Introduction to Methods of Applied Mathematics
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Module No # 06
Lecture No # 26
Fourier Series and Fourier Transform

Welcome to all of you in the next class of this course so in yesterday lecture we were discussing about Fourier series so let me continue with the same topic what is Fourier series?

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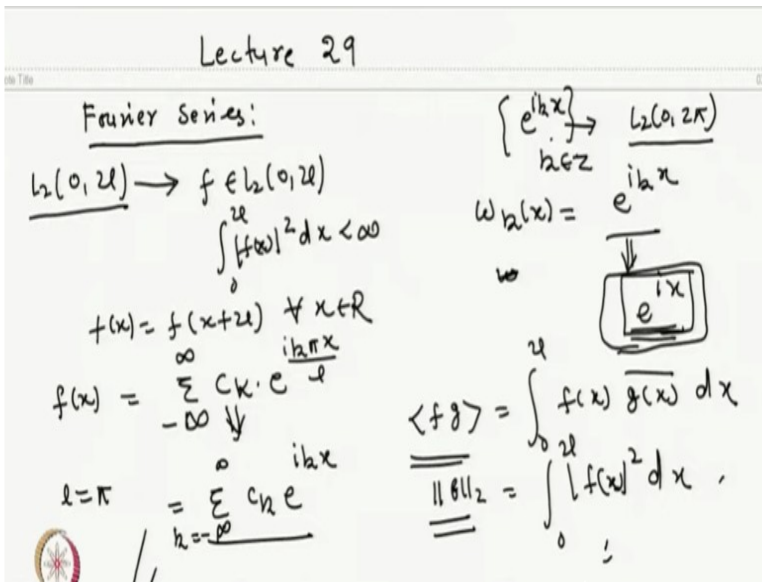

Fourier series so in last lecture we were taking function which belongs to this space. So let us have some knowledge about this space this is a set of a this is a space of all square integrable function it means if f belongs to L2 0 and dx is less than infinite okay. So more over one more condition which we have written in this space is means this function which belongs to this space is a periodic function of period 2l. All of us know what is the periodic function so it means f of x will be f of x +2l for all x belong to R.

Now in the last lecture I have written this as a Fourier series okay where ck's are called Fourier coefficients. So if you look at just for simplicity I am considering l = Pi in that case this expansion will become ikx okay. So here you look at one thing that e to the power ikx gives you an orthonormal basis for space 0 to 2 Pi for different k belongs to Z. Or here basically we are writing k is equal to 1 infinity sorry minus infinity to infinity okay k is minus infinity to infinity.

So if you look at this space more carefully how we are generating this space if it denote this function by this so basically this is let us say wx so this is a dilation of a simple one element what is that one element that one element is e to the iota x. So we are generating whole space by dilation of a single function which is this function okay. So this function is by dilating this functions we are generating the whole space l2 0 2 pi.

So you should notice that there is a dilation property later on when we will connect this to the this Fourier series to the wavelet series so you will see that dilation will also be in the wavelet series. So at that point of time we will continue with this dilation property and that is the motivation that right now I am mentioning to that property okay. So now as this L2 0 2Pi is equipped with norm and inner product what is the inner product on this space?

Inner product is given by this definition okay this is a inner product equipped on this space norm is this space. So this is the definition of an norm on this space this is the definition of a inner product on this space. Now the next question comes why we are considering function belongs to this space okay that is the questions I can so now can I write a Fourier series for a function which is not belongs to this space? Of course you can write down Fourier series.

