Introduction to Methods of Applied Mathematics
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Lecture-21
Fourier integral and Fourier transform

Hello viewers welcome back to the course, so in the last class we have discussed the Fourier series. So today I am going to start the another topic and that is called the Fourier integral.

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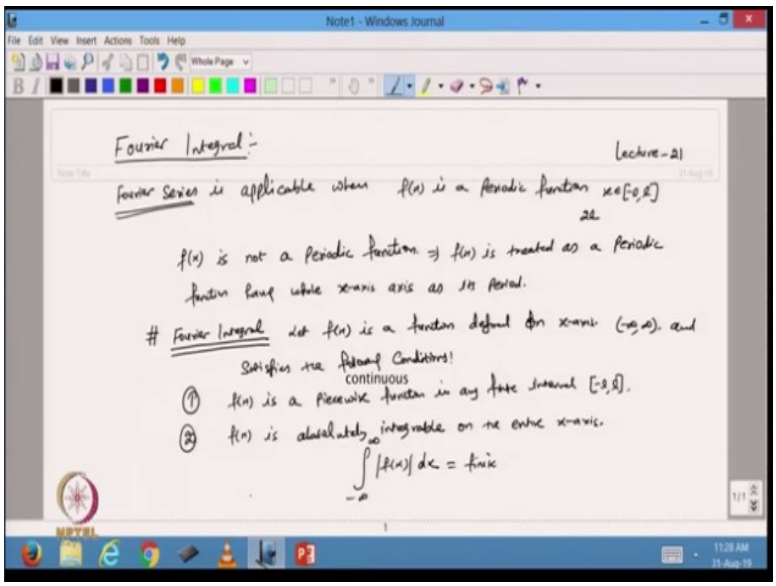

So as we know that the Fourier series is applicable whenever the function fx is a periodic function. So for the Fourier series I know that the Fourier series is applicable when my function fx is a periodic function and generally we define my x belongs to from – l to l. So in this case we know that the our function periodic function with period 2l. Now the things start that will happen when the function f x is not a periodic function.

So in that case so my I have a function my fx, so my function is fx and which is not a periodic function. So what will happen because the function is not a periodic then we I cannot write it is Fourier series. So in that case we generally that if the function is not a periodic we generally call it that in this case my function fx is treated as a periodic function having whole x axis as it is period.

So in this case we will generally say that ok if the function is periodic and period is completely the whole real line. So in that sense I can say that no function is periodic and period of the that function is a whole real line. So in this case now, so to start with a Fourier integral, so I assume that let my fx is a function defined on x axis that is from minus infinity to infinity and satisfies the following conditions.

The first one is that then our function fx is a piecewise function in any finite interval – l to l, so this is my first condition. That my function should be a piecewise type in any of the finite interval. The second one is that my function fx is absolutely integrable on the entire x axis. So what is the meaning of absolutely integrable function it means that my function fx I take the absolute value and I take the integration from – infinity to infinity and with respect to x.

So in that case it should be finite, so if it is happening then we say that my function is absolutely integrable on the entire x axis.

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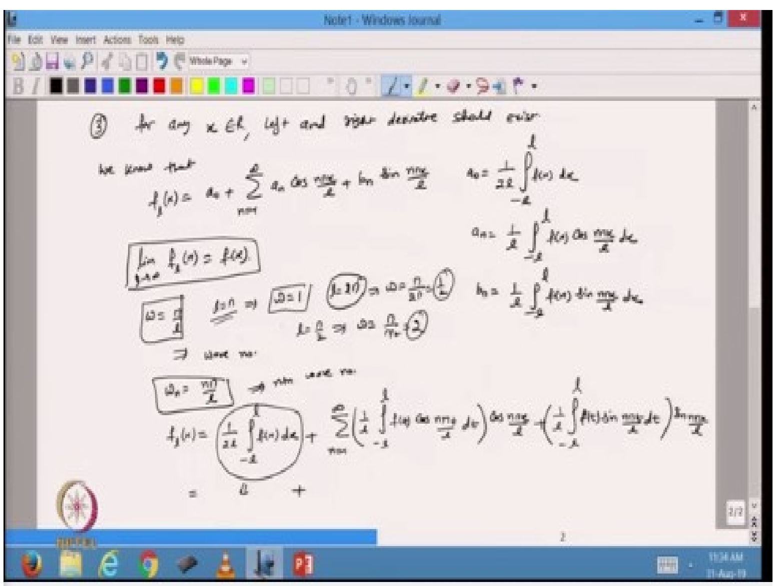

And the third condition is that for any x belong to the real line left and right derivative should exist. So in this case my function is defined and for any x I should be able to find it is left derivative and the right derivative. So if the all these 3 conditions are satisfied, then we are able to write the Fourier integral. So let us see that how we can we start with the Fourier series because we know that about the Fourier series.

