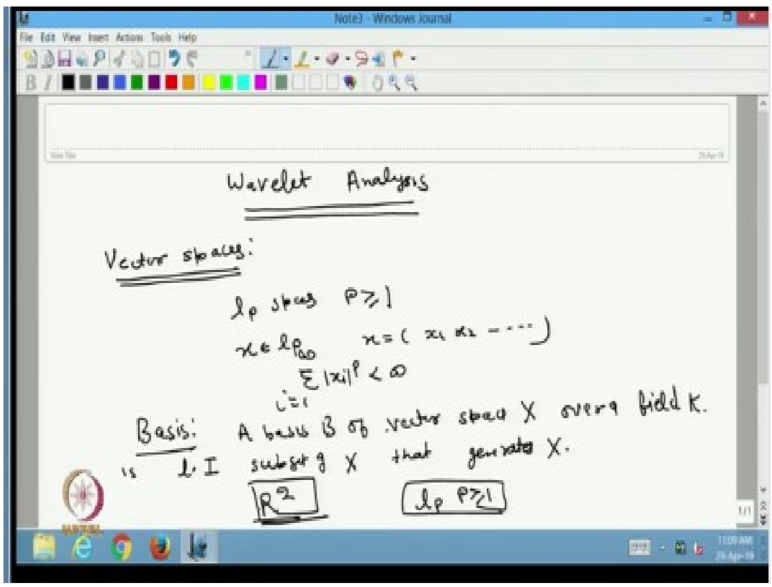
Introduction to Methods of Applied Mathematics

Prof. Mani Mehra and Prof. Vivek Aggarwal
Department of Mathematics
Indian Institute of Technology-Delhi/DTU

Lecture-24
Frames, Riesz bases and orthogonal bases

Welcome to all of you in this lecture today I will be speaking on very different topic of applied mathematics which is not taught in many courses but it has enormous applications if you really you are working in the field of applied mathematics.

(Refer Slide Time: 00:41)



So what is this topic is, wavelet analysis this is a very important topic in the area of applied mathematics as I have already said. But before going to say something about the wavelet, let me give you the mathematical foundation behind it. So for that reason I am starting with the very basic spaces which we call it as a vector spaces as all of you are familiar with the vector spaces. Because vector spaces play an important role in many branches of a mathematics and it has some operations on it is.

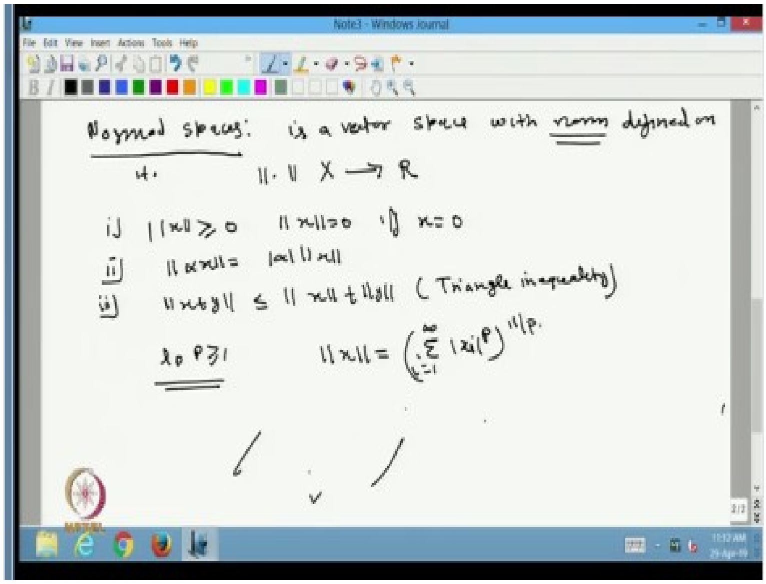
So vector space, the element of a vector spaces does not mean vectors which we see in the coordinate geometry element of a vector spaces can be a functions as well vectors as well anything sequences as well, as we all know already about it. So one of the vector spaces example let me give you l p space for p is greater than 1. So the element of this vector spaces can be written like this which is a basically sequence which satisfy this property.

