Introduction to Methods of Applied Mathematics
Prof. Mani Mehra & Prof. Vivek Aggarwal
Department of Mathematics
Indian Institute of Technology- Delhi

Module No # 05
Lecture No #25
Frames, Riesz Bases and Orthonormal Bases (Contd…)

Welcome to all of you in the next class so now let us continue again with the 3 concepts which we I have discussed in my previous lecture frames, riesz bases and orthonormal bases. So in the last lecture we have taken 2 examples.

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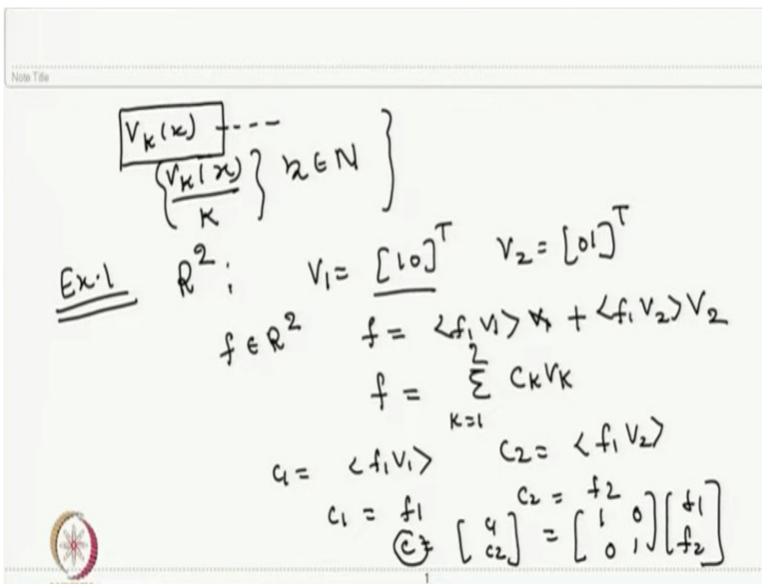

We are in first example I was trying to prove that Vkx is given orthonormal bases then Vkx / k where k belongs index set N will not be will not form a riesz basis. Why it will not form a Riesz bases? We have already seen in the last lecture because it does not satisfy the frames conditions the next examples which we took in the last lecture is if Vkx again is Vkx where k belongs to N are given orthonormal basis and if I repeat in fact one of the single entry in that case it will not be Riesz basis in fact.

Because then these vectors will be a linearly dependent vectors and therefore it will be a frames and that frames will be a different frame bound okay. So that was the summary of my last lecture and again in today's in last lecture whatever 2 examples we have seen that those examples was for infinite dimensional spaces and today I am again going to consider 3 different examples for a finite dimensional spaces.

So as first example which I am going to consider is on the space R2 which all of you are very familiar R2 space and what I am considering here that consider an orthonormal basis for R2. So V1 is basically 10 and V2 is 01 this is a standard orthonormal basis of R2 all of us know very well. So now I will give you more detailed explanation how you can represent a function of R2 with the help of this orthonormal basis.

So if f is any not function it can be a vector in R2 okay so f will be okay or basically we can write this way where C1 will be fV1 C2 will be fV2 okay. So now what is C1? V1 is this C1 will be you can inner product of the function vector f with V1 so what it will give you? This if I try to write in a matrix form this will be C1 will be f1 which is the first component of the vector C2 will be F2. So if wanted to write this whole C which is C1 and C2 I can write this way 1, 0, 0, 1 f1 and f2. So this is the way I am writing expansion coefficient C in the form of how to calculate expansion coefficients C.