And from the Fourier series I should be able to convert that Fourier series into the Fourier integral. So let us see that how it happens, so we know that my Fourier series for the function, so I call it l because Fourier series I know that it is applicable when the function is periodic. So I write the Fourier series the function having the period 2l, so this is given as a 0 + summation and from 1 to infinity a n cos n pi x by l + bn sin n pi x by l where my a 0 is I know that this is equal to the average value of the function dx.

And my a n is 1 over l from - l to l f (x) cos n pi x by l dx and my bn in that case is again 1 over l - l to l f (x) sin n pi x by l dx. So this is already we know, now what we want to do, we want to find out that what will happen if I take limit l tends to infinity f l (x) and this is basically our function whatever the function we are going to define. Because I told you that this function is a periodic function on the entire real line ok, so it has a period as a entire real line.

So this is I am going to do, so now what do you do let us see what will happen I define the parameter omega is pi by l so with this suppose I take l = pi in that case my omega will be 1, if I take l = 2 or 2 pi suppose I take. In that case my omega for this one my omega will be pi over 2 pi, so half ok, when I take my l = pi by 2 in that case my omega will be pi over pi by 2, so that is 2.

So it means it is telling you that how many waves are there from period - pi to pi, so in this case when I take l = pi, it shows that there is only one wave form - pi to pi, when l = 2 pi I am taking then only half wave is there and when l = pi by 2 I am taking then there is a 2 waves. So we generally call it that w or this omega is a wave number, and if I take for all value of n then we define omega n as n pi by l, so this is we call it the nth wave number ok.

So after this I can write my function f l (x) is equal to, so what I do I put the value of the coefficients here itself it is - l to l f (x) dx + summation and from 1 to infinity my n is this so I will call it 1 over l - l to l f (x) cos n pi x by l dx, so this is I have written there from here and this = cos n pi x by l ok. So generally I can because both the x are coming, so what I do I will just make it n pi t by l dt, just I change the indexing plus 1 over l - l to l ft sin n pi t by l dt.

And then I put sin n pi x by pi i , so this is the expression and I put my integral there. So now from here I can write my this factor I will keep it same, so this will be here plus.

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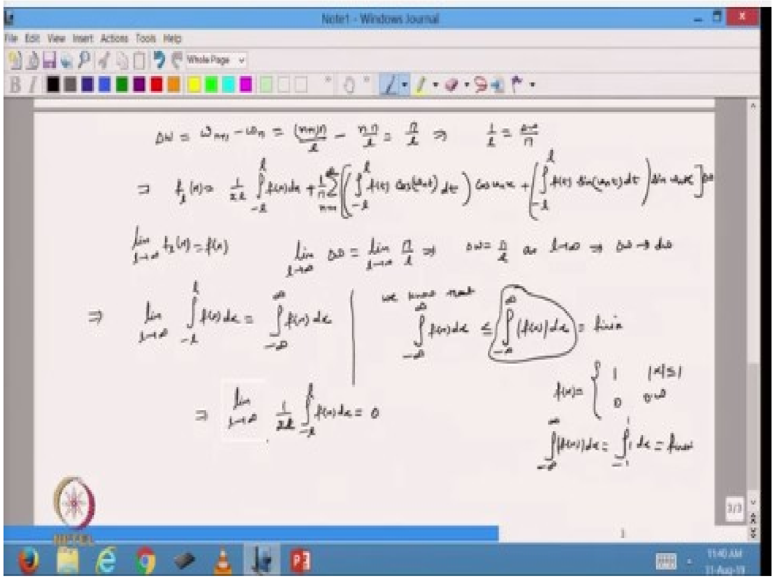

So from here I will try to find out what is my del omega to change in omega. So this will be omega n + 1 - omega n, so that will be n + 1 pi by l - n pi by l, so this will give me only pi by l. So from here I will write that my 1 over l, if I write so that will be del omega by pi. So from here I can write my f l (x) = 1 over 2 l f (x) dx plus so instead of 1 over l I will put in this n from 1 to infinity, I will put 1 or pi here and this is - l to l ft cos n pi t by l dt into cos.

