Introduction to Methods of Applied Mathematics

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Lecture-23

Fourier transform method for solving ordinary differential equations

Hello viewers welcome back to the course, so today I am going to take the last lecture of this first part of this course. So in the last class we have started with the Fourier transform and then we have discuss the various properties of the Fourier transform, so from the last class I remember that for a functions satisfy the sufficient conditions for the existence of the Fourier transform.

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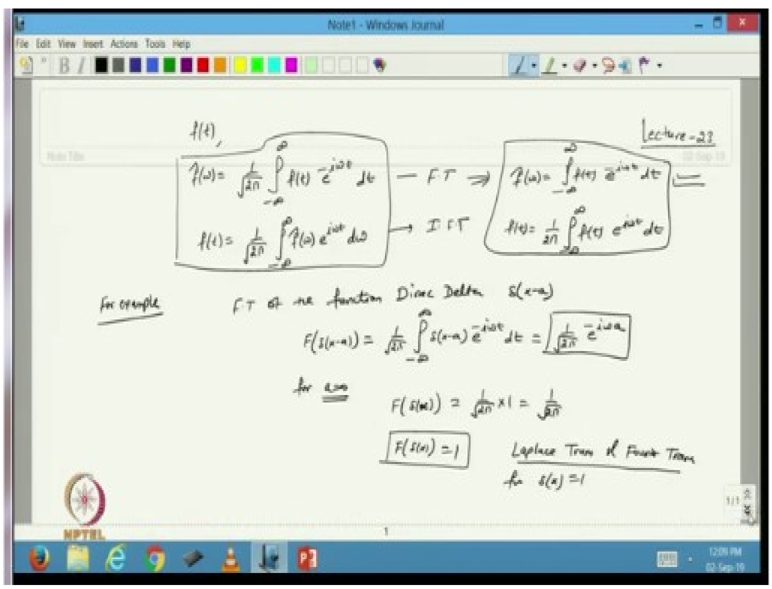

Then I know that for a function f(t) or sometime I also write in the form of f(x) my Fourier transform is given as f (t) e raise to power - i omega t dt, where omega is angular frequency and it is a any real number. So in that case, so this is my Fourier transform and then from there I can go back to my physical space by taking the inverse Fourier transformation, so that we have written like this one.

So **-** infinity to + infinity, then the f hat omega e i omega t with respect to d omega, so this is the Fourier transformation we call it F T and this is the inverse Fourier transformation I. F T. But if you see in the some books they have taken the Fourier transform in this form also, so generally they take the Fourier transform in the some books like this one. So they have taken the Fourier transform like this except 1 over under root 2 pi.

And then they take the inverse Fourier transform as 1 over 2 pi - infinity to infinity f (t) e i omega t dt 2 pi. So some books follow this type of the expression for Fourier transform in the inverse Fourier transform but in our case we are taking this one. So for the same value of the function you may have the different type of Fourier transform because of this factor 1 over under root 2 pi.

Now, so for example in the previous class I have discuss that the Fourier transformation of the function that is Dirac delta, so delta x – a. So if you remember then the Fourier transmission f of x - a it was from 1 over under root 2 pi from - infinity to infinity e raise to power - omega t dt and it is coming 1 under root 2 pi e raise to power - i omega a, so that was my the Fourier transformation for the Dirac delta function.

Now suppose I take for a = 0, so in that case my Fourier transformation of the delta function will be 1 over under root 2 pi and into 1, so in this case this will be under root 2 pi. But generally some books also they take that the Fourier transformation of Dirac delta function is 1. Because they are using sometime this type of expression for that, so in that case this value will be 1, but here we are using this 1.

So that is why the value 1 over under root 2 pi is coming, so suppose I take this value then you can see from here that if you remember from the Laplace transformation then the Laplace transformation and Fourier transformation for Dirac delta function is 1. So in this case both the transformation are coming 1.

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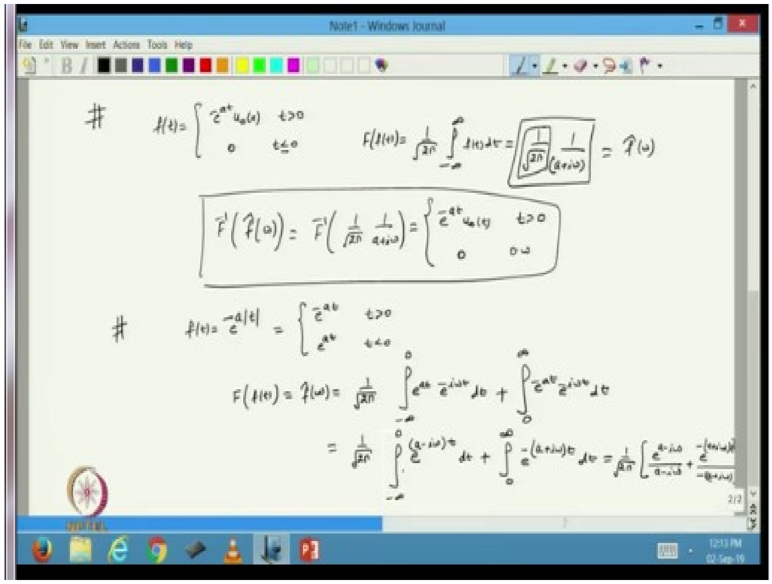

So that was the Fourier transformation for the Dirac delta function and then we can have a different type of functions. So in the previous class also then we have discuss the Fourier transformation for the function f(t) = e raise to power - at into unit step function when t is positive and 0 when t is negative or = 0. So that was the we have discuss with that and in this case if I take the Fourier transformation F of f(t) my f(t) dt.