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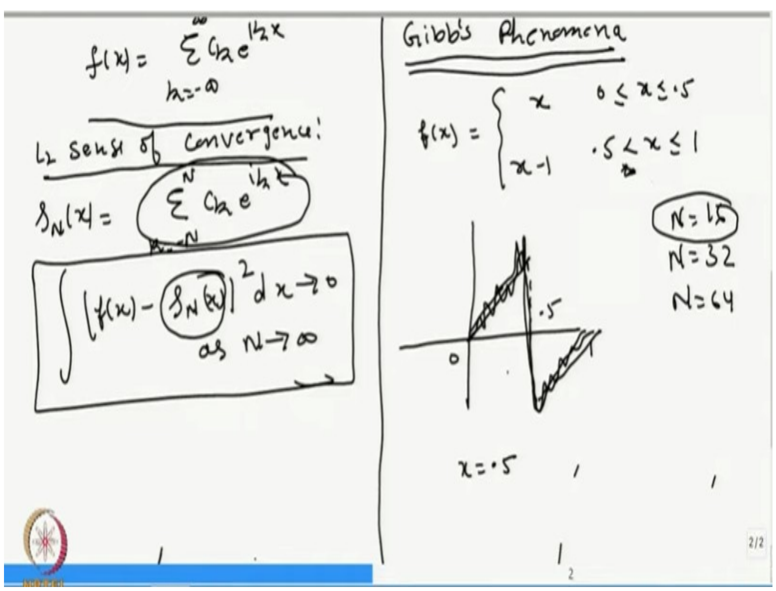

But the questions comes what will be because it is a infinite series f of x you know it is a infinite series okay. So it is a infinite series may converge or may not converge so and when we talk about the converge of infinite series the notions of the convergence are also different they are at different notions of the convergence of a infinite series uniform convergence point wise convergence L2 sense convergence.

So I am now mentioning all of that notion of a convergence in details but what I am doing right now I will tell you what is L2 sense of convergence? So for that reason I am denoting the partial sum okay so this series is set to converge in L2 sense provided this property this tends to 0 as N tends to infinity. This is the meaning of L2 sense of convergence so basically error is fx – snx is the error between the function and it is partial sum which is tending to 0 in L2 that is meaning which you could observe from the given formula.

So that is the reason and so if function belongs to this space L2 0 to 2l only then we can guarantee that Fourier series will converge at least in l2 sense. This is basically a theorem then proof of that theorem can be seen in any standard partial differential equation text we are Fourier series is explained in detail way so okay. I have given you very explanation this l2 sense of convergence because if you are writing a infinite series you should be sure in what sense it is converging okay.

So now the one of the very I should say disadvantage of Fourier series I am going to explain and what is that disadvantage is Gibbs phenomena okay. So what is this Gibbs phenomena let me explain you with the help of a one function is given like that okay this is the function if you want to look at the graph this will be x so 0.5, 0, 1 okay what you see in how if I wanted to approximate this function what will I do? I will approximate this function with the help of this partial sum snx which is given by this formula okay for different values of n.

So for different values of n means n 16 I can try for a n = 32 n = 64 so that you can also implement in a Matlab by Fourier approximations and once you compute snx for n = 16, 32, 64 what you will observe that as roughly I can give a plot for n = 16 this will be like this for n = 64 this will be more closer okay. So what as you increasing the n what will be the observation that wherever the function is as some smoothness it is okay function this approximation will be close to the approximation of the function.

But if function is non smooth near at x = 0.5 so at this point how large and you take some oscillation will always be there and this will be spreaded throughout the domain also. Means oscillation will be less for let us say n = 1000, 2000 but you will be not be able to remove this oscillation at any cost okay and why this happening? Because this is happening because of the non-smoothness this continuity at x = 0.5 okay.

So this should also be 0.5 or that also less than sin should also be okay because at X = 0.5 because there is a discontinuity in this function at x = 0.5. So when we are taking a infinite sum of this exponential functions then we are table to resolve this discontinuity that is what this Fourier series says but once I am working with a finite some k = - n to n we are not able to resolve this discontinuity proper and because you know fx is a discontinuous function and we are writing that function with a help of a sin and Cosine exponential iota x is sin and Cosine which are continuous functions.

So finite some of it continuous functions cannot given you this continuous function and that is one of the main drawback of the Fourier series which is called in terms of a Gibbs phenomena okay. So this is Gibbs phenomena I have already explained you so now what we do that I can tell you as n tends to infinity it will be more closer. So this as I am saying this is a disadvantage of a Fourier series and later on I will show you that this disadvantage is not in wavelet series that is why wavelet series is better than Fourier series.