So instead of n pi, so here I have define n pi by l, so I can replace this by cos omega n t this one and this one also I can write like this. So this would be omega n x + - l to l f (t) sin omega n t dt into sin omega n x and the whole del omega, so this is the expression I got from here. Now what I do is that, now what I want to do I want to put what will happen when I take the limit l tends to infinity, so you know from here that I want to check for this one.

So this is I consider that this is my function f (x) and also I want to check what will happen for del omega. So del omega is basically limit l tends to infinity it is pi by l right, so from here I can say that this is, so del omega is pi by l and as l tends to infinity from here I can say that this will goes to d omega. Because this is a just for discreet form and when I put the limit then my del omega becomes d omega.

Because when l is very very large this quantity is going to be very very small, so from here now I want to see that what will happen when I apply this limit for each of the term here. So now let us see that what will happen if I put limit l tends to infinity to this integral, the first integer - l to l f (x) dx. So basically it will come this is same as this one - infinity to infinity f (x) dx. Now we know from here we know that any integral f (x) is less than mod of dx.

So this we already know right, so from here and this is true for any integration and what is this, so this is already in the condition I have started with that my function is absolutely integrable it means this value is a finite value. Like for example I take a function I define a function f (x) is equal to suppose 1 when it is less than equal to 1, 0 otherwise. So in this case I want to check whether this function is a absolute integrable or not.

So I can take my limit from - infinity to infinity f (x) mod x dx and this is basically from - 1 to 1, 1 into dx, so that is a finite value. So I can say that this function is absolutely integrable, so I know that this is the condition I have started with and this is equal to finite so from here I can say that the limit l tends to infinity 1 over 2 l - l to l f (x) dx become 0. Because this quantity is finite and divided by the Infinity and tends to infinity so this becomes zero, so from here I can say that the first integral will be my 0.

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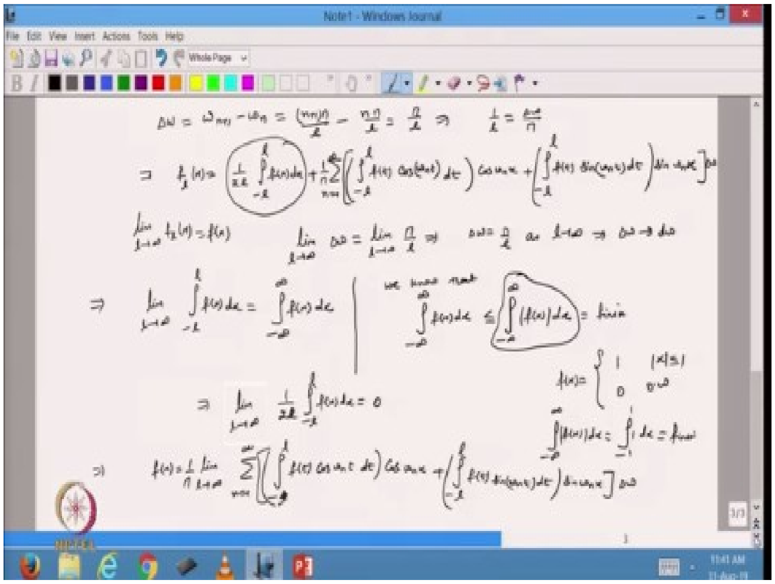

Then I start with the second one, now what will happen, so now I put the limit so left side I will get f (x) and on the right-hand side what I will get, I will get first integral is 0. So from here I can find my pi from here then I will get and from 1 to infinity and this quantity can be written as minus infinity, so - l to l f (t) cos omega n t dt cos omega n x + - l to l f (t) sin omega n t dt, sin omega n x and whole this one is del omega. But I just now we have done that when the l tends to infinity this del omega becomes the d omega.

