Introduction to Methods of Applied Mathematics
Prof. Mani Mehra & Prof. Vivek Aggarwal
Department of Mathematics
Indian Institute of Technology- Delhi/ DTU

Module No # 06
Lecture No # 27
Time-Frequency Analysis and Gabor Transform

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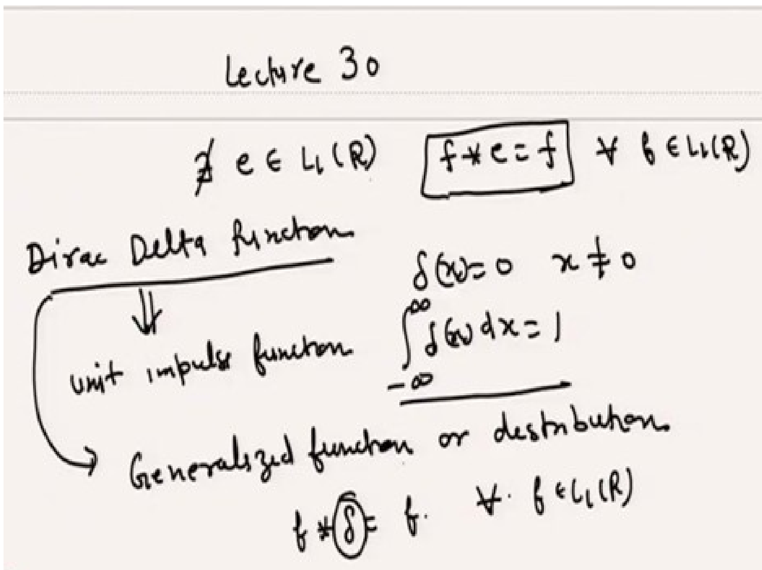

So welcome to all of you in the next lecture of this course the summary of last lecture of my course was that it means that they are we were not able to find e convolution identity which belongs to L1 R space such that this is satisfied. So there does not exist ok so we because using the convolution theorem we prove that there does not exist a convolution identity for which this is true ok.

This is a convolution operation that is why e is called convolution identity because using the convolution theorem if convolution exists there was a contradiction with some assumption. So that is why they are does not exist a convolution identity if it does not exist what is the next point to think. I will tell you that we can say that ok convolution if we are saying e belongs to L1 R so ofcourse it is a function which we have anyway said that it does not exist.

So now we are taking the concept of a function to the next level which is called generalized function or distribution. What is that concept all of you must have already studied what is Dirac delta functions let us recall what is Dirac delta function? Dirac delta is a function which is non zero only at 1 point ok. Rest where it is 0 moreover it is satisfied this property ok so integral is equal to 1 that is a definition of a Dirac delta function we see with respect to signal analysis the same function is called the unit impulse function ok.

So now and this Dirac delta function because we cannot think of any function with which is non zero only at 1 point is still it is area under this integral is 1. So we cannot think of that is why it is called generalized function or distribution. It is not called a function because if you look at the function property these kind of property will not be there.

So we have extended the concept of a function to a generalized function so with this extended concept of a function I can say that Dirac delta function will satisfy this property for off belongs to L1 R. So this is a Dirac delta functions so but this is as we are saying if this is not a function so what is out next step we will think of functions which could approximate this Dirac delta function.

We will think of those kind of a functions which will approximate this Dirac delta function because if we cannot achieve this function as such convolution identity as such because this is not a function at all. Something is better than nothing so we are thinking of approximation of this generalized function ok. So that is our next goal how we could think of approximating this convolution identity.