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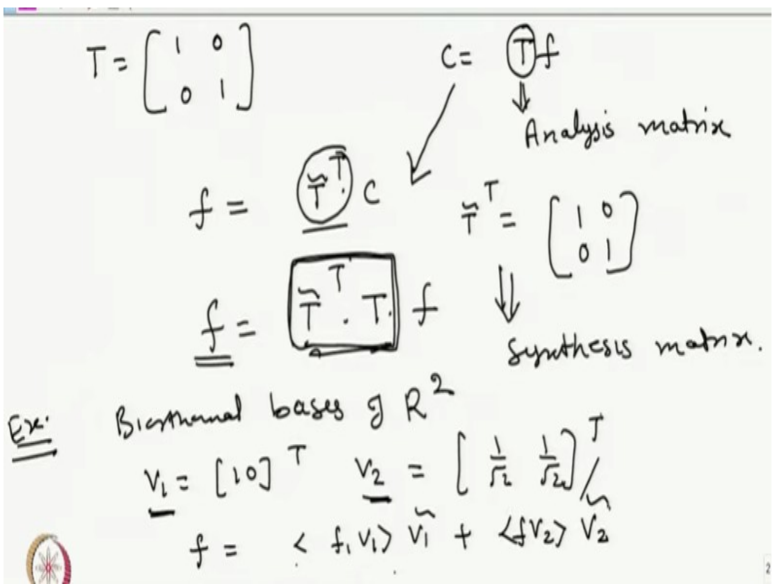

So if I am denoting this matrix T then I can write C = Tf okay this T which help us calculate expansion coefficient from a given vector or functions it is called analysis matrix. Now with the help of this C how I can write down this f in the form of C okay. So what will be this you could see in the previous page what will be this fV1, V1 and V2? So I am writing this in this way this is TC okay so what will be this matrix?

This matrix will be a again V1, V2 okay so now I can write T of f C is T of f which we can plug this value of C from here to here okay. So these examples sets the way how to have a non-orthonormal or redundant signal expansion that is idea behind this example. For a mathematician f can be a any vector of a any vector space but for engineers f is a signal okay. So this is the main idea behind this example that understand what should be this matrix T and this is called synthesis matrix which help us to reconstruct the vector f from here okay.

In this example we have understood 2 different kind of a matrix 1 is called analysis matrix another one is called synthesis matrix. So is it clear to everyone? Now we are considering another example the way I have said that this is the motivation this example is the motivation behind next 2 examples. So the next example which I am considering is by orthonormal basis of R2 okay.

So this time we are considering V1 as 10 V2 is okay V1 is this V2 is this so of course can you tell whether this will be linearly independent? So that you could think otherwise at the end I will answer V1 at V2 okay.

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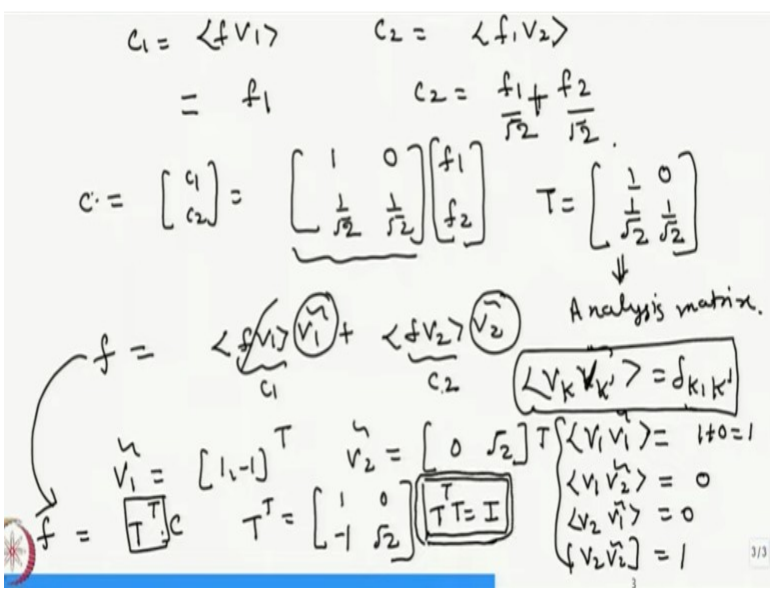

So now I am again using this expression okay so again C1 is f of V1 C2 is f of V2. So C1 will be f of V1 so this will be f1 C2 will be f1 by under root 2 into f2 by under root 2 sorry it should okay. So with this I can again write the same thing so this time this will be my analysis matrix T okay. So now this will be my analysis matrix and now to reconstruct f from this basically this is my C1 this is my C2 so first of all I have to choose these two things V1 tilde V2 tilde these are the vectors which satisfy this property okay.