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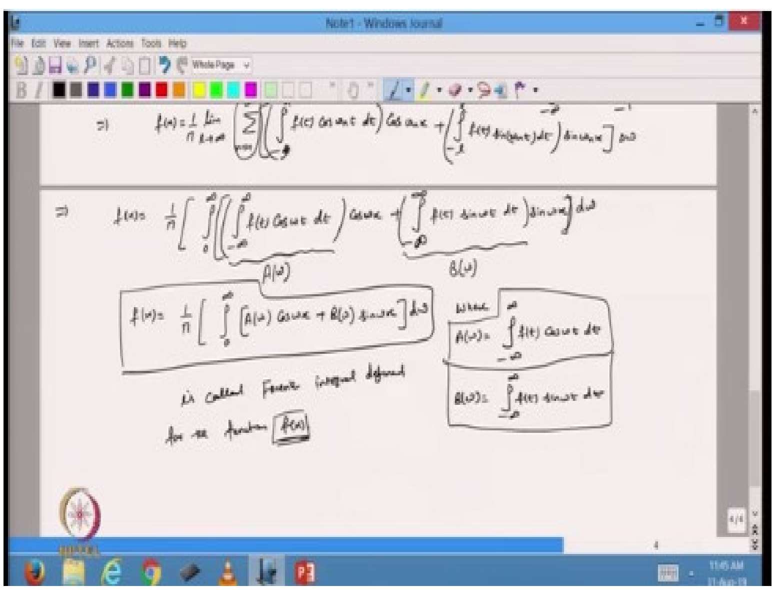

So from here I can write my expression as f (x) is equal to and if you see this one this is same as the Riemann sum of the some integrable quantity. So if I put the l tends to infinity then this sum will tends to become the integral from 0 to infinity. Because this is coming from n = 1, 2, 3, 4, so this summation become integral from 0 to infinity and from here I can write that this become equal to one over epsilon 1 over pi.

And then I can write this expression as from 0 to infinity and the inside quantity would be same. So this is - infinity to infinity f of (t), so omega n becomes omega in that case dt into cos omega x + - infinity to infinity f (t) sin omega t dt into sin omega x and the integration with respect to d omega. So this is my complete expression for this one, so from here what I do that I write this integral as because if you see this one if I do the integration with respect to t.

In that case I will the function of omega, so I write this as a A omega sum function omega and I write this quantity as B omega. So from here I can write my expression is 1 over pi from 0 to infinity A omega cos omega x + B omega sin omega x d omega. So this is the integral I am getting where my A omega is - infinity to infinity f of (t) into cos omega t dt and B omega is - infinity to infinity F t sin omega t dt.

So this expression whatever the expression is written here is called Fourier integral defined for the function f (x) and this function satisfy all the 3 condition whatever that I have written in the starting. So may if the my function is satisfying all the condition then the corresponding Fourier integral can be written like this one where these are my coefficients A omega and B omega, so this is called the Fourier integral.

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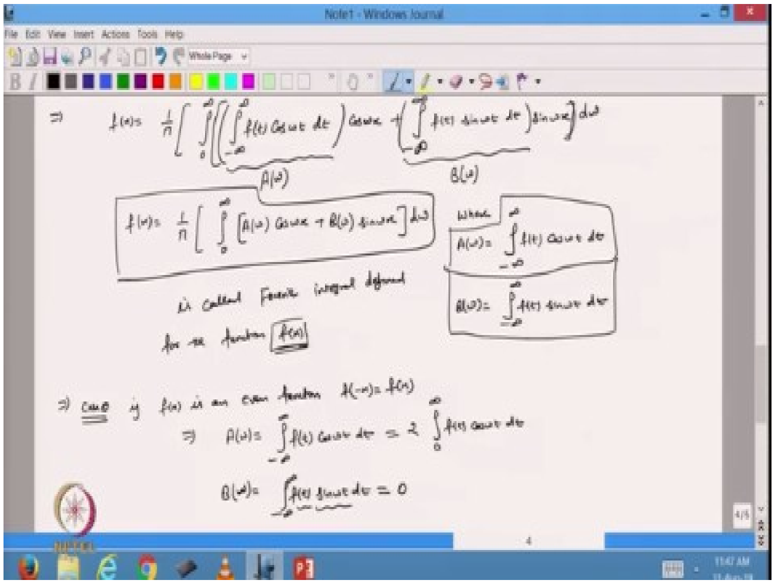

So let us do one example that how we can solve this one. Now so I want to write, so before that I just want to start with some case also, so let us do some cases as we have done for the Fourier series. The case 1 I take that if the function f (x) is an even function in the whole domain, that is f of (- x)= f (x). So what will happen in that case I know that if the function f (x) is a even function then the expression A omega = - infinity to infinity f of (t) I have written you can write x also no problem now, cos of omega t dt.