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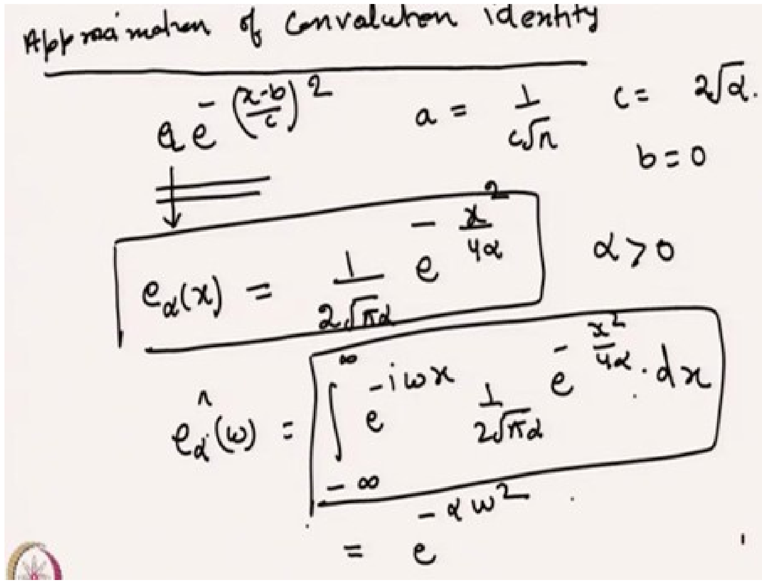

So for that reason our aim is to define approximation of convolution identity ok so for that purpose first I am defining a Gaussian function ok. What is a Gaussian function all of us are aware already what is a Gaussian function I am Gaussian function is basically I am writing in the form of let us say a e ok. This will be the form of Gaussian function which contains three parameters a,b,c what is a? a will determine the height of the peak b will be the will determine the position of the peak and c will determine the width of the peak.

These are 3 parameters to govern the Gaussian function I will show you that Gaussian functions in the figure also how it looks like but first of all just for simplicity we are taking 3 a,b,c everything we are writing in the form of some 1 particular parameter. So for that purpose I am taking a is 1 over c under root pi and c is 2 under root of alpha ok and b is 0 these are the 3 parameters I am taking a is this.

So with this what this will become I am denoting with e alpha x is 1 over 2 under root pi alpha e to the power –x b is 0 and the c square will become 4 alpha. So this will become my Gaussian function for alpha greater than 0 so now if you wanted to see this Gaussian function for different value of alpha that one could observe in the following figure.

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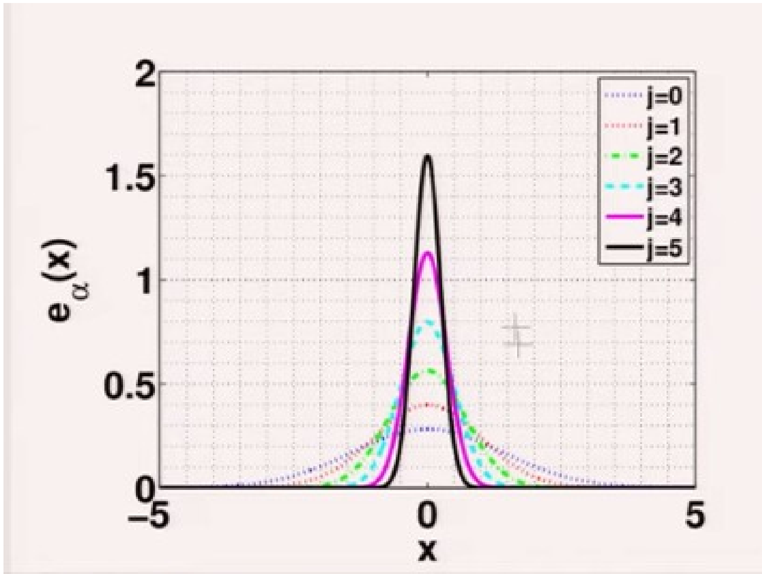

So you look at here what is the value of alpha? Alpha is 1 / 2j I am taking and then we are taking the different value of j. j is 0 to 1 to 5 alpha what is the value of alpha we are taking we are alpha = 1 / 2 j 2 to the power j. So what you could observe from the figure that as alpha is tending to 0 means alpha is 1 / 2 to the power 5. This width size is width of the peak is decreasing and you could see as I am keep tending alpha tends to 0 what will happen? It will converge it should converge to a delta function. That is my motivation behind defining this approximation of a convolution identity that also you could see from here.