So what are the choices of V k and V k dash we can have of V1 and V2 are this V1 is this V2 is this. So now we are choosing V1 tilde so now we are choosing V1 is 1 into -1 and V2 is this so of course you could check what will be this? This will be 1 + 0 1 V1, V2 what will be this? This will be 0 V2 V1 tilde this will also be 0 V2, V2 tilde this will be 1 so from this we have checked it satisfy this by orthonormal property okay we have already checked by these 4 things that it satisfy by orthonormal property.

So with this I am writing f this I can always write this expansion in this way okay so what should what will be this value this matrix. As we have said that this is the analysis matrix and this is a synthesis matrix so what will be this value? This value will be 1 – 1 0 under root 2 okay and you could check that the multiplication of this 2 matrix analysis matrix and synthesis matrix should be equal to the identity matrix.

So it means that this synthesis matrix is left inverse of the matrix T okay so this is a left inverse of this matrix T basically T transpose should be equal to I. So this time we started with non-orthonormal vectors but still we ended up with this relation so this is the idea behind by orthonormal basis for R2.

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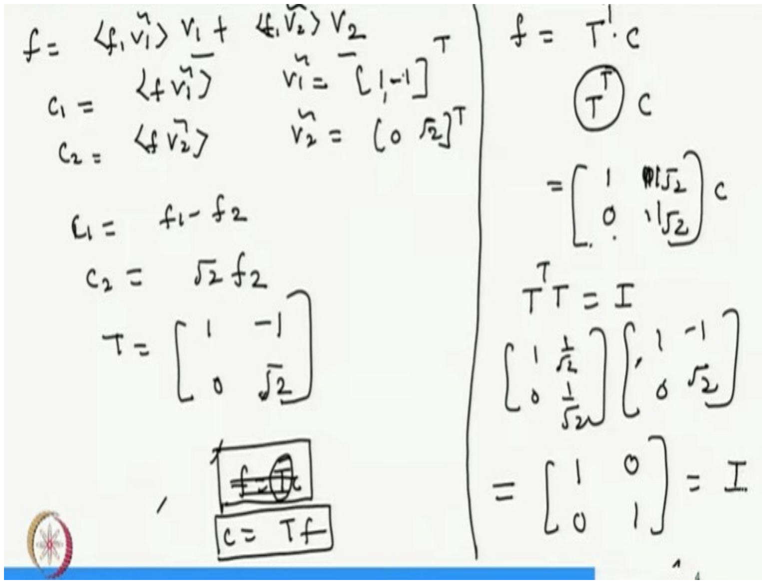

Similarly I can also use this expression in this case C1 will be f into V1 tilde and C2 will be f into V2 tilde okay. So V1 and V2 tilde we know as I have already stated okay so what will be C1? C1 will be f1 – f2 C2 will be under root 2 f2 so analysis matrix T will become 1 – 1 0 under root 2 this will be my analysis matrix in this case and I can write f = Tc. So this is the way with the help of a analysis matrix we are sorry just a minute this should be C = Tf not f = Tc c is equal to so with the help of analysis matrix T we are getting the expansion coefficient c.

Now and now you could rightly observe from here that what should be the synthesis matrix? f = T, T transpose c so now you have to calculate this matrix okay what will be this matrix? This matrix will be this 1, 0, 0 what are the choices of V1 and V2 we had? Choices of V1 and V2 was 1, 0 and 1 by under root 2. So now you could again check that this and this matrix multiplication of this matrix T should be equal to identity matrix because this is the way place where we are inducting that by orthonormal property because you could check once will be equal to 1 this will be 0, 0, 1 which is equal to identity matrix okay. So now the this is the second example we have seen.