In this case now this is also even function and this is also the even function, so it is a even function. So from here I can write that this is equal to 2 times 0 to infinity f of (t) cos omega t dt, and in that case my B omega will be from - infinity to infinity f (t) sin omega t dt. And this is the odd function and this is the even function the product is the odd function, so this is equal to 0.

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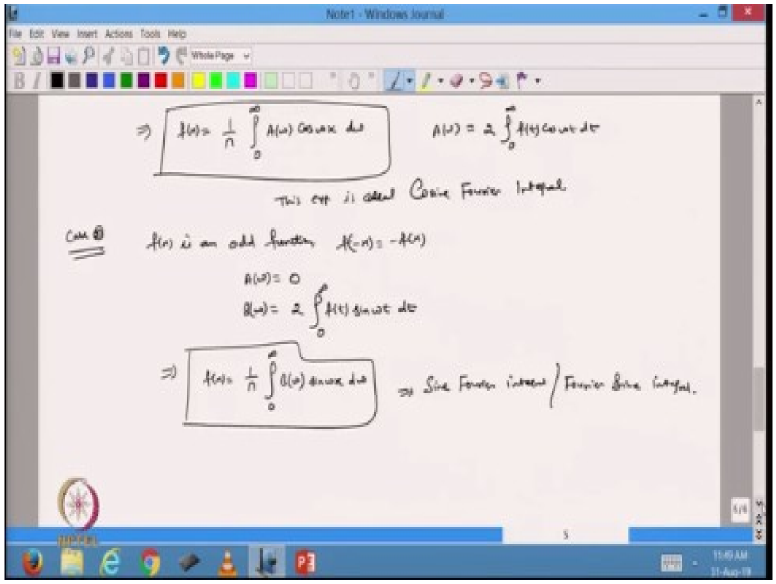

So in that case, so from here I can say that the coefficient would be 0 and then my Fourier integral will be 1 over pi from 0 to infinity A omega cos omega x d omega, so this will be there with A omega = 2 times 0 to infinity f of (t) cos omega t dt, so this expression is called cosin Fourier integral. So this is similar as the Fourier cosin series I had the case 2 when we function f (x) is an odd function.

So in that case my function I can write my function is equal to - of f (x) and you know that from there I can show that my A omega will be 0 in that case and my B omega will be 2 times from 0 to infinity f of (t) sin omega t dt. And then my corresponding Fourier integral will be 1 over pi 0 to infinity d omega, sin omega x, d omega and is called sin Fourier integral or Fourier sin integral. So this is the cases when your function is either a even function or the odd function.

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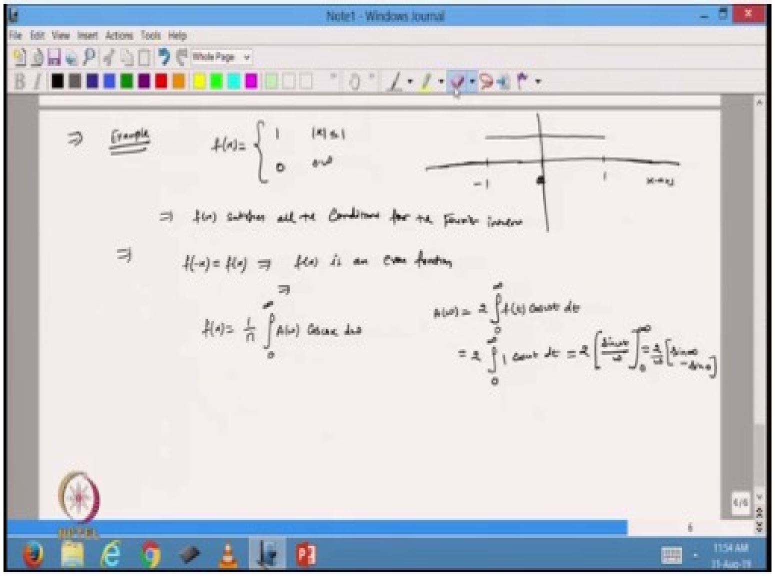

So let us do some examples, so let us take 1 example, so I take my function f (x) = so these are function I started with, so I just take the same function 1 when mod x is less than equal to 1 and 0 otherwise. So if you see this function this function is like this one, so this is my x axis this is 0 and this is x axis, this is my y axis I am taking. So the function from - 1 to 1, so this is - 1 and this 1, so in that case my function is 1 and after that this is 0.