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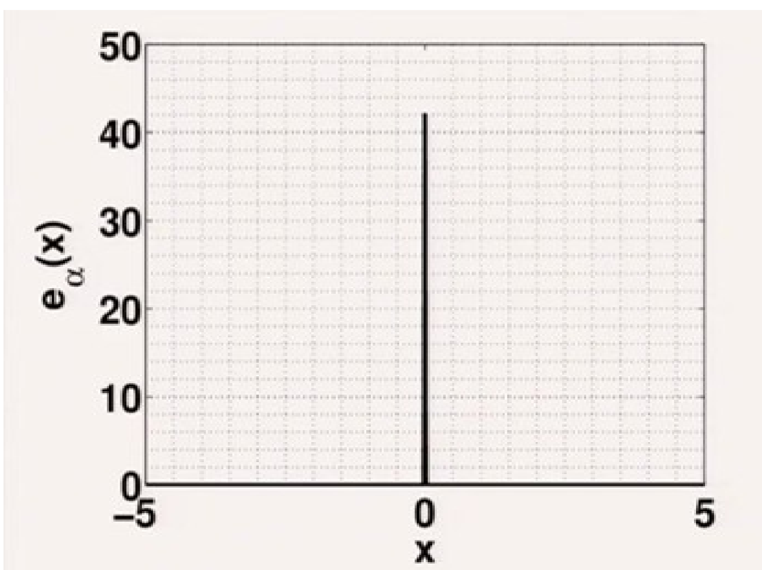

This is the figure I have plotted for alpha = 1 / 2 to the power 15 so really alpha is tending to 0 so you could look at this will look like a non zero value only at a single point and even you can work for a smaller alpha also. So what is we could observe from these two figures that as alpha tends to 0. This Gaussian function is tending to delta function ok. So now let us and a now the another thing we wanted to observe what will be the Fourier transform of this Gaussian function?

What will be the Fourier transform? e to the power –i omega x 1 over 2 pi alpha e to the power minus x square / 4 alpha ok. If you remember when we were discussing about Fourier transform we computed the Fourier transform of a Gaussian functions ok. So with that knowledge I am directly writing here which can be a question for the assignment also e to the power –alpha omega square so this will be the Fourier transform of a Gaussian function.

So that is the beauty and that is 1 of the reason we are choosing a Gaussian because the Fourier transform of Gaussian is again a Gaussian not same Gaussian but with respect to different with magnitude but the form of this is also a Gaussian. That is our that is why we are choosing Gaussian function so whatever we have summarized in this discussion I am writing in the form of 1 theorem.

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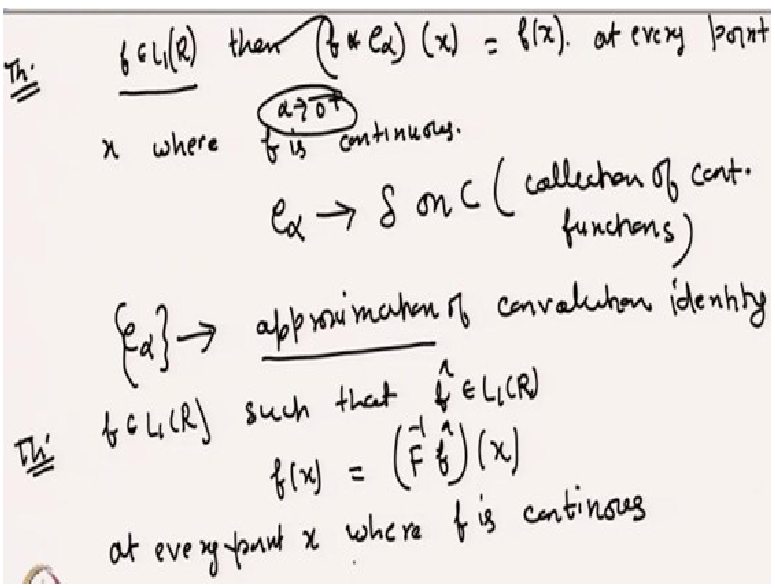

That if f belongs to L1 R then f convolution with e alpha as alpha tends to 0+ at every point x where f is continuous ok. So I have stated this theorem without proof if anyone is interested to look at the proof of this theorem it is given in the book of Charles Chui by academic press. So what is the statement of this theorem that if function is in L1 R then f convolution with e alpha which is a Gaussian functions will be equal to fx at every point x wherever f is continuous.