Then we have seen that this is equal to 1 over under root 2 pi into 1 over a + i omega, so this factors is multiply by this one ok. So this is my Fourier transformation and what will the inverse of Fourier transformation, so suppose this is my f of omega and now I want to take the inverse of this one. So inverse of this one will be 1 over a + i omega is equal to that will be always equal to e raise to power – a t u0 t when t is and 0 otherwise.

So that is the function we have, so that is the inverse Fourier transformation for the same function. Now we can have another function then also we have discuss that suppose I take the another function my f(t) = e raise to power - a mod t. So in this case this will be equal to e raise to power – a t when t is greater than 0 and e raise to power a t when t will be less than 0. So in this case also if I take the Fourier transformation of my function f(t).

Then this will be 1 over under root 2 pi and then I will split my integral from - infinity to 0, it will be e at e - i omega t dt + and then from 0 to infinity it will be e - at - omega t dt. So if you further solve this one, so then this will be, so it can be written as at, so I can write from here this will can be written as a - i omega t. So this one can be written like this + e raise to power - I can take common, so a + i omega t dt and if I further solve this equation.

So the integral will be e raise to power a - omega over a - i omega + a + i omega t divided - a + i omega, so this will be the expression for that, so which further can be solved.

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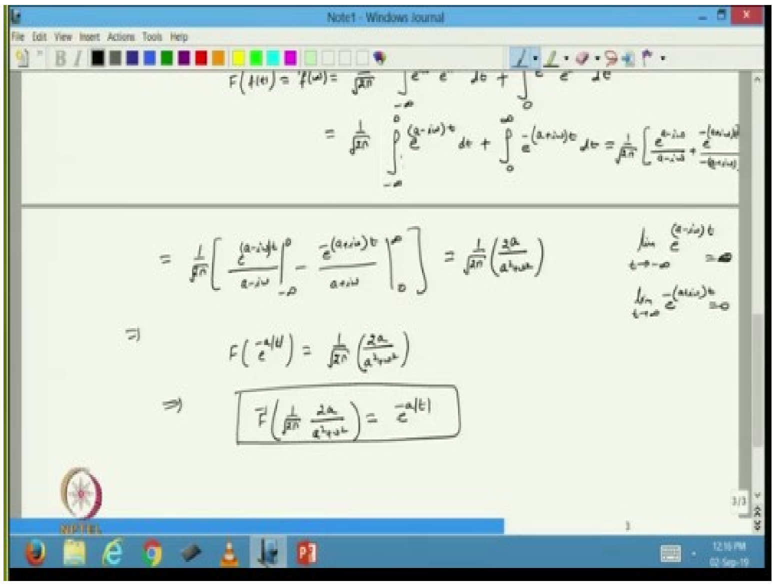

Now I will get a - i omega and that the limit I have to put the limit from here, so in this case, so this expression. So from here I can write that now this is equal to 1 over 2 pi e a - i omega from - infinity to 0 that was the limit – a + over a + i omega, so this will be from 0 to infinity. And now this expression we know that the limit t times to - infinity e a - i omega t that will be 0 and the limit t times to infinity e raise to power - a - omega t a + omega t that will be 0.

Because this is a decay function and when t tends to - infinity this value will be 1 over infinity and that will be 0, so in this case this will be 0. So from here if I solve it further, so I will get this expression 1 under root 2 pi, 2a over a square over omega square. So if I solve further from here I will get this value, so from here I can say that the F of e raise to power - a mod t will be equal to 1 over under root 2 pi, 2a over omega square.

And from here if I want to take the F inverse the inverse Fourier transformation of this expression then it will be - a mod t, so that will be the inverse Fourier transformation for the Fourier transformation of the function e raise to power – at, so these are the Fourier transformation of few functions.

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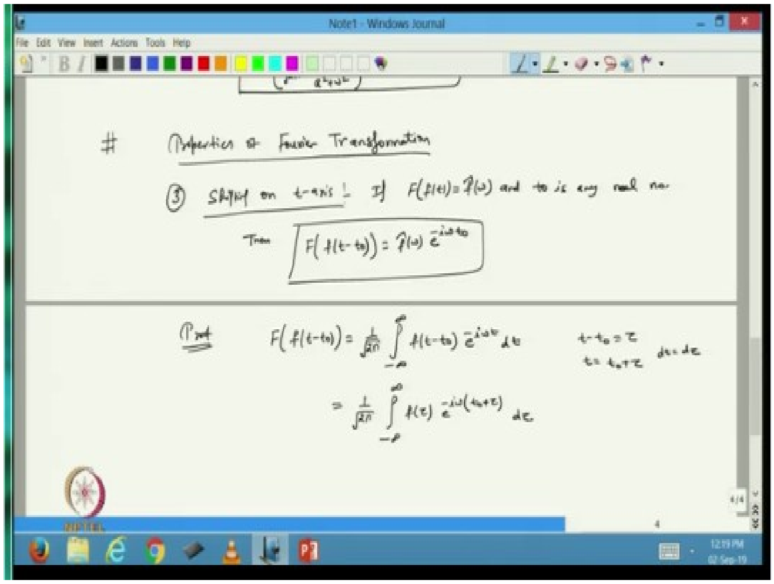

Then we define some properties of the Fourier transformation, so now few properties linearity property and the Fourier transformation of the derivative we have already done. So today I will start with the third one the third property and that is called the shifting on t axis, so it is same as the shifting theorem what we have discuss in the Laplace transformation, so shifting on t axis. So in this case if I take the Fourier transformation of the function f(t) and we write as f hat omega and suppose t0 is any real number then f of (t) - t0.