So it is satisfying whether I want to check whether it is satisfying all the 3 condition or not. So first condition is that this function is a piecewise continuous yeah it is a piecewise continuous function from - 1 to 1, so that is true. Second one is that I want to check whether this function is a integrable or not absolutely integrable and we know that this is absolute integrable that we have done.

And third one is that at each value of x the left derivative and the right derivative should exist. So here also we can see that the left derivative and the right derivative existing left derivative a right derivative from - 1 and left derivative and right derivative the x = 1, it is exist there and all other value it will be exist. So from here I can say that f (x) satisfy all the conditions for the Fourier integral, so from here I want to find this Fourier integral.

Now if you see from here what will happen if I take F = - x you can see from here that this function like this so it is a if you see that it is a even function because in this case it is same as f (x). So from here I can say that this function f (x) is an even function because the value of the function at x from greater than 0 and the value of the function at x less than 0 are the same just the image about the y axis.

So from here if I want to write my Fourier integral, so my Fourier integral for this function will be 1 over pi A omega cos omega x d omega and from here I can find my A omega so A omega is 2 times from 0 to infinity f of (t) cos omega t and I can write my dt. So this is equal to 2 and from 0 to infinity or does not matter in this case I can also take from 0 to infinity my f (t) from 0 to infinity is what it just the function 1 into cos omega t dt.

So this from here I can write and this is the sin omega t by omega 2 times from 0 to infinity because if I take the derivative sin omega t is the cos omega. So from here I can write this is equal to sin infinity - sin 0 not infinity from 0 to 1.

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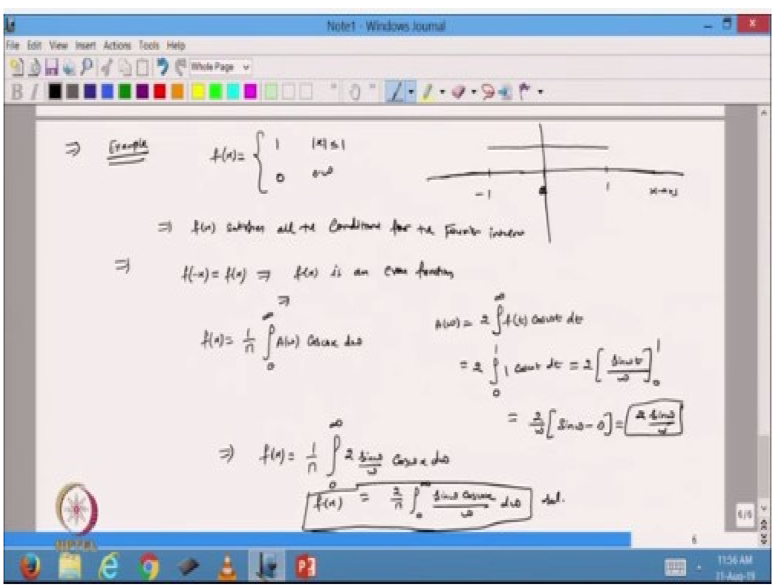

So this quantity I can define this one is from 0 to 1 otherwise it is value 0, so this is equal to this sin omega t over omega 0 to 1. So from here I can take my quantity as sin omega – 0, so this will be 2 times sin omega over omega m so that is my A omega. So from here after finding the value of A omega I can define my Fourier cosin integral will be 1 over pi 0 to infinity e omega this one 2 times sin omega over omega into cos omega x d omega.

So this will be d omega, so which can further be sold so it will be 2 over pi sin omega cos omega x by omega d omega. So this is my Fourier cosin integral for the function that the function which function I have taken, so this is the solution. So it can further be solved, so that is you can do yourself.