The main point which you should note as alpha tends to 0+ so basically the conclusion of this theorem you can also write that e alpha tends to delta on c on collections of continuous functions ok. So this e alpha is called because from here we have proved that e alpha will tends to delta on a collection of a continuous function as alpha tends to 0+.

So that is why e alpha we are calling as a approximation of convolution identity ok. So that is what we have to we could not think of a function which can act as a convolution identity but that is why this word is there. Because as alpha tends to 0 because Gaussian is not defined for alpha is equal to 0 so this a approximation of a convolution identity. So the next theorem we could write that if f belongs to L1 R such that its Fourier transform is also in L1 R.

Why am I stating this because it is not necessary that if function is in L1 R its Fourier transform is also in R we have already seen this when we were looking at the this Fourier transform examples we have found 1 example where in fact I have shown you that for some function belongs to L1 R Fourier transform was noted L1 R. So if so that is why we are making assumption function belongs to L1 R says that Fourier transform is also in L1 R.

Then I can define the inverse Fourier transform of this function ok at so this will true at every point x where f is continuous. Again I have stated this theorem without proof if someone is interested to know about the proof again the proof is in the book of Charles gache. So for detailed proof you can look at there so now this is the how we have defined the inverse Fourier transform. So now this was about the L1 R function so can we talk of any better space where we do not need to make assumption that if function is belonging to that space it is Fourier transform is also in that space.

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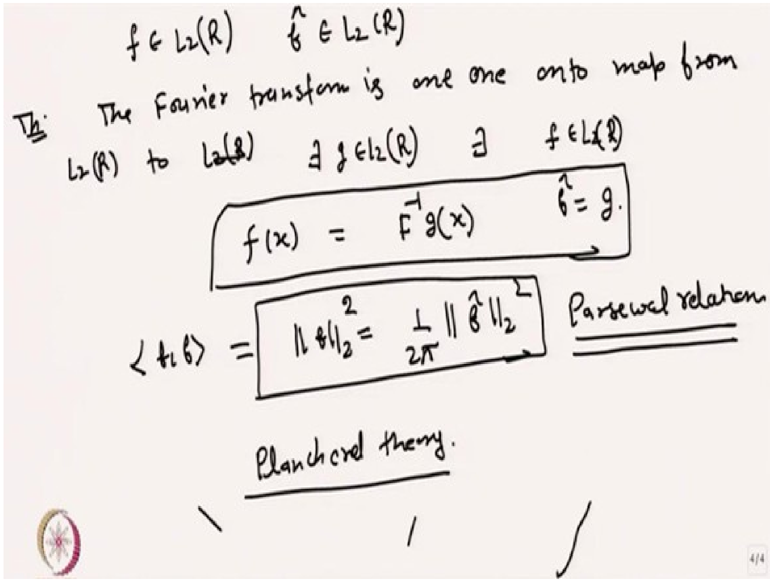

So for that reason we are taking this if function belongs to L2 R its Fourier transform will also be so now the if f is in L1 R to talk about the inverse Fourier transform I am saying f hat should also be in L1 R. So can we talk of any better space where we can say that inverse Fourier transform always exists yes the answer is yes and that better spaces L2 R space. So if f belongs to L2 R space is always guaranteed that f hat will also been L2 R one could check this.

If function is in L2 R Fourier transform is also in L2 R so it is not just true there is something more which I can write down in the form of a theorem and that theorem says the Fourier transform is one one onto map from L2 R space to L2 R space. It means for every g belongs to L2 R ok there exist one and only one f belongs to L2 R such that this condition will be satisfied ok so that is the idea that Fourier transform is a one one and onto map.