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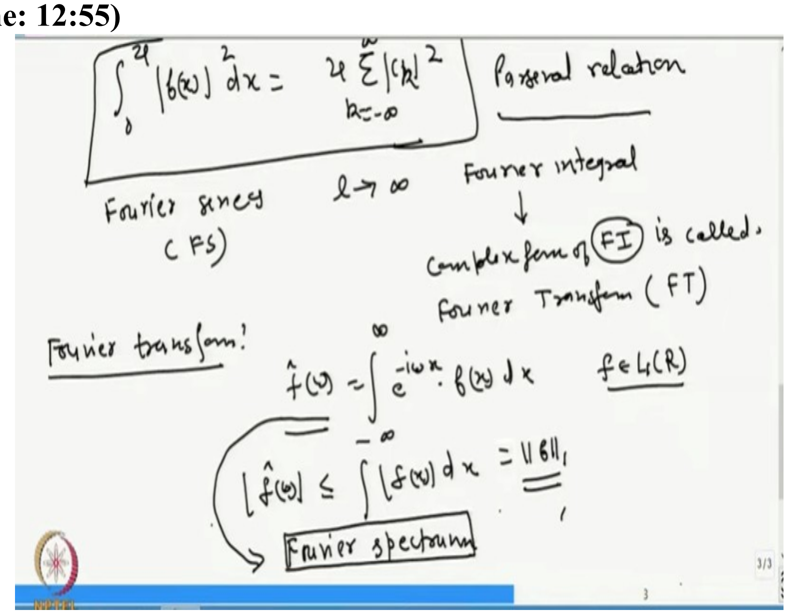

Because this Gibbs phenomena is completely removed in wavelet series okay so that is and now one of the other thing which Fourier coefficient satisfy this relation which is called Parseval relation which is called Parseval relation this is also very famous relations if you okay because if you remember earlier we were also talking some conditions but this is also one of the like if you remember when we were discussing orthonormal basis, Riesz basis we were writing some condition ck is basically when f comma f inner product of f with vk which vk's are exponential of ikx okay.

So this is also what is the coefficient of Fourier series are satisfying this relation is called Parseval relation. So now from Fourier series I was writing for a function which has a period to l but now L tends to infinity means period tends to infinity Fourier series is converted to a Fourier integral okay. And complex form of Fourier integral is called Fourier transform complex form of Fourier integral in short form is called Fourier transform.

Again in short form Fourier transform can be written with FT and Fourier series can be written with FS. So now let me define what is Fourier transform because first I will define you Fourier series, Fourier transform and then wavelet series, wavelet transform that is the sequence because only then you will be able to appreciate the beauty of wavelet analysis over Fourier analysis. So now what is Fourier transform? Now I am defining a Fourier transform by this formula for f belongs to L1R.

We have already seen this kind of a spaces L1R L2 LPR for P general greater than or equal to 1 so I do not to redefine again what is this L1R spaces. So this is a Fourier transform a function fx which belongs to L1 R this is a proper integral so if I am talking about infinite series improper integral I also have to talk about convergence of them. So for that reason I am taking f belongs to L1 R because if f belongs to L1R this Fourier transform always exist that I can show you also what will be this thing?

This thing will be so this is basically if you look at this is the this norm so if f belongs to L1 this is less than infinitive so it means Fourier transform exists this improper integral converges. So that is why we are working out with f belongs to L1R now we will study first of all let me give you physical interpretations why we study Fourier transform why we study sometime we want to analyze functions for mathematicians signal for engineers in a domain with respect to the frequency okay.

Because you know what Fourier transform is doing Fourier transform is decomposing the function into the various waves. So sometime if you want to analyze the function in different domain and that what is the different domain that different domain is called frequency domain and then Fourier transform is also sometime called Fourier spectrum okay Fourier transform is called also Fourier spectrum.

We analyze what is the need of it we analyze the functions in a different domain which is called a frequency domain and Fourier transform is also called after sometime Fourier spectrum. So now we will study different properties of different transform which will be used subsequently.