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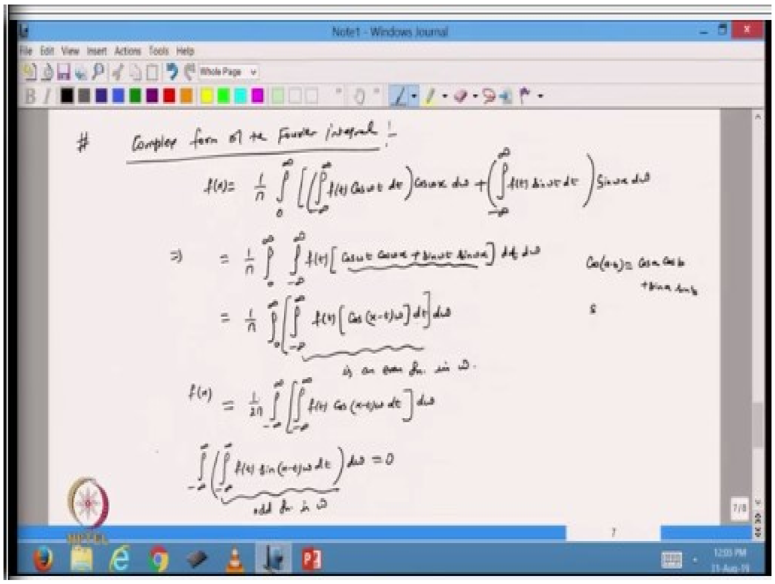

So after doing this one, now I will define the another form, so let us do complex form of the Fourier integral. So I want to write the complex form of the Fourier integral. So all the conditions are same, now I know that if function is a non periodic function and defined from - infinity to infinity and satisfying all this condition then I know that my Fourier integral is this one from 0 to infinity.

And then I so instead of A omega I just write it expression here, so from here I can write from - infinity to infinity f of (t) cos omega t dt into cos omega x d omega + - infinity to infinity f of (t) sin omega t dt sin omega x d. So this is the expression for the Fourier integral, now what I do is that from here I can write this expression as 1 over pi from 0 to infinity. So I will keep all the terms together, so from here I can write this 1 this x can be written as cos omega t into cos omega x + sin omega t into sin omega x ok into dt d omega.

So this is the quantity we left with, so from here I can write this expression, so what is this, so if I write cos A + B so in the expression is cos A cos B + sin A sin B. So from here I will write this quantity as this is - sign cos A - B cos A cos B + sin A sin B, so from here I can put this quantity here, so I can write this expression as a cos x - t into omega dt d omega, so this quantity can be written like this one ok.

So from here so this expression, the expression I have written is this become the this quantity. Now from here if you see from here that if I am doing this integration with respect to t here first and after doing the integration putting the limits this expression is a function some value some constant into the cos of x - t omega. And which is the even function, so from here I can say that this expression the whole expression is an even function in omega.

So from here I can write this quantity as so this is the even function with respect to omega and then I am taking the integration from 0 to infinity with respect to omega. So from here I can write this expression as from - infinity to infinity and inside expression will be same, f (t) cos x - t omega dt d omega and this will be 2 times. So this becomes, so I can write this as 2 pi because here I am taking from - infinity to infinity otherwise earlier it was from 0 to infinity.

So I have written this expression as this one, so this is what I have written, so here the expression for that become this one. Now what I do is that from here I can say that I will take the quantity like this one - infinity to infinity f (t) sin x - t omega dt d omega. So I know that the same terminology or the same will be used here because in that case the function is the odd function in omega. So if I do the integration from - infinity to infinity of the odd function then I know that this quantity will be 0.

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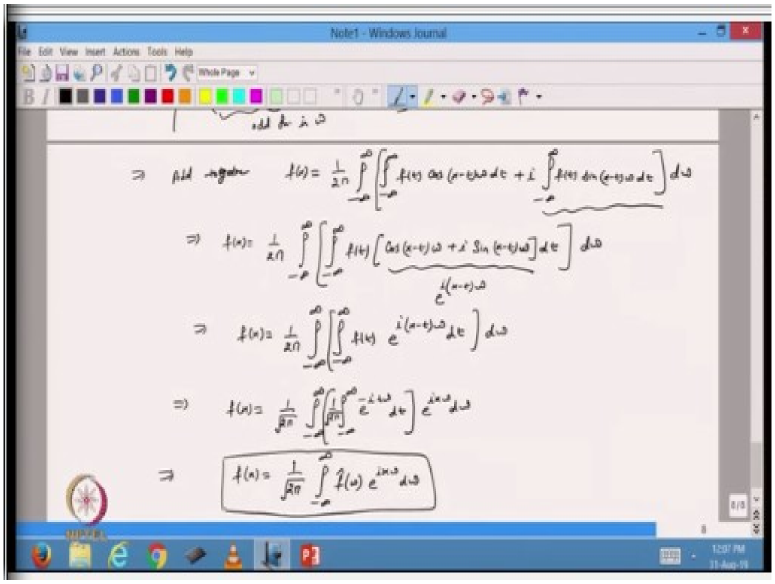