So l p is a example of a vector spaces which satisfy where the elements of a vector spaces are sequence ok. So is it clear to everyone that vector what is vector spaces I am not giving the basic definition of a vector spaces and I assume that all of you know this ok. If you now after vector spaces what is the main mathematical object defined on the vector spaces which we call it as a basis, a basis B of vector space X over a field K is linearly independent sets subset of X that generates X.

We all know what is basis, a basis B of a vector space X over a field K is a linearly independent subset of X that generate X. So I am not going to explain you what is linearly independent **set** subset and I am assuming that all of you know what are linearly independent subsets. So now if basis contains finite number of elements it is called finite dimensional spaces if basis contains infinite elements it is called infinite dimensional spaces.

So if let me ask 2 questions what will be the dimension of a space vector space R 2, so the answer should be 2, what will be the dimension of the vector spaces l p, p is greater than 1 which I have defined just now the dimension of this vector space will be infinite. So this is called infinite dimensional vector spaces this is called finite dimensional vector spaces, so it should be clear to you that what is the difference between finite dimensional spaces and infinite dimensional spaces ok. So now I am coming to the concept of a normed spaces.

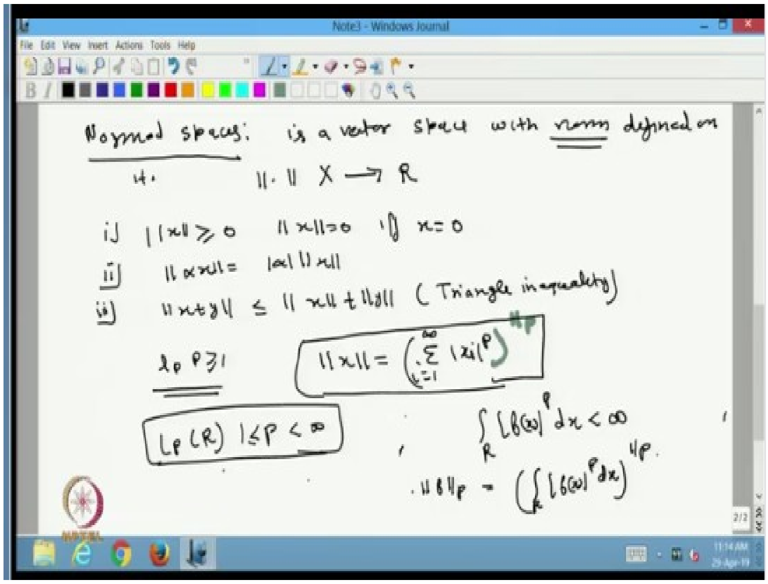
(Refer Slide Time: 04:37)



So what are normed spaces, in normed spaces a vector space with norm what is the concept of a norm, norm is which we define to measure the size of a element of a vector space. So basically normed space is a vector space with norm defined on it more mathematically the norm space is a function the function norm, norm is a function from vector space X to R which satisfy the following axioms and norm of x is 0 if and only x is 0 ok.

So norm satisfy these 3 definitions first definition is norm of X should be greater than 0 and norm of X is equal to 0 if and only if that element itself is a 0. Second definition is also if alpha is a element, any real number and the third property of a norm is called triangle inequality ok. So now the norm on a vector space l p which we have already discuss this vector space, so what will be the norm on this vector space, I am giving you the definitions this way we can define the norm ok clear to everyone.