So I just do the shifting in terms of t then this value will be equal to f hat omega e raise to power - i omega t0. So in this case they no need to solve the Fourier transformation again we can use this property. So in the proof for this one is I want to find the Fourier transformation of the function f(t) – t0, so this is 1 over under root 2 pi from - infinity to infinity f of (t) – t0 e - i omega t dt. Now I change my parameter, so I take this transformation.

I take t – t0 is equal to some tau, so from here I will get t0 + tau and also I get from here that the dt will be d tau and when the t will be - infinity tau will also – infinity, when t is infinity then the tau will be infinity. So from here if I apply this transformation, so I will get 1 over under root 2 pi f of tau, so this is the tau we have taken e - i omega and see f t I put t0 + tau and d tau.

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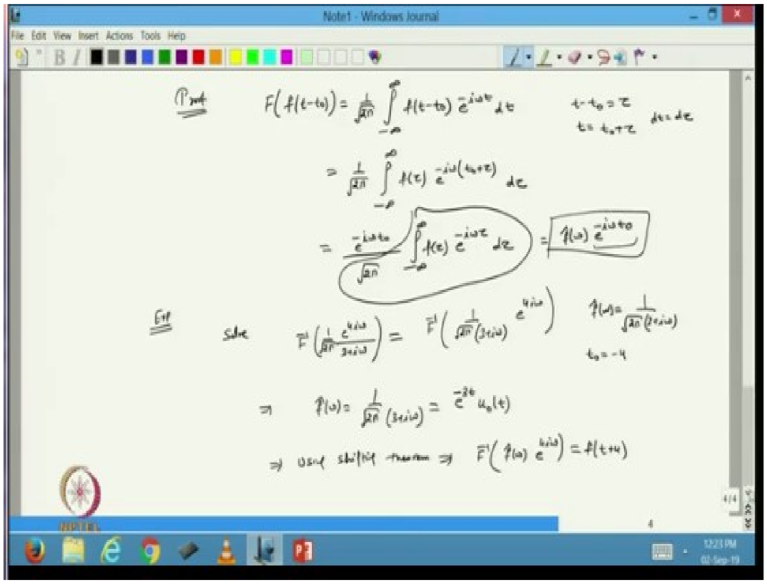

So from here I can take this part common, so because I am taking the integration with respect to tau now, so this is a constant. So I can take this one outside I will get - i omega t0 divided by under root pi from - infinity to infinity d tau and if you remember then this is the Fourier transformation of the function f. So from here I can write that this is equal to e - i omega t0, so that is the Fourier transformation of the function which is shifted on t axis by t0.

So let us do one example, so let us take the example, so suppose I want to take the inverse Fourier transformation of the function. So let us define, so let us solve f of inverse of e 4 i omega divided by 3 + i omega under root pi. So this one I want to solve, so from here if you see then I have to compare these functions with the shifting theorem because here I am getting e raise to power something.

So that is also here and have the linear Fourier transformation of the function f hat omega is somewhere here. So from here I can see that, so this one I can written as 1 over under root pi 3 + i omega I take separate and I write it here 4 i omega. And then I will compare here with this value, so from here I can say that I am getting my f omega is equal to this one 3 + i omega and from here I can say that t0 will be - 4 because t0 is – 4.

So from here now my f omega is 1 over 3 + i omega, so that is equal to if you remember, so this will be equal to e raise to power - a is 3 t into u0 (t). So that is my expression for which the Fourier transformation at this one because this one we have solved here in the, so this one. So from here I apply my shifting theorem, so using shifting theorem I can write that in this case the F inverse of f hat omega into e raise to power 4 i omega will be, so using this property I will write that this will be equal to the f of (t) – t0 sum, so it will be 4.

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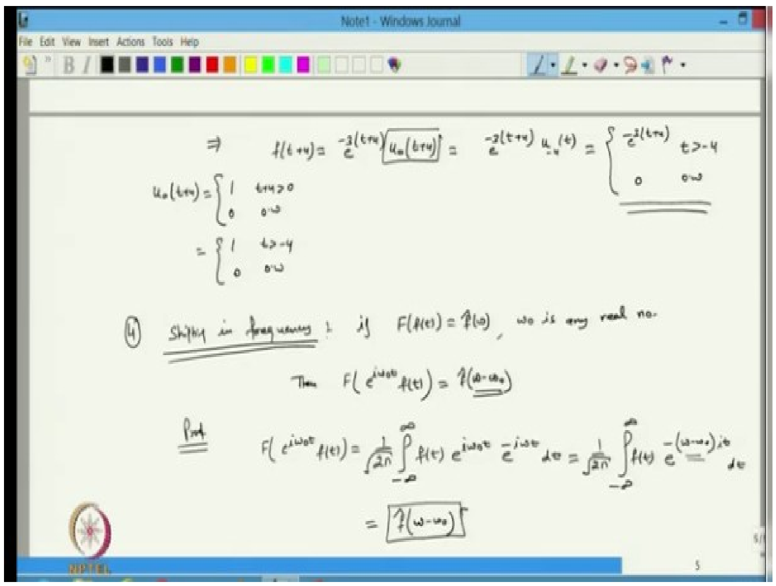