It is in fact something more can be said for this and what is more we have already studied parseval relation when we were discussing the Fourier series. Now parsewal relation is also true in case of it Fourier transform we will be using this relation later on in the form of inner product. That is why I have stated this here and this is called parseval relation ok because you can also write this as a inner product.

Because L2 R is a Hilbert space so always you can define the inner product so the theory of the Fourier transform of a function which belongs to L2 R space there is an elegant theory and that theory is called Plancherel theory. So one could study this theory in the course of a harmonic analysis that how the Fourier transform of a L2 R function behaves Fourier transform is in all 2 R L2 R as well as this parseval relation is satisfied in many more.

So that is the beyond the scope of this course that one could see in the course of a harmonic analysis but that is just because I will be using this relation later on that is why I have stated here. So now what was the sequence we followed to define the wavelet first we did some mathematical foundation then we defined the Fourier series then we define the Fourier transform and we were just recently we were talking about the Fourier transform and now I am going to the next thing which is called time frequency analysis.

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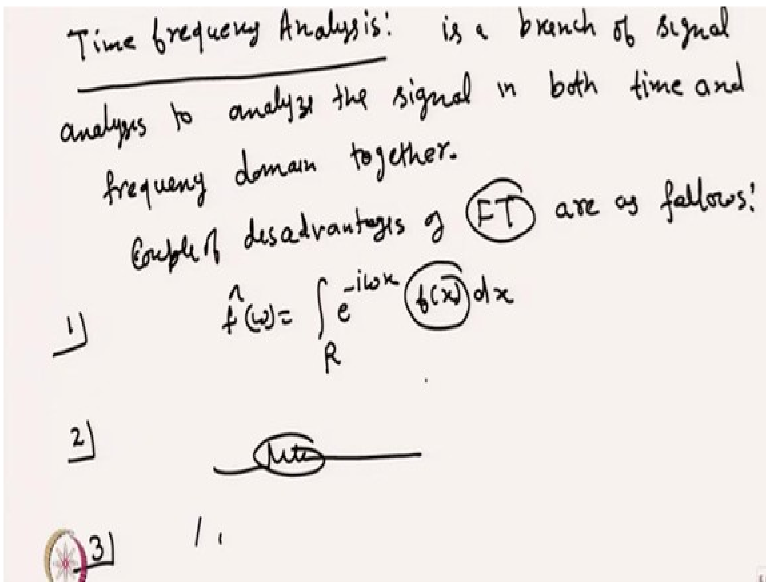

Time frequency analysis is a branch of a signal analysis to look at the signal in both and time domain and in frequency domain together. So time frequency analysis is a branch of signal analysis to analyze the signal in both time and frequency domain together. So ofcourse if from the Fourier transform also we can look at we can analyze the signal in both time and frequency domain and now we are introducing another concept time frequency analysis which is also used to analyze the signal in both time and frequency domain together.

So ofcourse then I have to state some of the disadvantage of the Fourier transform that is why we are going in this direction so with couple of disadvantage of Fourier transform I am mentioning here of Fourier transform in a short form I am writing FT are as follows what is the first disadvantage one could think of Fourier transform any idea you could think for a while?

Because if you look at the definition of Fourier transform which I have again recalled here ofcourse this is over R minus infinity to infinity. You need complete information of this function for a mathematician or if you are saying signal so you need for as a engineer you need complete information of this functions. In fact if you do not know any future behavior of the signal that also you know you should know to compute the Fourier transform.

So that is that can be considered as 1 of the drawback that you need complete information of the signal that is the first drawback. Second drawback is again second drawback I am mentioning with respect to definition itself if fx is changed in a smaller neighborhood sometime like if signal is changed I could if signal is changed only by small magnitude as well as in the small neighborhood.

But you know this Fourier transform will be changed entirely if you look at the definition so complete spectrum is changed. So basically Fourier transform is not based on the local information of the function it should have complete information which is also the second drawback of the Fourier transform. So third drawback you could think that if you look at if signal is periodic we can take Fourier series if signal is infinite signal is of infinite period then we could think of Fourier transform.