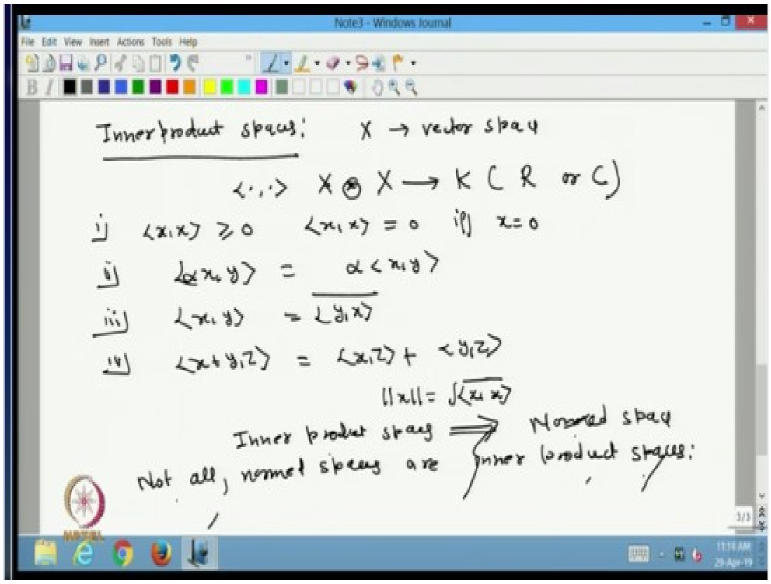
(Refer Slide Time: 06:54)



As I said that vector is the element of a vector space can be a sequence element of a vector space can be vectors, element of a vector space can be a functions as well. So this is the example of a different vector space the element of which will be functions ok l p R this is a very famous space which is a set of a all measurable function such that this property is satisfied ok, this is called l P R spaces this is a vector space.

And if I want to make it the normed spaces I have to define a norm on it, so the norm on this vector space is defined by this ok. So we have seen 2 examples of a vectors spaces, in this case the element of a vector space was a sequence. In this case element of a vector space was a function ok and like simple vector spaces we all know we have already studied in coordinate geometry like r2, r3, r4 they are called finite dimensional vector spaces.

(Refer Slide Time: 08:30)



So now let me give you the explanation of a inner product space first of all I wanted to ask you a simple question what is the need of inner product space ok that I will you could think when I am defining the definition of inner product otherwise I will answer at the end. So an inner product space X is a vector space, so you know normed space, inner product space every space we are defining on the vector space ok.

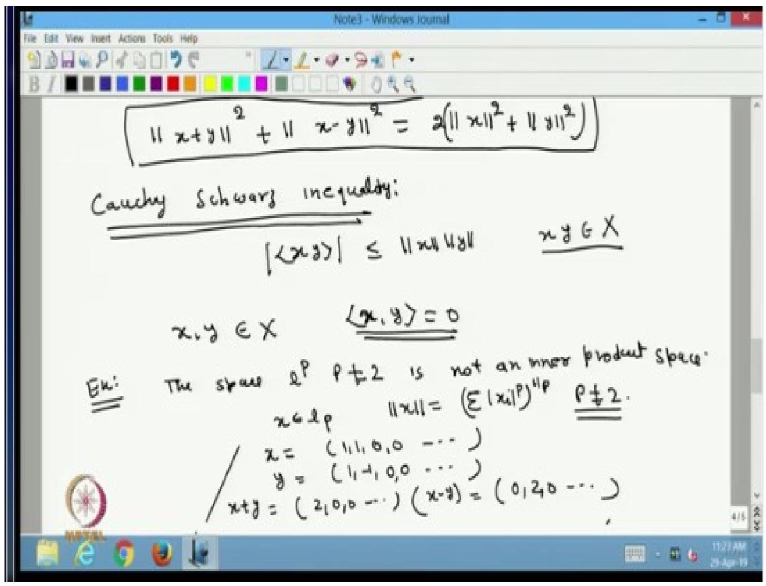
So inner product space is a vector space with the inner product to find on it like a normed spaces we were defining a norm on it. So inner product it is a function from X multiplications by X, so this is a multiplication sign and this is X is a vector space to the field K where this vector space X is defined. So this field can be R or it can be complex numbers, so this can be R or C. So C is a set of a complex number, R is a set of a real numbers.

So if you want to make a differentiation between the inner product function and norm functions, so it is a binary functions. Because it takes 2 operands on it and inner product function satisfy following properties which I am defining now 0 and x and x will be this will be equal to 0, if and only if that element itself is 0, the second property is alpha x into y should be third property is x ok and the last property is ok.