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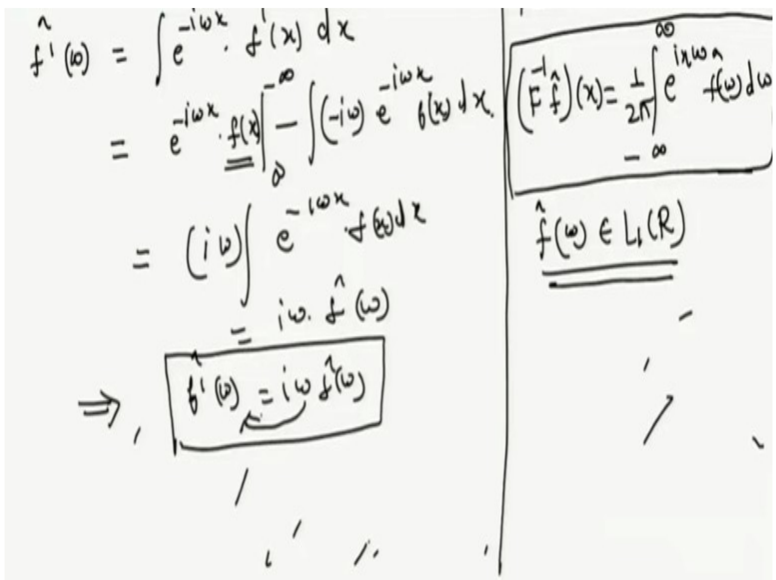

So if I say f dash x is also in L1R then one of the property i can prove this this what is this? e to the power –i omega f dash of x dx that is why chosen this f dash x is L1R because only then I can define this I can talk about the existence of a Fourier transform. So if I use integration by parts so we are using integration by parts where so this is i omega fx dx okay this is an L1 R functions so L1 R function always tends to 0 at x tends to plus minus infinity that is the property of a function which belongs to that basis.

So basically this is i omega f hat of omega of a okay so in a nutshell this is a famous property of the Fourier transform from this property we can deduce one more thing that effect of omega tends to 0 as omega tends to plus minus infinite this can be proved directly from here because once you write down this in the left hand side this is very much clear this property we will use later on that is why I have defined here.

And the next property of this Fourier transform is yes this is now if we know from one domain we are going to the frequency domain we should also know how to come back otherwise there is a no point of defining some integral transform. Fourier transform is also integral transform as you must have already studied in earlier. So now we are defining inverse Fourier transform so inverse Fourier transform we are defining by this formula provided f hat is omega is also in L1 R.

Because only then we can again talk the convergence of a this improper integral okay we can talk that is why we can choosing f hat omega belongs to L1 R. It is not necessary that if fx belongs to L1R f at omega will also be L1R that I can show by one example.

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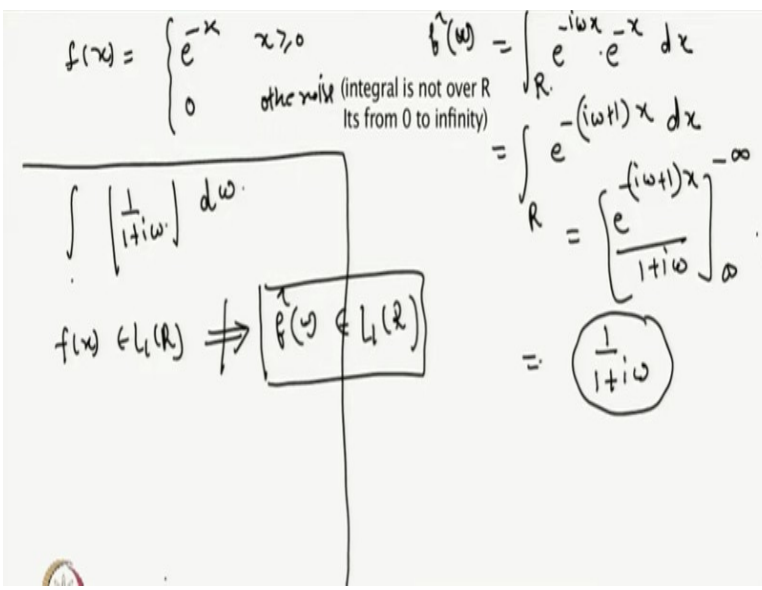