So from here I can write, that the answer will be f t + 4 and this is equal to e raise power - 3t + 4 u0 t + 4. So from here I can write this one as e raise to power – 3 t + 4, so now this quantity is u0 I can write as from here I can add u0 t + 4 will be 1 when t + 4 is positive 0 otherwise. So from here I can say that this is equal to 1 when t is greater than - 4 and 0 otherwise, so from here I can say that this is equal to the function u – 4 (t).

So this function can be written like this one and which further can be written as e raise to power - 3 t + 4 when t is greater the - 4 and 0 otherwise. So that is the inverse Fourier transformation for this function and the answer is this one, so this is the answer for that. So now we can take the another one the fourth property and that is called another shifting theorem, so shifting in frequency, so in this case if the Fourier transformation of F f (t) is f hat omega.

And let us we choose omega0 is any real number then if I take F of e i omega0 t f (t), so I want to take the Fourier transformation of the function multiply by this 1. So this will be equal to f hat omega – omega0, so this is we have done the shifting in terms of omega, so the proof is straightforward. Now I have given by this, so I want to find the Fourier transformation for the function e i omega0 t f (t), so by the definition I will write like this.

So my f (t) is there e i omega0 t e - i omega t dt, so from here I can write this expression from - infinity to infinity f (t) e and from here this i can take common. So I can take i common and t common, so from here I can write - omega – omega0 i t dt because it will be e raise to power - i omega - omega i t - omega i t and this negative will be go inside set will be this one. So from here, so if you remember this one, so this is same as the Fourier transformation of the function except this factor.

So from here I can add that this should be equal to f hat omega – omega0, so that is the proof of this one, so that is called the shifting in the frequency.

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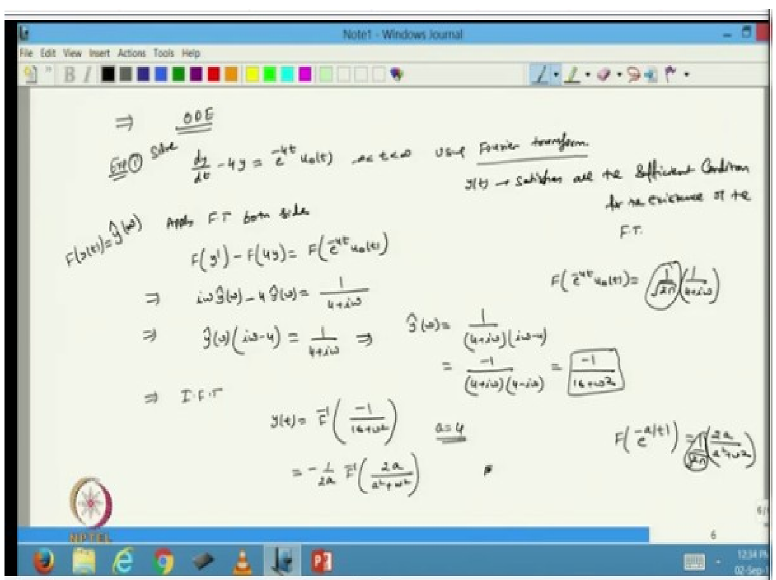

Now so after solving this problem, so let us solve few differential equations, so in this case I want to solve some ordinary differential equation with the help of Fourier transformation. So let us take example one, so one example I have taken I think in the previous class, so the same I just take the same type of example. So let us take this one dy by dt – 4 y = e raise to power – 4 t u0 t where t is defined from - infinity to infinity.

So one thing you should keep in mind that in this case suppose I want to solve by the Fourier transformation, so the question is that solve this one solve this using Fourier transform, so that is the question. So and we know that the Fourier transformation can be applied only when the functions satisfy some conditions that the function is piecewise continuous in the finite interval it is absolutely integrable.

So in this case we assume that the y (t) satisfy all the sufficient condition for the existence of the Fourier transforms. So let us take the Fourier transformation, so apply Fourier transformation both side, now from here I can write that F of so this be can we written as y dash basically I am taking - F of 4 y = Fourier transformation of e raise to - 4 t u0 t and this is equal to I know that this is omega, so let us take that F of y (t) is this one.

So from here I can write that this will be equal to this one - 4 times and this one will be equal to 1 over a is 4, so 4 + i omega, so we know that this is 4 equal multiply. So if you remember then F of e – 4 t u0 t is 1 over under root 2 pi over 4 + i omega but whenever we are solving the differential equation we ignore this factor. Because on the left side also we are taking the Fourier transformation, on the right hand side are also we are taking the Fourier transformation.