So if any function satisfy these 4 properties it is called inner product space, so this inner product an inner product space on x defines in norm on x by this. So if this is the case I can say all inner product spaces are normed spaces but converse is not true not all normed spaces are inner product spaces. So inner product space it will always be normed space but not all normed spaces are inner product space ok clear.

So now the question comes when normed space can become a inner product space, so for that reason I am defining if norm introduced by inner product space satisfy the parallelogram law means parallel what parallelogram law says.

(Refer Slide Time: 12:06)



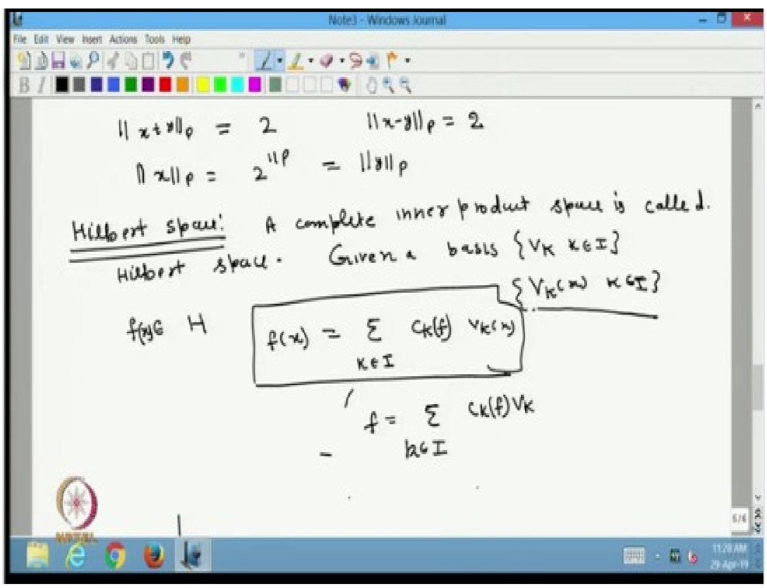
So if any norm satisfy this parallelogram law then it will be generated by inner product otherwise not. So in norm which does not satisfy this property will not be generated by inner product ok clear. Now moreover one another important property which will be used later on is called Cauchy Schwartz inequality. So if x and y are 2 element of a vector space then this property will be satisfied.

I am stating this property without proof and this is a very famous property which is called Cauchy Schwartz inequality. If x and y are 2 elements of a vector spaces ok. Now as I am answering my postponed question which I asked you earlier that why we need inner product space. So this is the time I can answer that question to you and some of you might have already thought about it, 2 vectors if I need a concept of orthogonality.

I have to introduce the concept of inner product, so for that reason 2 vectors x and y are said to be orthogonal, if x this inner product is 0. So to define the concept of orthogonality we need the concept of inner product clear, now let me give you a 1 example the space l p, p is not an inner product space. So how can we prove this example the space l p, p is equal to not equal to 2 is not a inner product space.

So of course it is a normed space we have already defined the norm on it, so if x belongs to l p norm will be defined this way ok. So we have to prove that for p not equal to 2 it is not a inner product space. So of course if I can prove that this norm does not satisfy the parallelogram, I can say it is it will not be generated through inner product, so then my job is done. So what will be for that reason I am taking 1 particular x which is this sequence, another sequence I am taking 1, - 1 ok, so what will be x + y, x + y will be, what will be x – y, clear.

(Refer Slide Time: 15:53)