I am taking one function 0 otherwise okay this is the function I am taking so I am now I am defining what will be Fourier transform for this function with this will be 1 + 1 + i omega okay. So if you look at now we have define whether this function is L1R of not this is the Fourier transform of this function so I have to choose whether this function is in not in L1R. So I will I can take this function separately. Suppose this is not an L1R function okay so I have given you this by one example that if function is L1R does not necessarily means f at omega may not be L1R okay.

So that is why we are we whenever we were defining a inverse Fourier transform separately we were making this assumption. So now we are looking at the Fourier transform of a one anther functions which all of you must have seen earlier and that function is called as a Gaussian function.

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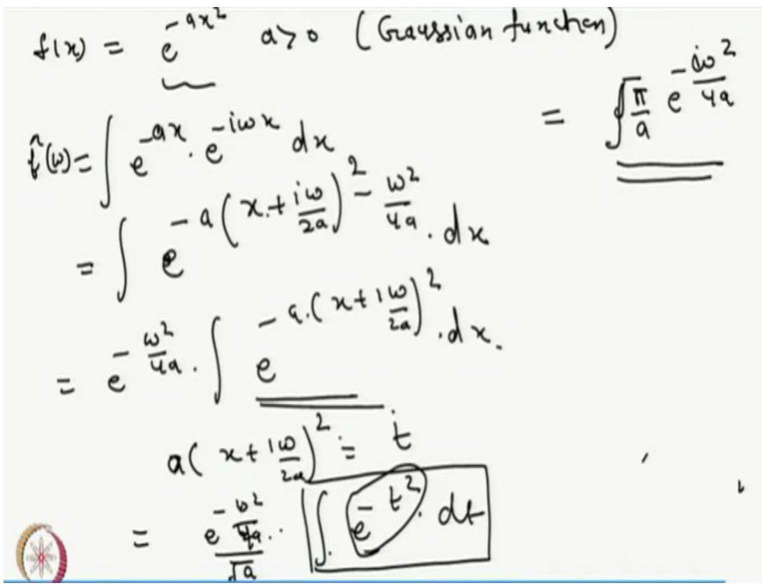

This is called Gaussian functions we are defining a Fourier transform of this function. So how I can write down ax + i omega I can always add and subtract something so using this terminology I have done that. So now I can take this term common okay so what now we have to simplify this integral as well whole thing x + i omega / 2a whole square I can take t so then after substituting this we will get e to the power –t square dt okay.

So this is now we have to calculate this integral we all know this is under root Pi but if you will ask me how to calculate because we do know the anti derivative of this function that is why in that case some special techniques are device to calculate this kind of a integral and we will based on Taylor's series etc., that is just for your knowledge. So f hat omega will be under root Pi / a e to the power minus omega square by 4 a okay.

So this will be the Fourier transform of a Gaussian function sorry this under root sin will be because this value is under root Pi by under root a. So what do you observe specifically about the Fourier transform of a Gaussian can you think for a while the main point is that Fourier transform of a Gaussian function is also in the form of a Gaussian this is also a form of a Gaussian that is the specific point to observe which we will exploit later on that Fourier and we will work out with this Gaussian function because the Fourier transform of Gaussian is again a Gaussian function.

So this is a important point which all of you should note for a future lectures now I am coming to define a another concept which is used in some of the analysis which is called convolutions okay. So I am defining what is called convolution?