And we considered that this factor 1 by under root 2 pi got cancelled, so whenever we solving for the differential equation we generally ignore this factor it is just the multiplication by this sum factor. So in this case I will take my Fourier transformation for the function as this only, now if I go further and this will be equal to i omega - 4 = 1 over 4 + i omega, from here I can write my 4 + i omega and then i omega – 4, from - 1 I can write this as 4 + i omega and 4 - i omega.

So from here I can write this as 16 + omega square, so this is my Fourier transformation, now I have my solution in the y (t). So now I have to take the inverse Fourier transformation, so inverse Fourier transformation y (t) will be F of inverse of this function - 1 16 + omega square.

And if you remember we have taken the Fourier transformation of this function e - a mod t it was coming 2 a over a square + omega square 1 by under root 2 pi, so that is I am ignoring.

If you just remember this one, so that is the Fourier transformation I am going to use, now from here I can write that. So suppose my Fourier transformation with this one, so this expression can be written as -, so 2 a I will take. So in this case my a is 4, so a is 4, so I will get my 2 a F of inverse 2 a, a square + omega square where a is 4, so it would be 8 a by 16 + omega square and this one I have taken the factor outside.

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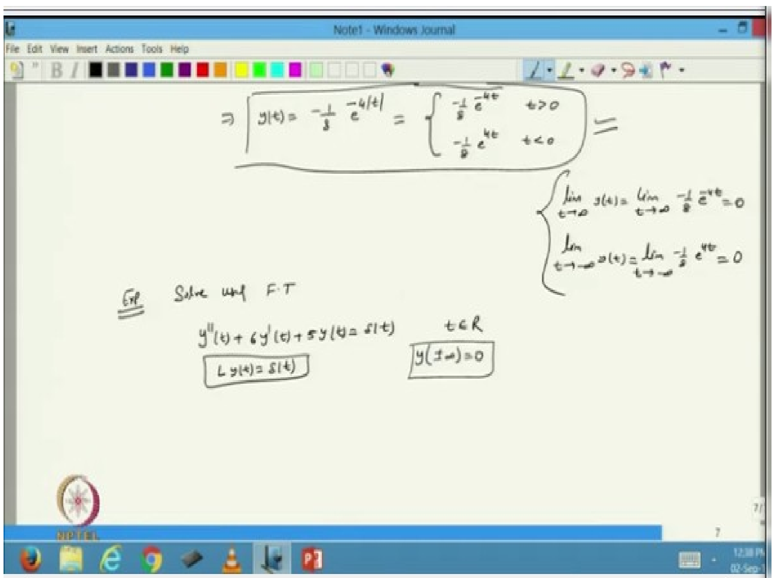

So from here I can write that my y (t) will be - 1 over 8 and this one I can write as e raise to power – a is 4 mod t which can further be written as, so this is equal to - 1 by 8 e raise to power - 4 t when t is greater than 0 and this will be - 8 e 4 t when t is less than 0. Now from here I just want to, so that is the solution, so you will get the solution in the terms of piecewise function. Now from here I just want to verify that from here you can verify that if I take the limit if I define this function then this function is a decay function and it is absolutely integrable.

So I can take the integration of this function and that will be finite the absolute value and also I want to check that what will happen if I take limit t tends to infinity. So if I take t tends to infinity y(t), so this will be limit t tends to infinity - 1 by 8 e raise to power - 4 t and that will be 0, also if I take limit t tends to - infinity y(t). So I will take this limit t tends to infinity - 8 e 4 t, so t tends to – infinity, this is – infinity, so then it is value will be 0, so this 2 properties are satisfied.

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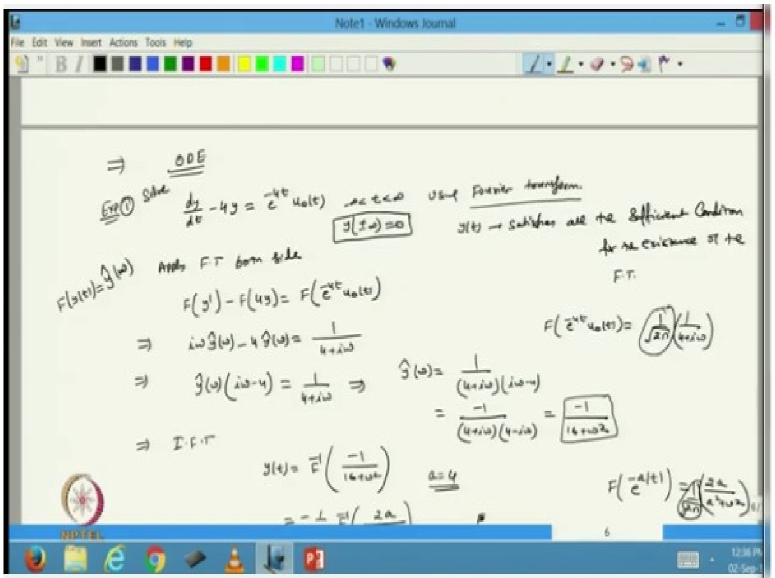

So from here whenever we are solving any of the differential equation ODE with the help of Fourier transformation it is understood that my function satisfy this condition ok. So that is the solution for the first order ODE, so let us take another example, so after solving the ODE I take the second order ODE. So solve using Fourier transform I will take the second order ODE y double dash (t) + 6 y dash (t) + 5 y that is a function of t = delta t.