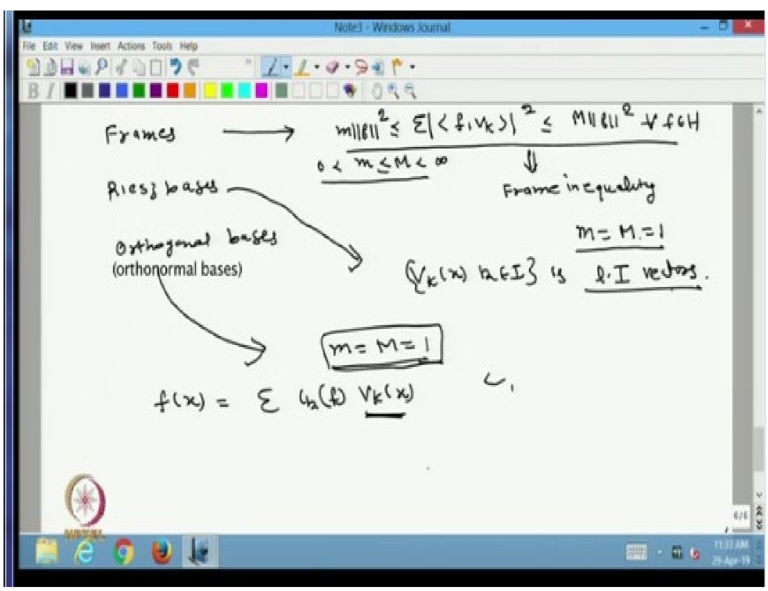
So what will be this norm, this norm will be 2, you can compute it with the formula which I have given you, similarly this will also be 2. But what will be norm of x p this will be 2 to the power 1 by p norm of y will also be p, so if you look at parallelogram law which is this you will be able to see that for this particular example of x and y, this parallelogram law is satisfied only in case of a p = 2, so that is why it is a inner product space if P is 2 and if P is not equal to 2 it is not a inner product space.

So is it clear to everyone, so this was the basic about vector space, normed space and inner product space. Now I am going to define another important spaces which is called Hilbert space ok. So in a very simple language I can say a complete inner product space is called a Hilbert space ok. So given a basis V K where K belongs to I C, so what this V K can be a sequence this can be sequence of a vector sequence of a functions ok.

So in case of a functions we could write V K (x) K belongs to I, so suppose we are considering F is any function which belongs to x ok H, for Hilbert space we are writing H ok. So if V K (X) is a basis for H we can write function which belongs to the Hilbert space like that, if you want to associate this coefficient with f, that is why I am using this notation C K multiply by x ok. So if f is a vector we could write this way ok that is just a notations ok.

So why I introduce the concept of Hilbert space as in a definition wise as I said a complete inner product space is called Hilbert space and most of the time you will see later on where wavelet is developed in l 2 R spaces and l 2 R spaces is a Hilbert space ok. So bases are this, now I will introduce with 3 different concepts of a bases, bases all of you already know because all of you already know this linear algebra stuff. But now at the advanced level basis also categorized into a different category and what are those bases I am going to explain you.

(Refer Slide Time: 20:09)



One of the thing I am frames Riesz bases and then I will come to orthogonal bases ok. So I will give you definitions step by step, first of all let me introduce ok what are called frames. Let me connect this frames definition with the previous thing, so if I wanted to say that this V K (x) are frames for a Hilbert space H, what are the conditions I need that then I will be needing that if f can be written like that ok.

Moreover this V K (x) satisfies some additional property what is that additional property, I am writing that additional property here ok for all f belongs to H, what is this is small m and capital M, a small m and capital M are constants which satisfy this inequality. Moreover when V K satisfy, this condition is called ok frames inequality and a small m and capital M are called frames bounds ok.

If m will be equal to capital M it will be a tight frame ok and if m = capital M = 1 what it will turns out to be I will come to that point later. So all of you are clear with this frames definitions frames are basically V K (x) which satisfy this frames inequality will be called frames. Now this VK x can be linearly dependent vectors in frames but to come to the definition of a Risez bases 1 additional assumption which I am making that this is a linearly l I linearly independent vectors ok.