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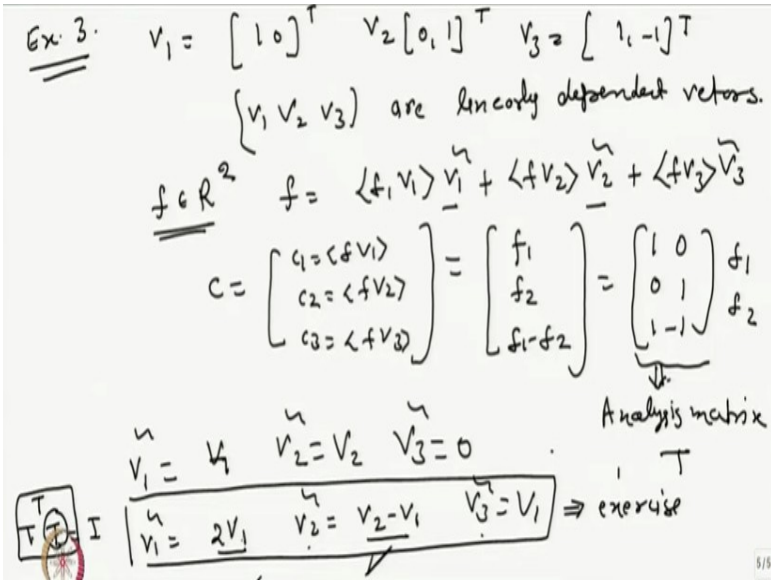

Now in the third example which I will consider in next page example 3 okay so this time we are consider V1 is 1, 0 V2 is 0, 1 V3 is 1 -1 transpose okay. So V1 and V2 are L.I set and if I have added V3 also in R2 anyone can make out that this will be a linearly dependent vectors because in R2 more than 2 vectors will always form a linearly dependent set of a vectors okay. So we now we know these are V1, V2, V3 are linearly dependent vectors now we will see how to expand a element of a vector f belongs to R2 okay. So again there are 2 ways so c will be equal to f1, f2 f1 – f2 so this thing we can write down 1, 0, 0 and 1 – 1.

So this time this becomes analysis matrix T now you tell me what can be choices for V1 tilde and V2 tilde V3 tilde because I can always choose this V1 tilde is V1, V2 tilde is V2 and V3 tilde is 0 because we know this is a V1 and V2 R an orthonormal basis for space R2 another choice of V1, V2, V3 tilde can be this choice of dual vectors you can verify yourself that how I have chosen this V1 tilde is 2 V1, V2 tilde is V2 – V1, V3 tilde is V1 this I am leaving it as a exercise you have to verify yourself okay.

So but what is the message you are getting out of this example that you can choose dual vectors in 2 different ways okay or in fact these dual vectors can be chosen in a infinitely many ways okay and this is supporting my statement which I have given in the last lecture that dual basically frames are linearly dependent vectors which increases the robustness to the noise Because here I can reconstruct my function in many ways I have a freedom to choose this dual vectors okay which is basically it means I have design freedom I can choose f in multiple ways.

You could say it is design freedom you could say it adds robustness to the noise whatever way you wanted to say you can say this mathematical things in many ways and but how to choose particular dual frame that is the scope of a sampling theory how to choose and that depends on the problem at hand. How to choose a but in principle you can choose many different varieties of this dual frames because you know that if this is a analysis matrix T.

We are calling this as a left inverse of matrix T they are infinitely many ways to define the left inverse and if you look at this left inverse is associated with this dual vectors. So both the things are equivalent that you have to observe so as i said in my first example that basically this is the idea which we will be using in my next 2 examples whether vectors are in fact even vectors are non-orthonormal but is still we are able to retain this property.

Why we are able to retain this property? Because of this and we are choosing dual frames which satisfy biorthogonal things okay. So these 3 examples which I have discussed just now are were the examples in finite dimensional spaces in the last class I have considered 2 examples in the infinite dimensional spaces okay.