So this one I am defining and t is belongs to the real line, so if you see that I have taken, so this is my differential operator. So L of y(t) = delta t and this one I want to solve, so if you remember that whenever we have a right hand side function as a delta function and the solution of this equation become the green function. So basically here I want to find the green function of this differential operator, so this is my differential operator Ly L.

So I want to take the L y (t) to find out the green functions, so from here and in this case I know that my function y (t) is a well defined function satisfying all the sufficient condition and also there will be 0, so this is understood that I am taking this one.

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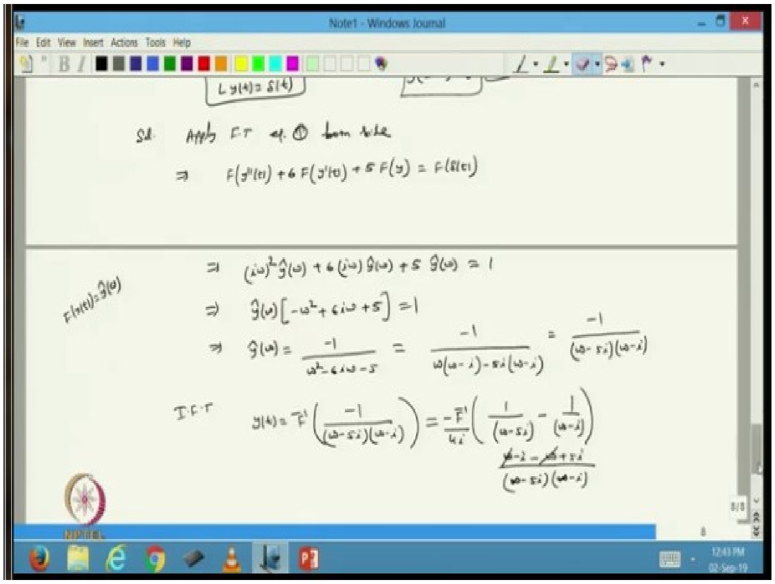

Because this is not given any of the differential equation but whenever it is been told to solve by the Fourier transformation then this is already satisfied no solution. So let us take the apply Fourier transformation to the equation 1 both side, so in that case I will get the F of y dash (t) double dash (t) + 6 of F y dash t + 5 of F (y) = F delta t, so this one I have taken. Now from here I know that this will be equal to, so I am considering here that F of y (t) is F instead of f I just take y omega.

So from here I can write that this would be equal to i omega square that we already know, so this will be equal to this + 6 times i omega + 5 times equal to this is equal to 1. So that is the Fourier transformation that the Dirac delta function, so that is 1, so from where I can write that this will be can be written as, so I take y omega common. So this can be written as - omega square + 6 i omega + 5 = 1, so this one I can further write and I can take - 1 outside.

So this will become omega square - 6 i omega - 5 which can further be written as, so this one I will just take the factorization of this. So I can write this as omega omega - i, so this one - 5 i omega – i, so this can be written as because when it will be omega square - omega i - 5 omega i so it will be 6 omega i and - 5 i square. So it will be + and i square, so it will be - 5 i, so from here I can write then this can be factorized.

So factor would be omega - 5 i and omega - i, so now I want to take the inverse Fourier transformation, so I want my solution in terms of y (t), so this will be F inverse of - 1 into omega - i. So this one I want to take the inverse Fourier transformation, so if you remember the same thing used to happen in the Laplace transformation. So we have to simplify this one by taking the partial fraction, so this one I can be written as if I do the taking the partial fraction.

So you further solving this one, this can be written as 1 over, so if I just I will take the partial fraction it will becomes 1 this will become 1 over 4 i omega - 5 i taking the - sign outside + - 1 over omega – i. So this one be equal to, so this one will be - over 4i because if you take the LCM of this it will be omega – 5 i omega - i then it will be omega - i - omega + 5 i. So this will cancel out this will be 4 i and 4 i will cancel out and will get the same value ok.

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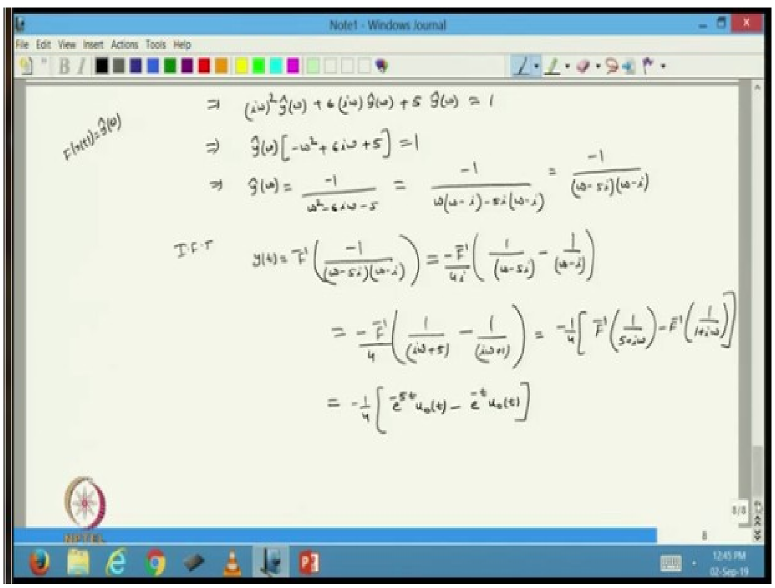

So from here, so that is verified now, so from here I can write that this will be equal to - of F inverse and this 4 i can take inside, so it will be, so I can take just 4 here. So this one can be written as I am just taking the i inside it will be i omega + 5 - i omega - i square, so it will be +, so this is - 1 by 4 and this is satisfying the linearity property. So this is equal to 5 + i omega –I know that it satisfy the linearity property.