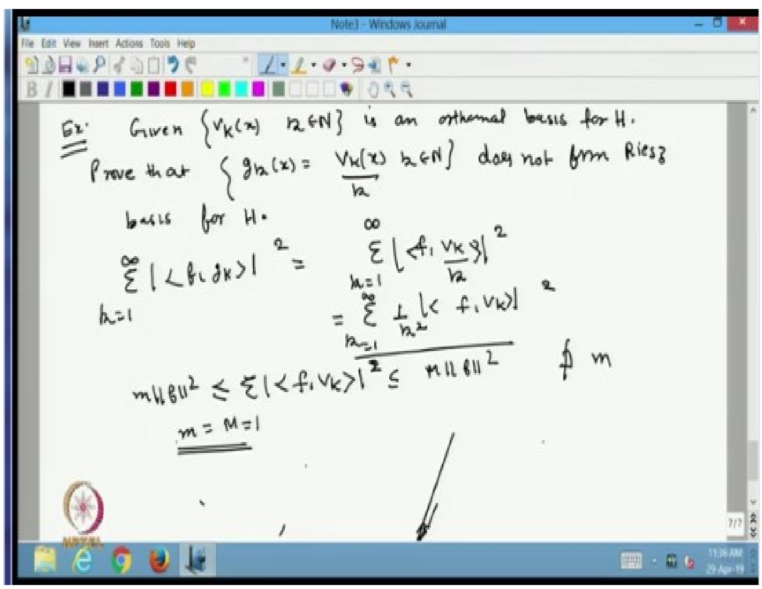
Of course again let me make one more point this vector does not means that really these are vectors, vectors means these are a element of a vector space which can be a functions as well. So what is the difference between frames and Risez bases, Risez bases also satisfy the frames inequality the only difference this time is that V K (x) will be a linearly independent vectors, while in frames V K (x) can be a linearly dependent vectors ok.

So now the third concept is orthonormal bases, in orthonormal bases as I said when m is equal to this, this is moreover V K (x) is again a linearly independent vectors then we call it as a orthonormal bases ok. So now I will explain these 3 things initially according to their properties and then I will give you some examples ok. So let me give you a 1 example, so every orthonormal basis is a frames with Capital M = 1 which is a frame bound ok and of course it is also a tight frame.

Because this a small m = capital M it then we call it as a tight frame ok, so in each case we can write this way but in case of a frames this can be linearly dependent vectors. However in case of these bases and orthonormal bases this will be a linearly independent vectors. Moreover frames will turns out to be orthonormal bases if this condition is satisfied, so this should be clear to everyone. Now why we need frames, why we need Riesz bases, why we need orthonormal bases ok.

Because so far we use to a study orthonormal bases as some of you might have noticed in case of a Fourier series e to the power I K x gives you an orthonormal bases for the corresponding space. But sometime we should notice that orthonormal bases are difficult to achieve that is why we are relaxing those criteria to the Riesz bases and frames ok. So but there is some advantage of frames also that I will tell you later on with respect to signal representation.

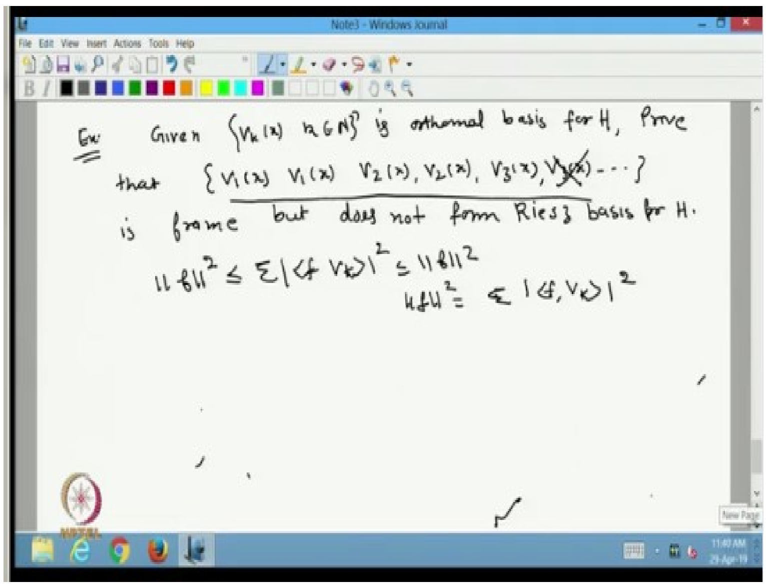
(Refer Slide Time: 25:51)



Now let me work out the one example given V K (x), here I am working out with index at N which is a set of a N natural numbers is an orthonormal basis for H then prove that does not form Riesz basis for H ok. So first of all I have to if V K it is given that V K (x) is an orthonormal basis, so it will be linearly independent vectors therefore this will also be a linearly independent vectors. Now I have to look at the frames inequality or sometimes it is also called frames conditions, whether frames conditions are satisfied or not that I am going to check what ok.

