also calculate?

So if you look at the product of time window and frequency window that is 2 that you can calculate. So the area of this time frequency window is 2. Now similarly we are defining what will be the time window with the respective window Fourier transform. Time window will be x star + b- Del Omega 1 x star + b + Del Omega 1. Because then x star is the center of w1. In that Gaussian case it was 0 that is why we were not writing this ok.

But in this case this is a general window function so let us define x star is the center of w1 then this is w1 x - b so center will become this and we are assuming that Del Omega1 is the radius of this function w1. Based on that concept we are defining time window. Now let see what will be the frequency window? Frequency window will become Omega star because that you could see from here + Omega - Del Omega 1 hat into Omega star + Omega + Del w hat ok. So here you could see what is w star? w star is the center of w1 hat so shifted this is the shifted so then this will become the center this way we are defining the frequency window. So, now if you look at what will be the area of time frequency window?

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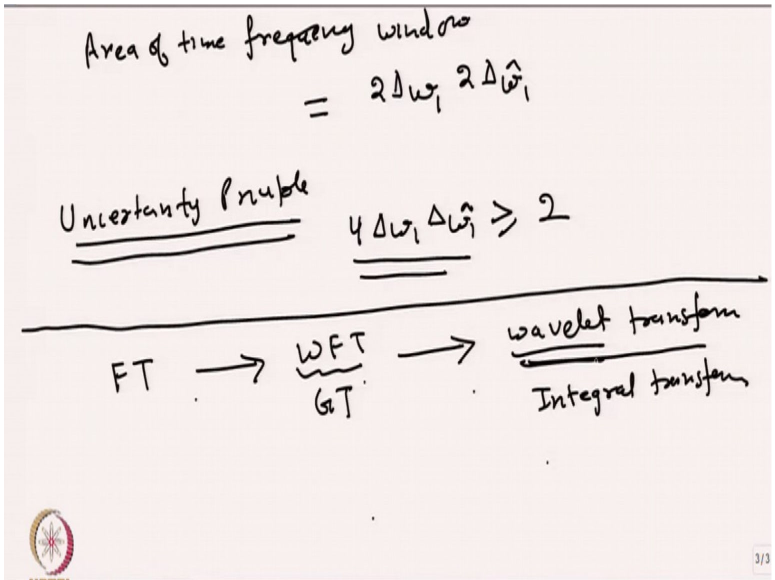

Area of time frequency window will become area of 2 Del Omega 1 2 because this will be the width of time window 2 this will be the width of frequency window. So this will be the area of a time frequency window. While in the case of a time frequency window which corresponds to Gabor transform the area was 2 we have just calculated. So there is one more thing which this principle tells us as that is called uncertainty principle.

So this is one kind of a principles which tells us that you cannot achieve a window smaller than Gabor Window. So Gabor is the Gabor window is the smallest window which one could achieve the proof of this principal is also given in the book of Charles but I am insisting this without proof. So basically it means that area of a any general time frequency window is this.

So which is then this will be always be greater than 2 ok because so means 2 is the area of the time frequency window which is corresponds to the Gabor window and that is the smallest window one could achieve and this equalative will attained if and only if w1 will be a Gaussian function that is what we have seen also. So whatever I have done here to support that argument theorems and I have also plotted that thing let me show that figure to you.

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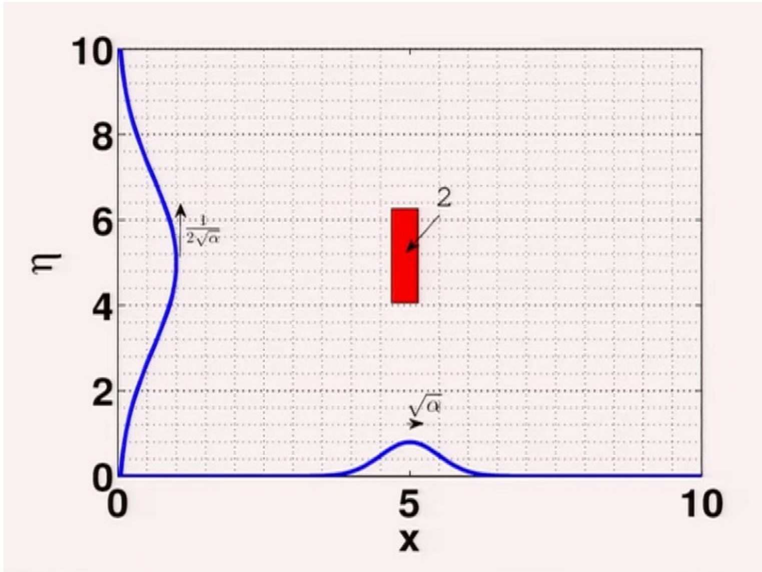

So this figure you could look at more carefully. So this is you could say time window where the under root alpha is the radius total width will be 2 under root Alpha and this is the frequency in the frequency domain 1 / 2 under root Alpha and the area of this time frequency window is 2 which is a constant. So whatever I have said graphically you could look at that thing with the help of following figure that this is corresponds to time domain this is corresponds to frequency domain.