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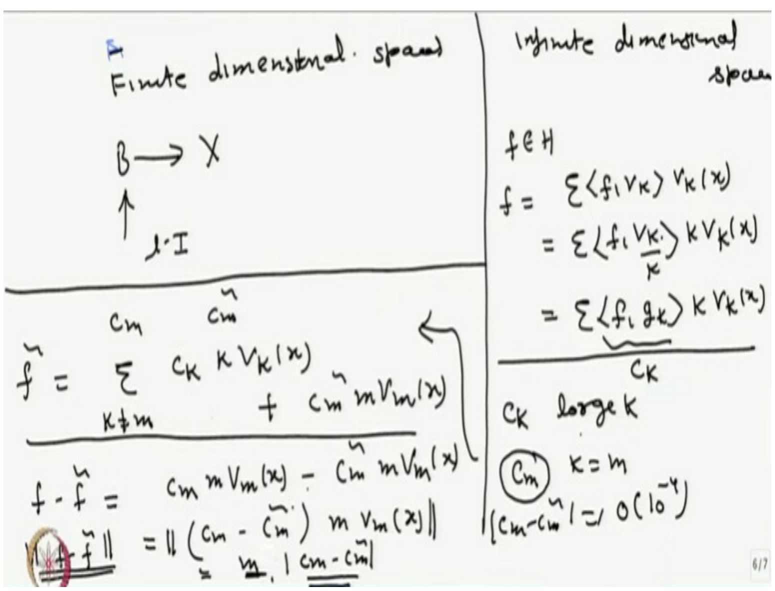

So what is I am telling that in finite dimensional spaces and infinite dimensional spaces. So the key idea in finite dimensional spaces for a basis is that if B is a basis it spans X as well as the element of a this set should be a linearly independent while in this case we said just spanning is not necessary spanning is not the only thing to say that this is basis we are adding one more condition frames conditions if you remember from my last lecture we said that Vk will form a Riesz bases if it generates X it is linearly independent vector moreover one extra conditions which we added in the last lecture was frames conditions okay.

So frame condition is must in infinite dimensional which is why it is saying must it will that you will see in my next example that will support my statement that otherwise you will not have a stable reconstruction. If you are not putting that frame conditions you will not get stable reconstruction while in this case finite dimensional case that extra condition is not needed linearly independent vector conditions itself guarantee the stable reconstruction.

So for that reason I am taking one example suppose f is a any element of Hilbert space H I am writing this in this way okay. Now what we are doing I can always do that because scalar multiplication I can take inside the inner product f into gk into k into Vks okay this is my ck. Suppose there is some error in one of the ck for large k and let us say that error in cm when k = m.

I am continuing here okay so if they are what kind of error you can have in this cm is the exact but there is some error that error can be measurement error which you can call it as a noise that error can be round off error which you get in the computation. So means any kind of error can be cm tilde so if that error happens at any coefficient for a large chain that what will happen. So then f tilde is basically okay and f is and f is this f tilde expansion is this.

So if I what will be this value? This value will become cm m Vmx – cm tilde m Vmx okay. So now if we take norm of that particular Hilbert space okay So this will be equal to m because Vn x are an orthonormal basis so that is why this 2 norm is 1 which was given here so this is if you have introduced is a small error cm – cm tilde but that you have introduce a small error but m is large because you have made the error at for a large m then this small error is magnified by m in fact m is square.

So in even if order of this error is 10 to the power 4 okay what do you understand by order that should be clear to all of you I am not going to explain what is order you must have already seen. So if this error is order is 10 to the power -4 but the m is let us say 1,10,000 so you could yourself see what will be error you must have introduced in the total reconstruction process. So this is reconstruct f minus this what is the error in the reconstruction?

That error is magnified and now I am asking you that can you answer can you think that why this happening why we are not ending with a stable reconstruction these are the question which you have to think and the answer to that question is because if you remember this gk's are Vk / k and we have proved in the last lecture that this is a frames we have proved this is sorry this is not a frame because it does not satisfy the frame condition we have proved in the last lecture.

So it means that frame condition is necessary for stable reconstruction if any set of a vector will generate X but still it does not satisfy the frame conditions we will not have a stable reconstruction so that was the idea behind this example. So now I suppose it should be clear to all of you that what is the difference between frames Riesz basis and orthonormal basis. Frames as certain advantage it adds robustness to the noise which I am keep saying in my last lecture as well as today's lecture at the cost of high computation.