So now V K is an orthonormal basis, so it will satisfy this thing, of course here you could consider both has a 1. But in this case if you plug this inequality here, can you achieve a small m here. So the answer is, of course no there does not exist m for which this inequality will be satisfied in case VK (x) I will replace V K by K. So that is why we because it does not satisfy the frame conditions that is why it does not forms a these spaces clear to everyone.

(Refer Slide Time: 28:40)

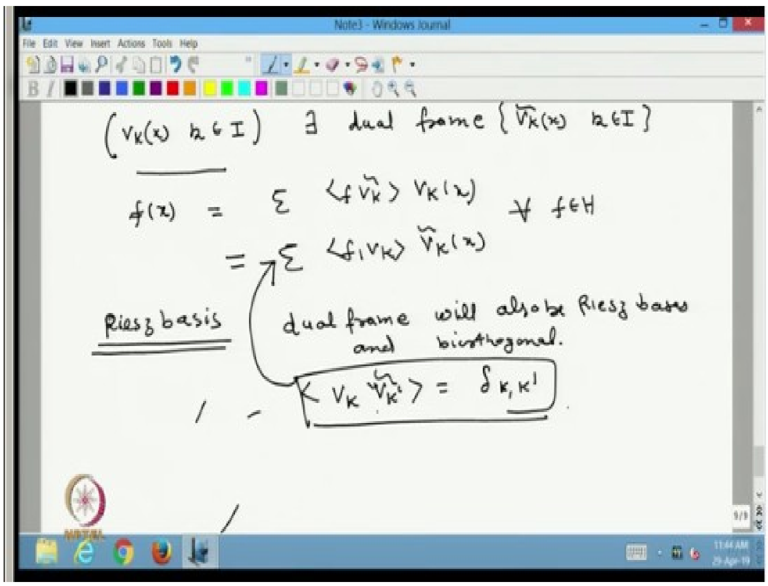


Now another example let me take given V K (x) K belongs to and is orthonormal basis for H prove that V1 (x), V1 (x), V2 (x), V2 (x), V3 (x), V3 (x) is frame but does not form Riesz basis for H ok. So this is the example we are working given V K (x) is an orthonormal basis for H prove that this set where we have repeated each entry V1 (x), V2 (x), V3 (x) is a frame but does not a Risez basis for H, why it will be a frame if V K (x) is an orthonormal basis it is also a frame with frame bound 1.

But the so it will become frames with frame bound 2 you could prove easily that if this set satisfy this conditions, so basically norm of f square is ok. Then in this case what this frame bound will be 2 ok, so but it is a linearly dependent set not a l I set therefore it does not form a Riesz basis for a Hilbert space H, clear that is the example. Now if I give you another example, where I cut this term.

Now you have to tell me whether it is a frame or not, it will be a frame, next question is will it form a Riesz basis no it will not form a Riesz basis. Because again it is a linearly dependent some of the vectors are linearly dependent like V1 (x) is repeated, V2 (x) is repeated. But what will be the frame bound frame bounds might be different ok, so given V K (x) is an orthonormal basis for H prove that is a frame but does not form a Riesz basis for H ok. So now another example I can take if there is a frame V K (x) not example another property of a frames let me mention in another page.