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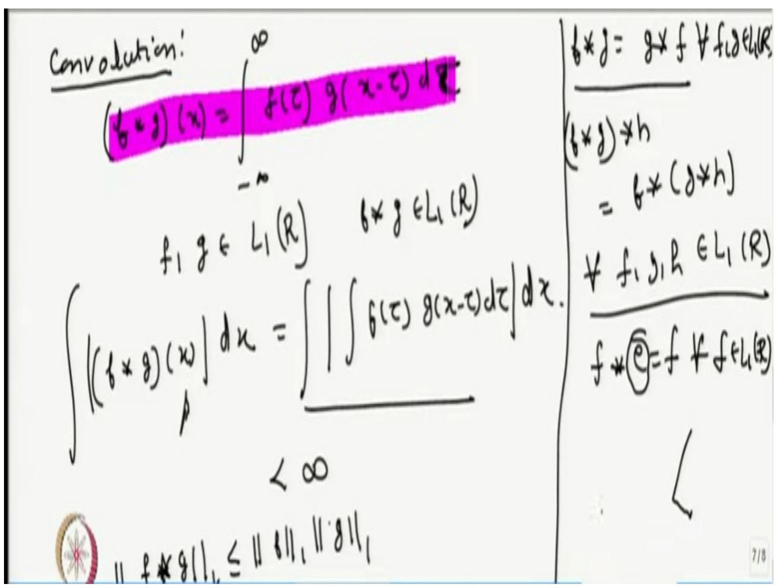

Okay convolution of a two functions is defined in this way okay so what we are doing we are multiplying the function f with the function g not exactly with delayed by parameter tau sorry this should be tau okay and then we are multiplying this function with that delayed by tau to the another function g and then we are integrating with respect to this delay parameter. This is how convolution in this defined convolution is very important it has lot of applications in electrical engineering probabilities statistics etc., many applications and I assume all of you must have a heard about this already.

So this is how we define convolution but again this is a improper in integral so I have to talk when it exist. So for that reason I am saying if f and g belongs to L1R then convolution will be also be a L1R this is just a 1 line proof because this will be equal to f of tau g of x – tau d tau dx. Now using Fubini's theorem you can very well because f and g both are in L1R you can very well prove that f and g will also be L1 R it means f and g norm as sorry this is a convolution okay.

So the next property of the convulsion which we can prove that convolution is also commutative that is also this you can take it a exercise and you can prove it by change of variables that convolution is a commutative property for all f and g belongs to f and L1R it is also distributive will be equal to f star of g of h for all fgh belongs to L1R. Now the next question is identity operation is also exist for a convolution it means f for all f belongs to L1 does identity exist for a convolution operator this is my next question okay.

So can you think for a while for this does identity exist for a convolution operations you think in the mean time I am proving one another theorem which I will be needing to answer to this questions does identity exists for a convolution operation and that theorem is called convolution theorem.

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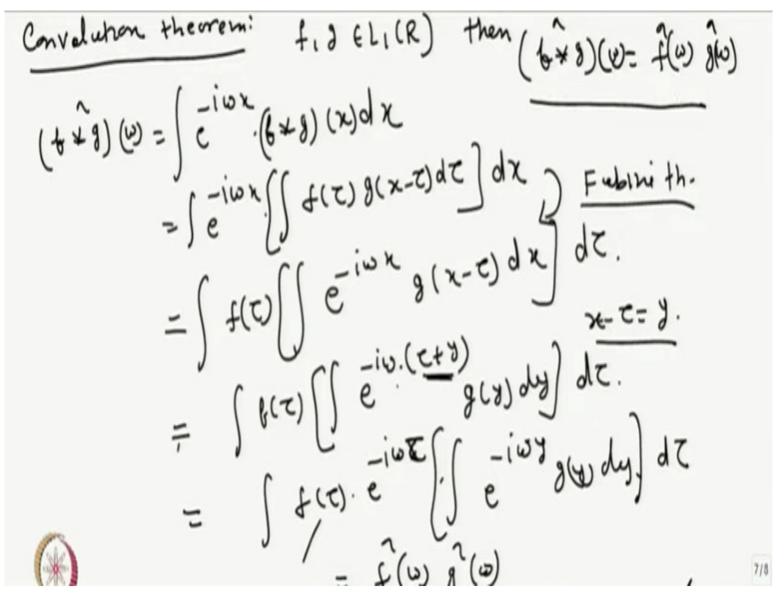

Convolution theorem what this theorem says? This theorem says it f and g belongs to L1R then this will always be the case i will prove this statement okay. Now I am using Fubini's theorem f of tau dx into d tau now I am using a change of variable x – tau = y from this step to this step we have using the fubini's theorem how order of integration basically in a nut shell this theorem says that how order of integration can be changed okay.