But mostly real life signals are neither periodic nor infinite you could observe mostly signals what we see in the real life. If any signal is coming from the musical instrument so that is neither they are for a very short period of time. So neither they are infinite moreover if you must have definitely you must have heard some signal of the musical instruments. So they gradually increases and then they gradually decay so they ok they are neither infinite as well as neither periodic.

So this also motivate us for a time frequency analysis because these type of signals you cannot handle with the Fourier transform. So again in a nutshell why we are going in the time frequency analysis directions because a couple of disadvantage of Fourier transform that you need complete information of the function.

Moreover if function is changed in a smaller neighborhood complete spectrum is changed which I have said as a disadvantage number 2 and the third disadvantage is that to define the Fourier transform of Fourier series I should have either function is of in periodic of finite period or infinite while real life signals are not that.

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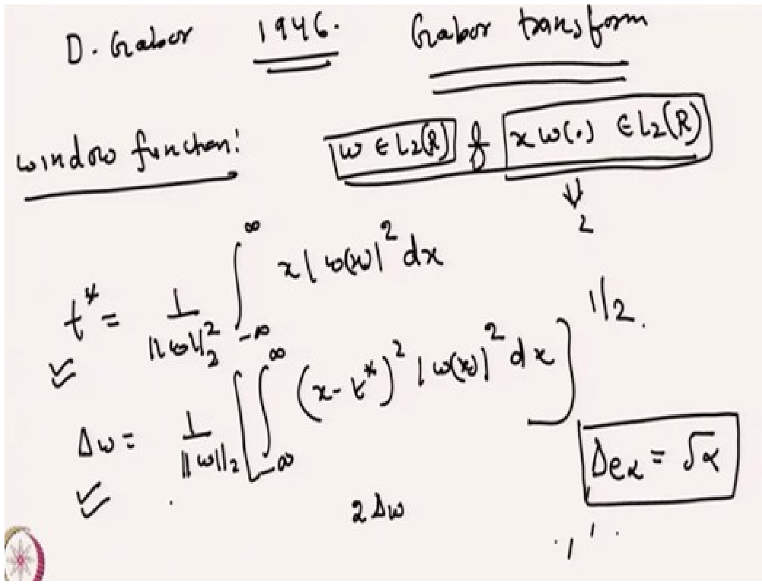

So because of that reasons there was a scientist D Gabor who discovered Gabor transform in 1946. So first he observed the drawback of Fourier transform and then he introduced a transform which is called Gabor transform ok. So before going to the definition of a Gabor transform let me introduce the concept of a window function. What is window function? If function which is in L2 R such that x wx will also been in L2 R then it is called window function.

If it satisfies both the property this one means it should be L2 R as well as this property so what is the implication of this property on a square integrable function. Implication of this property if you observe if you know how the behavior of this function if it is x wx is also in L2 R it means that it will decay faster. As if you just take only this property if you do not consider this second property you will consider only square integrable function.

Then what will be the comparison? A function which is satisfying this additional property will decay fast and that is why you know this is called a window function it has a resemblance with the window which we see our in every room. Because window are you know they are we call it as a window because it is in the smaller part of the wall otherwise we do not call it as a window.

We call it as a wall itself or door itself so that if you borrow that idea of a window with that idea it is defined as a window function so because it will decay very fast. So can you think of any example which will qualify the properties of a window functions. Of course one of the example which we have been keep doing is a Gaussian function. Gaussian functions will be a window function because you can see it belongs to L2 R as well as this property will also be satisfied.

And otherwise with the property whatever definitions I have given otherwise if Gaussian functions also decay very fast that is why Gaussian function falls in this category. So if it decays very fast so we are defining 2 notions with respect to this window function and what are those 2 notions are I am defining the center and radius of the window function. So center of the window function will be defined by this formula which I am stating now ok and radius will be defined by this form ok.