And now I should answer that question why this high computation cost issue is coming in frames why? Because we are working with LD vectors if you look at here in this example we were working with frames because these were linearly dependent vectors so where this high computation issue is coming? This high computation issue is coming because there is one more set of a vector so of course the size of matrix is increased so this leads to the high computation.

Because it is not just adding one more vectors you can in this way you can keep n more vectors but then size of the matrix will also be increased so that is the difference okay. And but now okay orthonormal basis are very good you could say but why we are going to frames it has very efficient computations but we are because many times for any Hilbert space it is difficult to construct orthonormal basis which satisfies certain additional properties.

Because sometimes we need some additional property of the basis and those properties those kind of basis are difficult to generate if you retain this orthonormal property so that is why we move from orthonormal basis to Riesz basis. We relax this orthonormal T2 by orthogonal T and then we went to the frames in fact there we said that you can have linearly dependent vectors also but at the cost of high computation. The benefit you are going you are adding robustness to the noise. So that was the end of my this concept that what is the difference between these basis and finite dimensional space and infinite dimensional space.

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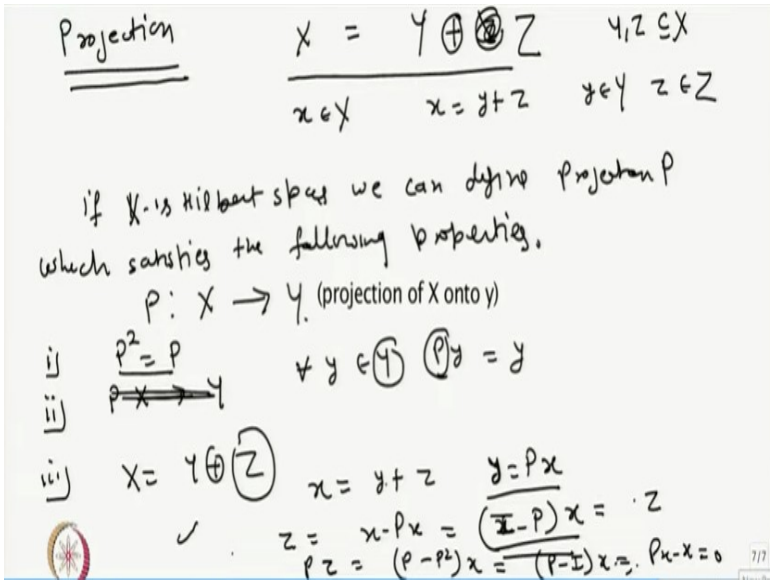

Now I am starting new concept which is required that is called projection okay. So a Hilbert space X is set to be a direct sum of 2 subspaces Y and Z sorry Y and Z are 2 subspaces of X then it will said to be direct sum if for each X belongs to X you could write Y + Z where Y will belongs to Y and Z will belongs to Z. Moreover so Y is called the algebraic component of Z and vice versa.

Now if this is in fact you do not need to say X is a Hilbert space X is X can be only a vector space is also to define this direct sum but if X is a Hilbert space we can define projection P which satisfies the following properties okay. So P is a map from X to Y okay if X is Hilbert space we can define projection P which satisfy the following property. P satisfy this property which is called idempotent matrix P is a identity matrix on is space Y it means for all Y belongs to Y PY is Y.

So not I should not write here okay P is a identity P works like a identity on the space Y P is idempotent and if it has a direct sum then every X you could write this where Y is Px okay Y is Px so it means Z is x – Px. So if I use this property P square = P what can I write here so basically it maps Z on to null set it maps Z on to the null set that is what we can deduce from here because what will be PZ?

PZ will be P – P square x which is P – I into x Px – x okay so which will be a PZ will be 0 so P maps Z on to null set so this was the idea behind projection which we will be needing later on. So if P is any mapping from space x to Y where x is Hilbert space we can define projection P which satisfy the following property in first case it is a idempotent matrix if this satisfy this thing in next case it work since it is a identity operator on the space Y and if X is direct sum of Y + Z it maps the space Z to the null set. So that was the idea behind projection now what we are considering I will tell you I will start with Fourier series.