So this I can split into 2 terms and then f of tau e to the power –i omega tau i omega y gy dy and d tau. So basically you can see this is equal to f at omega g at omega okay. So we have proved convolution theorem it is a very beautiful theorem because it has many applications one of the application is it helps in computing the convolutions with the very reduced complexity because otherwise how will you compute the convolution? You will compute this convolution with the this definition which I have given you here okay but with the help of this convolution theorem what you are doing?

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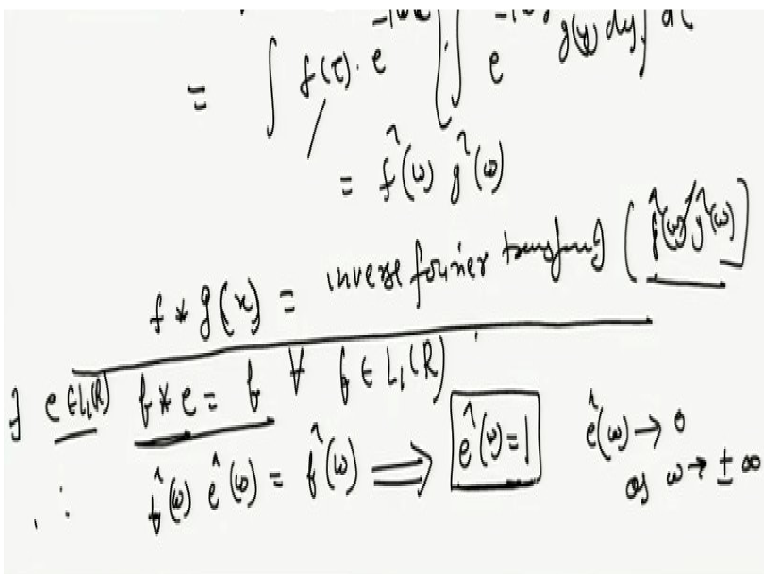

I can compute the convolution with this inverse Fourier transform of f at omega g at omega okay. So complexity reduced if I am computing the inverse Fourier transform of multiplication of 2 function f at omega and g at omega and then I am computing the conversions instead of going with the direct definitions. Okay so and convolutions as many application and we will using this convolution theorem to support my answer of my previous question that does identity exist for a convolution operations okay.

So how because now in the left hand side I will be using convolution theorem so it means I am taking the Fourier transform of this both sides so this will give me this by using convolution theorem which implies e at omega is 1 so is it possible? Can you think is it possible? No, existence of identity means they are exist e belongs to L1R existence of identity means there should exist e which belongs to L1R which satisfy this property but here we are saying e at omega is 1 which is not true why is it not true?

Because if any function which belongs to L1R is space it should tends to 0 at x tends to plus minus infinity. So means if e at omega is 1 it means if you remember I have used not I have used I have proved 1 property here f at omega should tends to 0 as omega tends to plus minus infinity okay. For a function which belongs to L1R so they are e belongs to L1R so Fourier transform should and because of that reason I proved this property as I said I will use this property later on.

So Fourier transform of this should tends to 0 as omega tends to plus minus infinity but they are it is just a minute but you look here this is not this according to the previous property of a Fourier transform this should tends to 0 as omega tends to plus minus infinity. But this is contradictory so it means their does not exist convolution identity or we have to say something more how we can approximate this convolution identity and with this words.

I am closing my toady's lecture an whatever exercise I have asked you to prove about convolution operations you could look at the proof of them because it really helps and in next lecture what we will do we will tell you how to work out with the approximation of the convolution identity okay thank you very much with this I am closing my lecture.