So area is constant which we have just proved also this is just the graphical version of the same which I have said there mathematically. So now this gives you a complete picture how you could look at any function with the help of a time frequency domain. You can look at the function in time domain as well and in the frequency domain as well by using the local information of the function because this the window function which we are using are localized function or window function windows are always localized function.

So now let me give you the how this integral transform are evolved with respective analyzing the function. First, we were defining Fourier transform then we were defining window Fourier transform or particular form of window Fourier transform is Gabor Transform and then later on we will define one of the most promising wavelet transform which is also a integral transform.

When I will define the wavelet transform I will show you what is the drawback of window Fourier transform which is removed in wavelet transform. So that is also one of the integral transform. But it has advantages over window Fourier transform and that is how it is evolved. But before defining the wavelet transform I have to define the concept of a what is wavelet ok. So for that reason we are changing our topic and now we are defining what is wavelet that ok.

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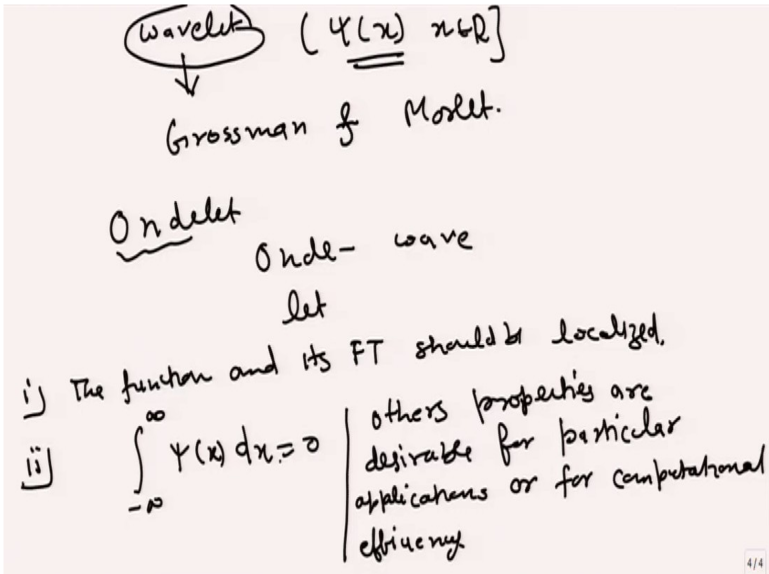

So wavelet is basically initially discovered by two scientist Grossman and Morlet ok. And both of them were physicist French physicist it is not a legacy of this mathematician. This is the concept which were discovered by a physicist and when it was invented with the name of Ondelet. Later on it was converted to the English by saying Onde means wave and let means as we all understand what is the meaning of let in English is small wave ok.

So that that is how it is invented. Later on it becomes a very popoular after doing a complete mathematical analyses by many scientist like Grossman and Morlet again Stromberg etc ok. So now to define wavelet in a very layman language what kind of functions you can call it as a wavelet. I am giving two properties which any wavelet should satisfy ok. So the wavelets most of the time it is denoted by function Psi x in a one dimensional if x belongs to R ok.

So which kind of functions you can call it as a wavelet the function and its Fourier transform should be localized ok. There is a difference between localized function and compactly supported function that you should look at. Gaussian is localized function but it is not a compactly supported function. So please do not misunderstood that localize and compactly supported functions are can be interchange changed ok.

So the function and its Fourier transform should be localized that is the first assumption if you are calling any function as a wavelet. The another assumption is Psi x dx should be 0. It means waving above and below the x axis. So these are the two property which is you could call must to define a wavelet. There are other properties also one could see in any wavelet but those property is are desirable for some particular application or for the computational efficiency.

So these two properties are basic and other properties are desirable for particular applications or for computational efficiency ok. So because later on you will see many more properties which is suited for some particular application or for computational efficiency. So now if I ask you two function which we have used in Fourier transform and Gabor transform one of the function is called Gaussian and another is e to the power i Omega x that we were doing in Fourier transform so which is basically combination of sin and Cosine.

So if I ask you to look at the Gaussian function the function and its Fourier transform is localized. And so first property is satisfied but the second is not satisfied for a Gaussian function. That is why you cannot call Gaussian as a wavelet clear. The next function you could take becaue sin function satisfy this property but it sin is not a localized function. So that is why you cannot call it sin x also as a wavelet.