So I can take this as 1 + i omega, so which can further be solved and can become now I have to take the inverse Fourier transformation of this factor. So I remember that this is equal to e raise to power - 5 t u0 t where u0 is the unit step function - e - t u0 t, so that is the inverse Fourier transformation which can be written in the form of a piecewise function.

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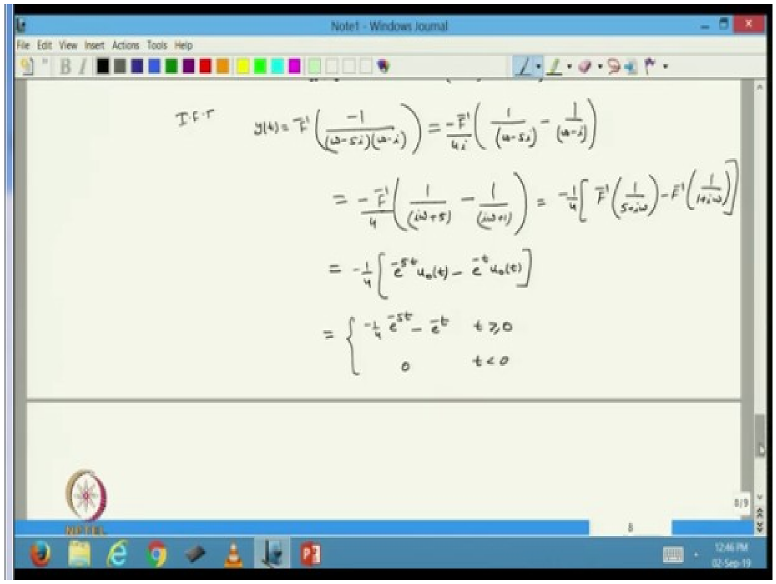

So this become equal to the solution become - 1 by 4 e – 5 t - e - t when t is greater than 0 and otherwise this will be 0 otherwise or I can say here that this is equal to t is less than 0. So that is your solution and when I put t = 0 in this case, so this value will be 1, this value will be 1 so it will be this factor or it maybe I can take equal to sign also here. It depends upon that how you are defining your unit step function.

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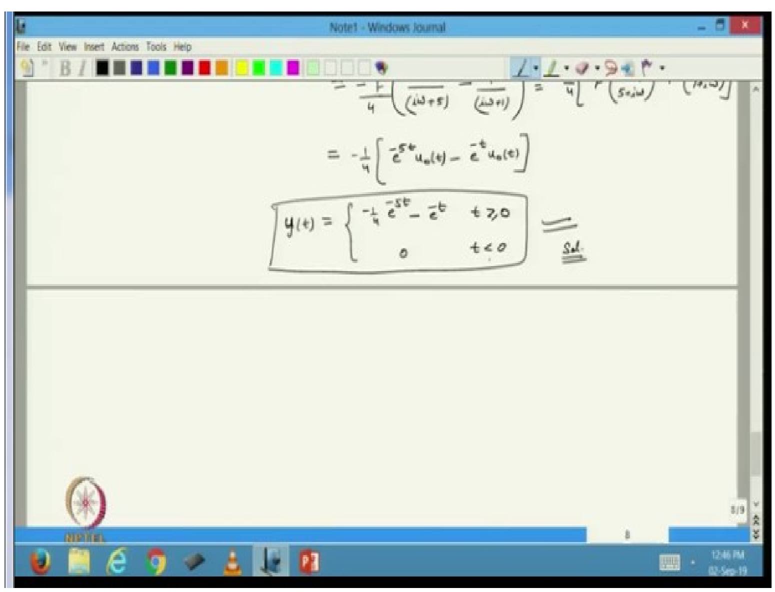

So that in my green functions for my given differential operator and the value of the green function is, so it is splitting at the 0. So this is my solution of the that is my solution of the equation, so this is the solution and also it is satisfying that when t tends to infinity I will take this will go to 0 and t is less tending to - infinity this is 0. So based on this type of equation we can find the solution or we can solve the given differential equation with the Fourier transformation. So let us take another example a little bit the same type of second order differential equation.