So this that is how we have defined the center and that is how we have defined the radius so what will be the width of the window function will be 2 del omega ok. So now it is clear to everyone that how we define the window function and as I said that we will be needing this concept to define the Gabor transform. So Gabor used this definition of a window functions to define the so when he defined window transform he took the particular example of a window function which is Gaussian as we are keep saying.

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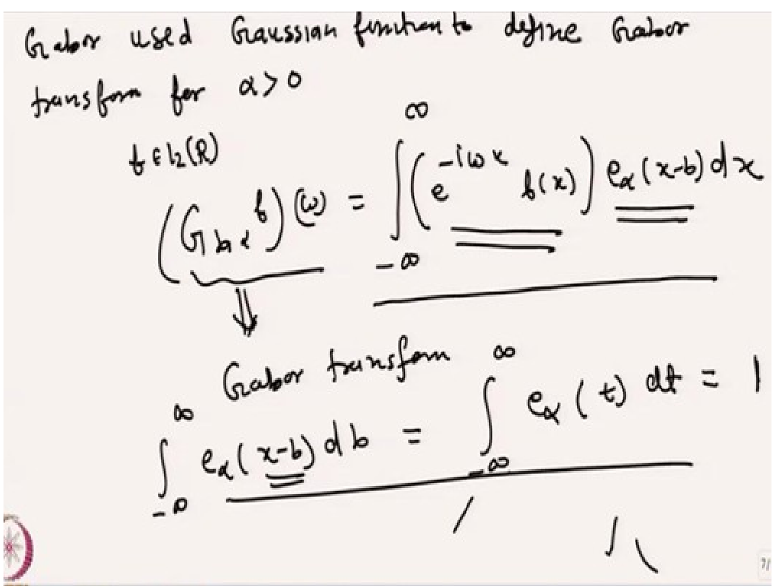

So Gabor used Gaussian function to define Gabor transform for alpha greater than 0. So if function belongs to L2 R I am giving you the definition of a Gabor transform ok so this is the definition of a Gabor transform which you could see from here. Here we have used e alpha as a Gaussian function which is a window function later on I will extend this idea of a Gabor transform to window transform.

But this time we are taking some particular example of a window function which is a Gaussian and that is why this is called Gabor transform. So Gabor transform so if you look at the right hand side of this Gabor transform carefully what do you observe from this. So you could observe that e alpha which was qualifying the properties of a window functions it is localizing the Fourier transform this way ok.

It was localizing the Fourier transform which was the drawback of a Fourier transform as I have just stated few minutes back. So it was removing the drawback of the Fourier transform so now you could because we know this is the property of a Gaussian I am just writing x-b = t ok so this will be equal to 1 ok.

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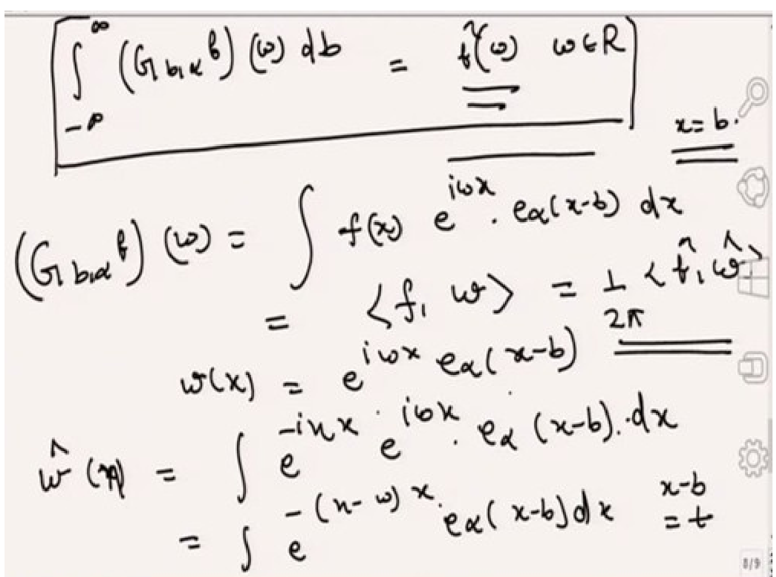

So if this property is true if I am integrating this with respect to db this will give you f at omega when omega belongs to R ok. So you can also observe this results very carefully ok that basically if I am integrating this Gabor transform with respect to this b then I am getting this Fourier spectrum. So it is it means that Gabor transform is localizing the Fourier transform around x = b.