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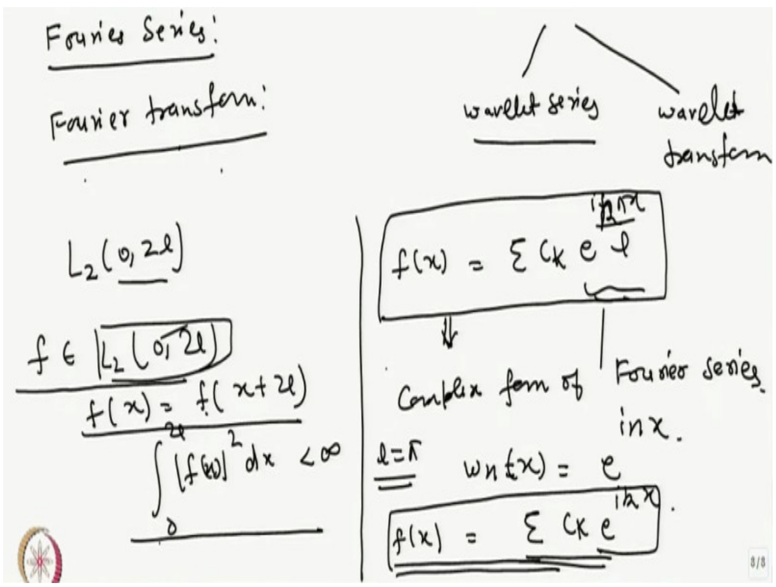

And in this course I feel you must have already have an idea what is Fourier series? But I am just giving a brief idea so that I can continue for the wavelet series because you know wavelet I have given you a mathematical foundation behind to constructive wavelet. Wavelet analysis also contain 2 components wavelet series and wavelet transform. Similarly Fourier analysis also contains 2 component Fourier series and Fourier transform okay.

So what will be the motivation for my next few lectures how wavelet series is superior to Fourier series and how wavelet transform is superior to Fourier transform. So for to appreciate the beauty of wavelet series and wavelet transform you should understand what is Fourier series and Fourier transform which I feel you must have already studied in this course earlier. But so continue to with that first of all to define a Fourier series I need the concept of this space.

We earlier we have already studied if you remember R2 spaces now I am saying L2 spaces on the intervals 0 to L2 l so what are those spaces let me specify if f belongs to L2 0 to 2L is spaces means it is a set of all square integrable function which is periodic okay as well as it satisfy this conditions. So this is the meaning of any function which belongs to this space will be periodic function of period 2l as well this property will be satisfied why I am defining this space because this space later on I will be needing to write a Fourier series and why I need Fourier series?

I will appreciate why wavelet series one you know the disadvantage of Fourier series so for that reason I am just writing complex form of the Fourier series complex form of the Fourier series. So if L is Pi this vector can be this function can be inx. So in that case fx will be c and ck e to the power i sorry this should be k ikx okay. So in my next lecture I will explain you how you are generating Fourier series and then the disadvantage of Fourier series and then I will define Fourier transforming and then I will go step by step in constructing wavelet series and wavelet transform.

So just I have I am written one step of the Fourier series which you could look at more carefully that basically how Fourier series is generating if you look at the concept what is the need of Fourier series because Fourier series is the series which by which you can express any complicated function in terms of simple function. Because this is a Iota exponential function which can Cos kx + Iota sin kx.

So basically any complicated function can be written in terms of a simple function sin x and Cosine x and many algebraic property many things you could do in a very easy way over sin and Cosine rather than on a complicated functions. So that is was the idea why Fourier series we need that is the motivation behind Fourier series okay and one another thing Fourier series when Fourier series was invented?

Fourier series was invented when someone was trying to solve a heat equation which is a partial differential equation for it became popular later on for a engineers but how Fourier series is developed? Fourier series was developed for solving a partial differential equations particularly heat equations okay. So with this statement I am closing my today’s lecture and I will discuss in my next lecture how basically you are generating this Fourier series? So thank you very much.