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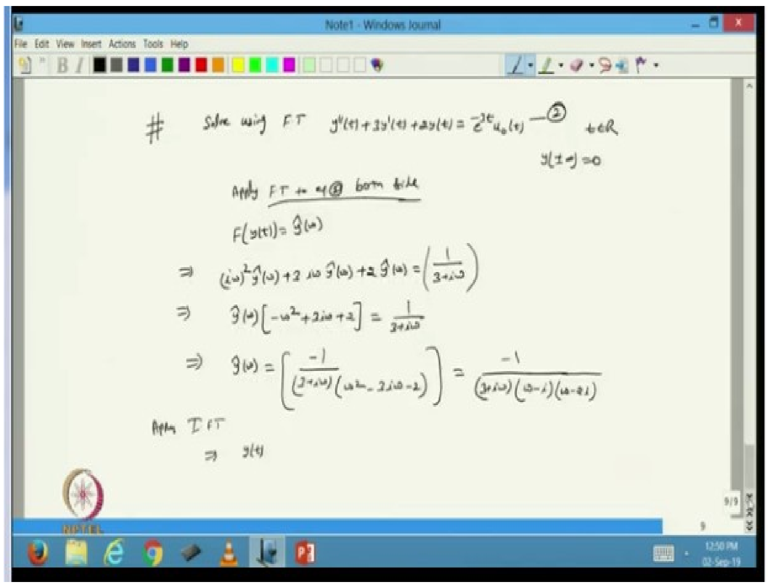

So let us solve another example, solve using Fourier transformation y double dash (t) + 3y dash (t) + 2 y (t) = e raise to power - 3 t u0 ok we had t belongs to realign and as I told you they understood that y + - infinity will be 0 and satisfying all the sufficient condition for the Fourier transformation. Now solution that, so call it equation number 2, apply Fourier transformation to equation to both side only I have taken the equation number 1 yeah, so this is the equation – 2 i.

So I am applying the Fourier transformation to the equation number 2 both side, now the same way I can find, so let my Fourier transformation for my function F of y (t) is y hat (t), y hat omega. So from here I can write directly I can write that this will be equal to i omega square + 3 i omega y hat omega + 2 y hat omega is equal to, so this is equal to 1 over a is 3 3 + i omega, now if I solve further.

So this will be equal to y at omega it will be - omega square + 3 i omega + 2 = 1 over 3 + i omega, so which can be further solve. So from here I can write this function as 1 over 3 + i omega and I take the - sign here. So it will be omega square – 3 i omega - 2 which can be further factorize, so it becomes 3 + i omega and this one I can write the factor form, so it will be omega - i omega – 2 i because when omega square this will be omega - 2 i.

And - i omega – 3 i omega and - i into – 2, so it will be – 2, so I can get the this factors. Now I have to take, so now apply inverse Fourier transformation from there my solution y (t) can be written as f of inverse of this factor - 1 3 + i omega omega - i omega - y.

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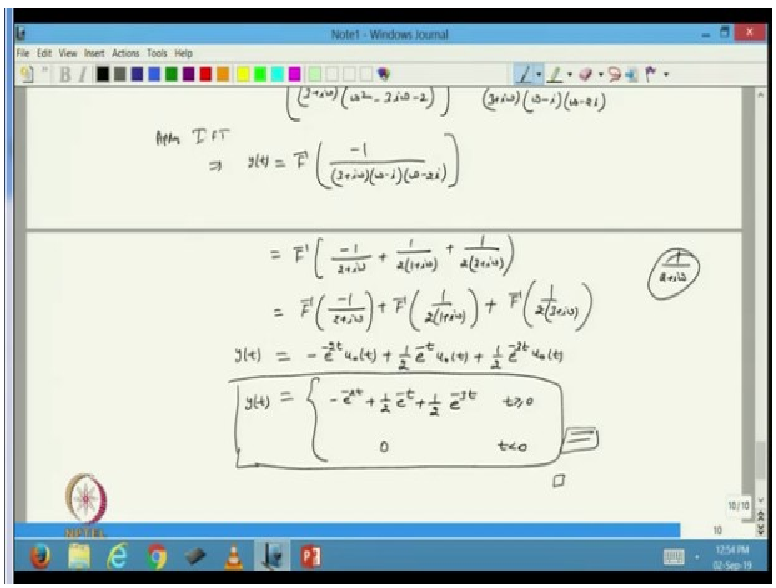

So now the same way we can apply my using the help of partial fractions I can write simplify this function, so if I take the partial fraction of this expression then I will get this value. So this is equal to - 1 over 2 + i omega, so this is coming + 1 over 2 1 + i omega and 1 over 2 3 + i omega because here we have multiply by i. Because we want to take, so here I am taking with i, so it is 2 + 3 i ok 1 + i 1 + this one.

So in this case I just multiply by i, so I will get 1 + omega i here also I multiply by i I will get this one, so this factors are because I know that I want to take the Fourier inverse and Fourier inverse is applicable only for this form whenever I have this form a + i omega. So if I solve this further taking the LCM I should get the same value, so this is the factors we are getting and now I apply the my inverse Fourier transformation.

So from here I can write using the linearity property I can split this to this form 1 + i omega 3 + i omega, so this is equal to - e – 2 t u0 t + 1 over 2 is to - t because a is 1 here u0 t + 1 over 2 e - 3 t u0 t. So that will be my solution which can be written as it would be equal to - 2 t + half e - t + half e - 3 t and this t will be greater than equal to 0 and 0 when t is less than 0, so this is my function and that is the solution.

So this is also given in the piecewise form and when I t tends to infinity this factor will vanish to 0 and when t tends to - infinity this vanish to 0. So that is the solution of the differential equation, **so** so the today class we have discussed that the various application of the Fourier transformation and the inverse Fourier transformation to solving different type of first order differential ODE and the second order ODE.

So and I told you that today is the last class for the first part of the course of this course, so I hope that you enjoyed the course good luck for this course thank you very much.