So now if you look at this what will be the if I am asking you what will be the center of a Gaussian function if in my previous slide this is the definition we have used for defining the center of the window function. So what will be the center of the Gaussian function that you can take it as a exercise the center of a Gaussian function will be 0.

And the radius of the Gaussian function will be del e to the power alpha will be that will be the question for your assignment that radius of a Gaussian function is under root of alpha ok that is your assignment question ok. So now again I am restating the definition of a Gabor transform in this way I can write because e alpha is a real function. So that is why this thing I am writing so that I can write in the form of a inner product.

Ofcourse you should make a difference this is an omega and this is a w function w is the notation which I am using for a window function. So wx will be equal to e to the power i omega x ok so now you could use the parsewal relation which I have stated earlier using that relation we can always write this thing ok. So for that reason we have to compute the Fourier transform of this window function.

What will be the Fourier transform of this window function of w ofcourse I am this time I am taking eta because omega is already here. So this will be e to the power i omega x e alpha x-b with respect to dx ok so by rearranging the terms I can write down this and then by substituting x-b equal to some parameter.

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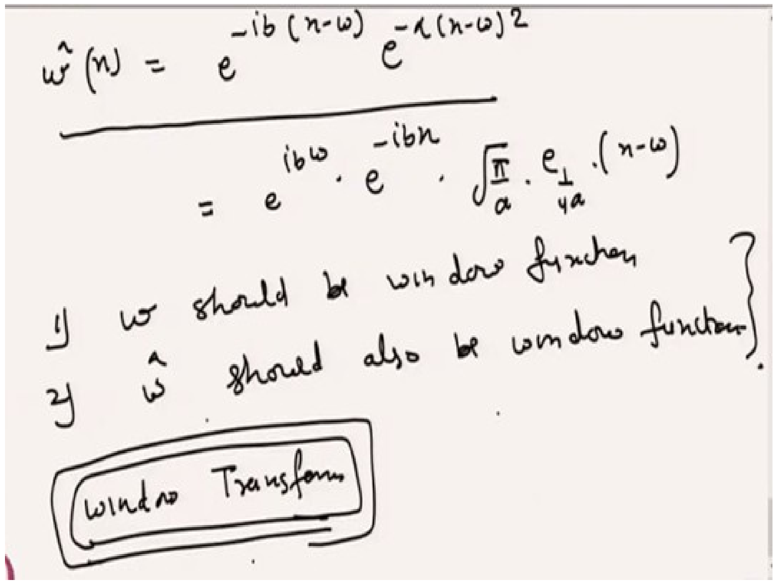

What is at the end I can get that w hat eta n will be = e ib eta – omega e to the power -n omega. So this is the best part of Fourier transform of window function is that this is also a Gaussian as I have already said that Fourier transform is of a Gaussian function itself is a Gaussian with different parameters. So that is what we are getting this and that we can write down again in this form ok.

So you could observe the signals in both the domains in this domain so any function which you want to use in time frequency analysis. What is should be the assumption that w should be a window function as well as w hat should also be a window function and we have a plus point with the Gaussian. Because Gaussian is a window function and it is Fourier transform is also in the form of a Gaussian which it is which will again be a window function and that is one of the reason that Gabor found that Gaussian will be the best example to choose as a window function initially.

Because so any function which or you could say which function you should choose in time frequency analysis. You should choose a function when w should be a window function moreover it is Fourier transform should also be a window function that is the idea. So now we will extend this idea of a Gabor transform to define window transform ok that I will do in my next lecture how to extend the idea of a Gabor transform.

To define window transform when you are not choosing as a special window function like Gaussian when you are choosing any general window function which should satisfy these 2 criteria ok so thank you very much.