(Refer Slide Time: 32:03)

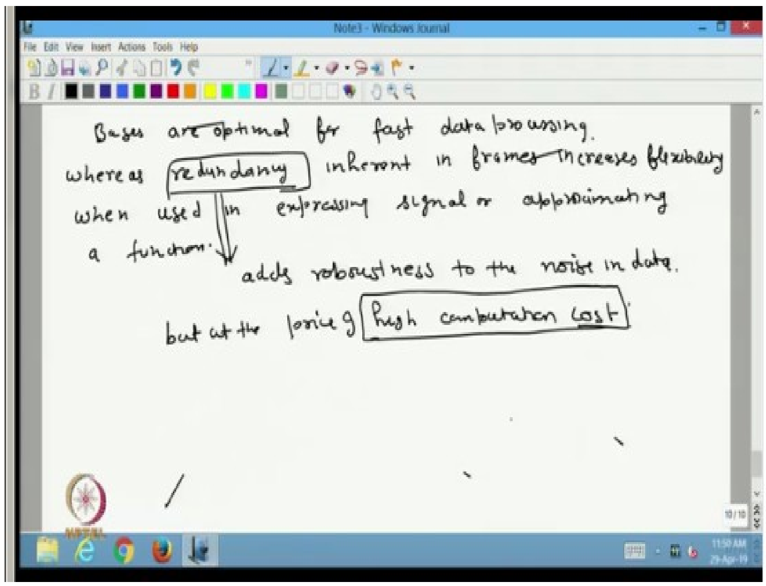


If V K (x) K belongs to I index set or you can also consider that as a natural numbers is accompanied by dual frame. So if this is a frame then there exist a dual frame V K (x) tilda K belongs to I, so whenever there will be a frame there will be a dual frame it will always be accompanied part. So in this case, I can write this or this can also be written like ok, for all f belongs to H, so again if frames are linearly independent set for us then frames gives basically a Riesz basis ok.

So now in case of a Riesz basis also dual frame will also be a Riesz basis and biorthogonal dual frame, if frames are Riesz basis dual frame will also be Riesz basis and biorthogonal ok. And what is this property biorthogonality, biorthogonality property or dual frame will always satisfy this property ok. In case of a Riesz basis these dual frame will also be a Riesz basis and biorthogonal.

Because of this property we are able to write this if you look at, so now the question comes up what is the advantage of frames ok. Because if you feel the difference in the definition, frames are allows us to write the expansion of a function f in terms of a linearly dependent vectors. So it helps in representing the noise of a signal ok, so in a nutshell, frames are like increases flexibility in representing the noise of a functions or basically whereas the basis so in a nutshell or maybe I can write on new page.

(Refer Slide Time: 35:33)



Basis are optimal for fast data processing, whereas redundancy inherent in frames increases flexibility when used in expressing signals or approximating a function it also adds robustness to the noise in the data but at what cost. It is means as I have already said basis are optimal for fast data processing we are as redundancy inherent in frames, this redundancy means in terms of Ld vectors you could measure, whereas redundancy inherent in frames increases flexibility when used in expressing signal or approximating a function.

So this linearly dependent vectors are also helping in some way to us ok, it acts robustness to the noise of the data this redundancy adds robustness to the noise in data. But at what cost that is a been concern for us at the high computation cost. So the answer to that concern is at the price of high computational cost it added robustness to the noise in the data but at the price of high computation cost ok.

So this anyway this later on I will take 1 example, where I will explain you how bases are helping in fast data processing and how frames are helping in representing robustness to the noise in data at the cost of high computation ok. So the difference between frames Riesz basis and orthonormal bases should be clear to you this is the main motivations behind this lecture, because later on to wavelets I need these concepts ok.

Because many people are like very furious or like they do not understand what is basically wavelet, wavelets are simple one of the bases functions. Sometimes wavelets are frames, sometime wavelets are orthonormal, sometimes wavelets are Riesz bases, so that is why it becomes very important for me that you understand these 3 things frames Riesz bases and orthonormal bases ok.

So wavelets are simple bases function nothing else and if you look at the application of a wavelet to in applied mathematics they are enormous ok enormous I could should say. But to understand it more clearly first of all you should have a grasp on 3 different things frames, Riesz bases and orthonormal bases and that was the motivation behind this particular lectures. In the next lecture I will take some more examples to explain you how bases are helping for fast data processing.

And how frames are helping in representing signals of course at the price of high computation cost that I will consider in my next class so thank you for today’s lecture.