So I have given two examples of a functions which are not satisfying these property and that is why we are not calling them as a wavelet. So now the next question comes can you tell me any function which will satisfy these two property?

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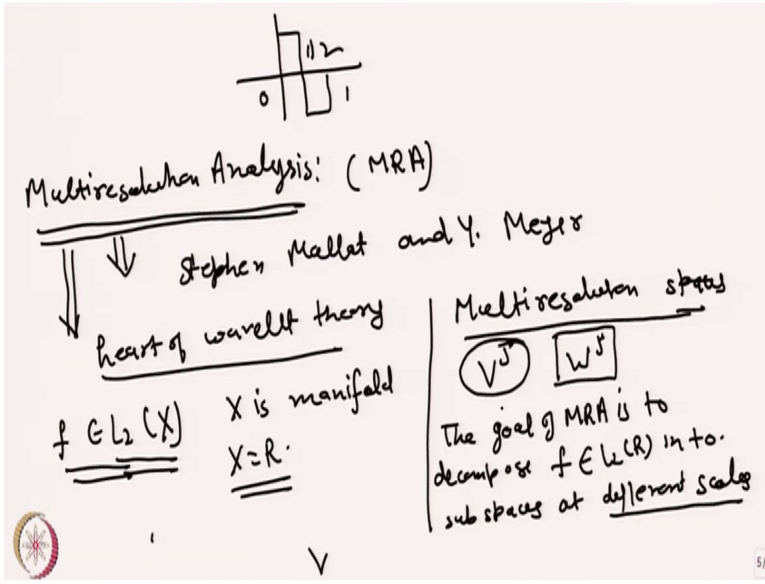

Of course if graphically I define this kind of a functions ok. So let say this is 0 this is half and this is 1. So in 0 to half it is positive and half to 1 is negative. So of course it will satisfy the second property and it is a compactly supported function. So every compactly supported function is localized. So then it is also satisfying the first property. So these kind of functions you could call it as a wavelet.

Of course this is I have given you the example but how to construct the wavelet that itself is a non trivial task. So for that reason how to construct wavelet mathematically we are going to define the main theory of the wavelet that is called multiresolution analysis ok. Multi or in the short form it is called MRA multiresolution analysis initally multiresolution analysis was invented by Stephen Mallat and Y. Meyer ok.

They are the one who discovered multiresolution analysis for the construction of the wavelet mathematically how to construct wavelet mathematically? So basically it is a this theory is also called heart of the wavelet theory. What is the goal of this MRA? The goal of MRA is to express any arbitrary function ok which belong to this space L2 X is any manifold or just for simplicity as far as this course is concerned we are taking X = R.

So basically we wanted to analize the function which is x square integral function over the real line. So we want to approximate this function with the help of multiresolution approximation spaces. So you could say multiresolution is analysis is defined with the respect to 2 spaces and they are called multiresolution approximation spaces. What is the symbol of those spaces that I am defining that later on.

So but multiresolution analysis is explained with the respective two spaces which are called multiresolution spaces ok. One of these space I am defining Vj and another space I am defining Wj. So initially we will concentrate on this space later on we will also bring this space ok. So what is the goal of MRA? The goal of MRA is to decompose the whole function ok which belongs to this space into a sub space at the different ok. The goal of MRA is to decompose the function f belongs to into sub spaces at different scales.

What is the meaning of this different scales? I am coming to that and because of that meaning different scales it is called multiresolution. So these are synonyms you can understand multiresolution or multi scales. Resolution are basically synonyms of the scale ok so sub spaces at different scales. So as I have already said that first I will focus on Vj and then I will focus on Wj ok.

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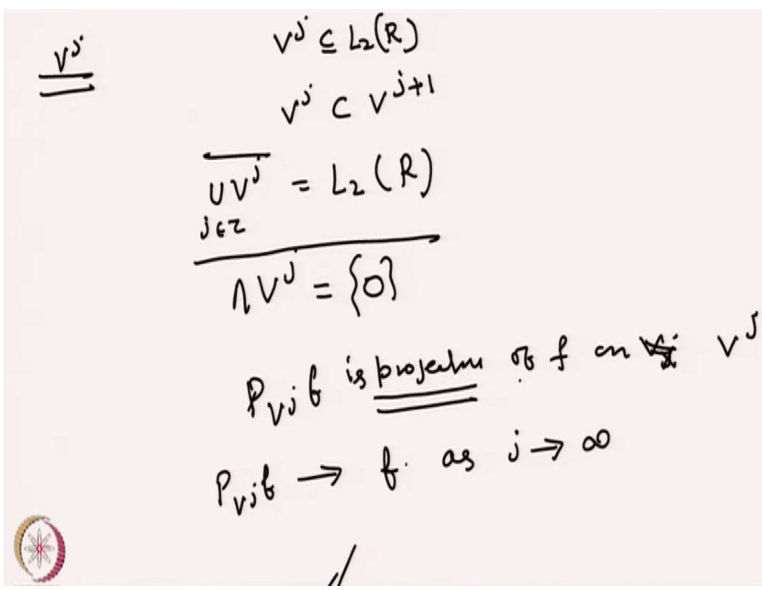

So what are Vj's? They are closed sub spaces of L 2 R. So Vj are close sub spaces of L 2 R. That these spaces are increasing it means Vj will always be contained in Vj + 1. So these sub spaces are in an increasing direction it behaves like this Vj is contained in Vj + 1. And later on this property will be called as a nestedness property of the spaces. Now if we go in the increasing direction of the j we will have the completeness exist L2 ok. So basically what is the meaning if at j tends to infinitive basically it will Vj it tends to L2 R. And if j is decreasing means j tends to minus infinitive so with that property we are calling it as a intersection of Vj is 0 null set ok.

So that is how these approximation spaces will behave in the increasing direction of j and in the decreasing direction of the j and so basically it means P Vj f is the projection of f on Vj sorry it should be Vj. So I have when I was doing the mathematical foundation I have already said what is the projection so I do not need to recall that it again. So if Pj is the projection of f on Vj so then P Vj f will tends to f when as j tends to infinite.

Ok that is what we have just said in the if j tends to inifinitive Vj will be like L2 R and because of that reason this completeness axiom is satisfied ok. So now when we are moving from space Vj to Vj + 1 How we are moving? So for that reason we are defining one another property which is called invariance to dilation.

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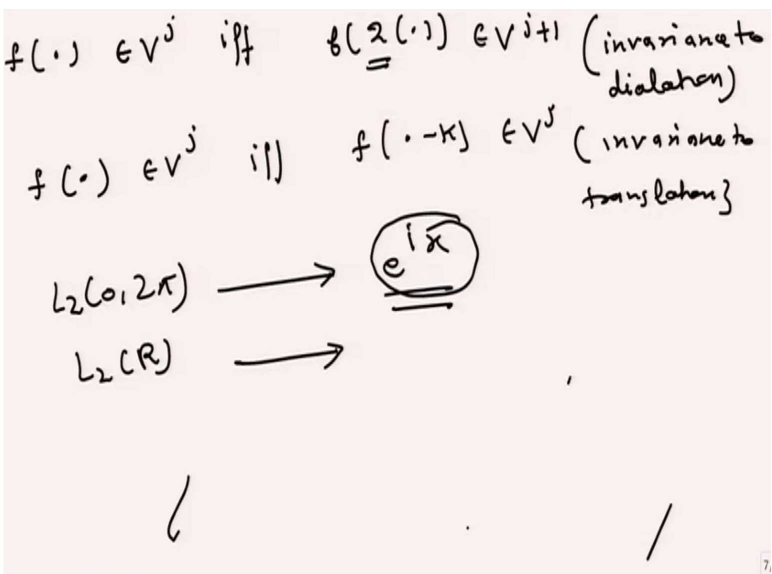

So if f belongs to Vj if and only if belongs to Vj + 1 ok. So all that dilated element of the Vj are in Vj +1 and now What is the dilation factor we are taking? The dilation factor is 2 ok. So this called dilation invariance to dilation ok. What is the other property if f dot belong to Vj if and only if f dot -k will also belongs to Vj. This property is called invariance to translation ok. Invariance to translation and this is invariance to dilation.

So one of the natural question which can come to your mind that why I am using only 2 I can use any other integer which it is like 3, 4 so What should be your answer for that? Yes you can use any other dilation parameter also but we are using 2 for computational efficiency which you will see later on what kind of computational efficiency we are talking. The next thing is invariance to translation.

So if you look at these two property my question to you is that when you were defining what were the Fourier basis function e to the power i Omega x. So at that time also if you remember I said basically whole L2 0 to 2 Pi space was generated by one function by dilation of a one function ok. If you remember I said but in case of a L2 R it is generated by which function it cannot be here we are using dilation because when we are defining wavelet we are saying it should be localized.

So if you wanted to cover the entire space with the help of a localized function. So of course translation will also be needed ok. While in this case we are not having any translation things. But to define a wavelet we need 2 concept invariance to dilation and invariance to translation. Translation property was not in Fourier basis function but dilation which you see here because we were dilating it single like Sine 1x Sine 2x Sine 3x that is the dilation of a one single function Sine x ok.

So these are some of the background i have made to define MRA more mathematically in the form of a assumption if you see any book on the wavelet MRA are always explained with the help of assumptions and what are those assumptions? Those assumptions are basically whatever I have said in this place or previous place here and there. Now I will collect all of them in a to put in a more mathematical form and they will be called MRA assumptions. Ok that I will do in my next lecture. So thank you very much.