So from here I can say that this quantity is 0, so what I do so I take this quantity as equation number 1 and 2 then I add together. So what I do that I write f (x) = 1 over 2 pi f (t) cos x - t omega dt + i time i is the complex number, I write this expression - infinity to infinity f (t) sin x - t omega dt and the whole expression d omega. Because I know that this quantity taking the integral from minus infinity infinity is 0, so I can just add this one 0 quantity on the both side + i the complex number I am taking.

So from here my f (x) will be 1 over 2 pi this quantity I am taking and this quantity I will write - infinity to infinity f of (t) this quantity is there and this expression will be cos x - t omega + i times sin x - t omega dt and the whole d omega. So I will get this expression and this quantify if I want to write it is earlier form in the complex form a complex number this will be equal to e raised to power i x - t omega.

So from here I can say that my function f (x) will be 1 over 2 pi - infinity to infinity f of (t) e raise to power i x - t omega dt. And of the whole expression can be written as d omega, so after doing this one what I do is that. Now I separate this integral, so I just instead of 1 over 2 pi I write 1 over under root 2 pi and I put this - infinity to infinity here and I put 1 over 2 pi here under root 2 pi - infinity to infinity here.

So in side I will take this value - i t omega dt and I will get here i x omega d omega because I have separated this expression e raise to power i x - t omega. So I will write this one as e raise to power - i 2 omega inside with respect to t and e raise to power i x omega with respect to the outer integral. From here I will write this quantity as f (x) = 1 over 2 pi under root 2 pi from - infinity to infinity and I call it a f of omega e i x omega d omega.

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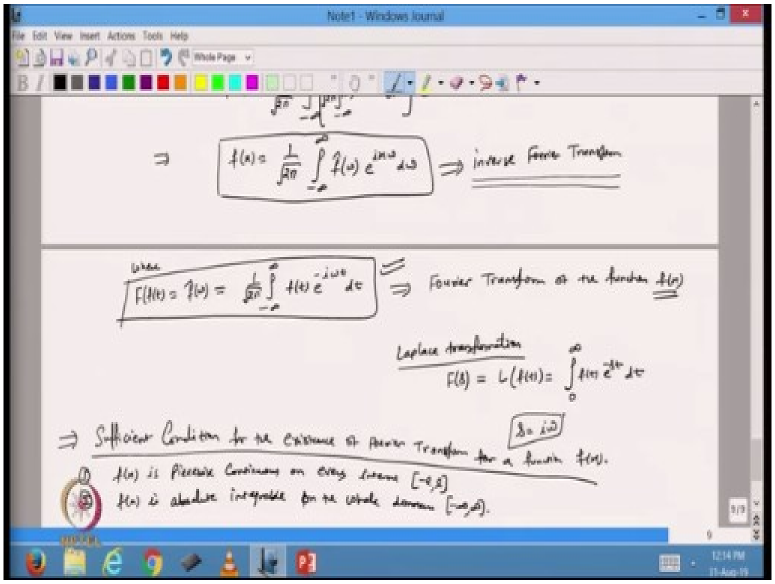

So this expression I have written here where I call this expression - infinity to infinity 1 over under root 2 pi f (t) is there. So it will be f (t) e raise to power - i omega t dt, so this expression I call it f hat omega and we call it capital F of f (t). So this is called Fourier transform and from the Fourier transform if I take the Fourier transform because I am taking the transform of the function f (t) and the expression is coming in the terms of omega.

So it is called the Fourier transform of the function f (x) and this then we call it inverse Fourier transform, why we call it inverse Fourier transform. Because after getting the Fourier transform see this is a initially I have the function in x and then this space will move from the x to the frequency space that the omega. And from omega I am going back to the x space.

So that is why if this is the Fourier transform then I call it that the inverse Fourier transform, so this is the expression. And now after defining this Fourier transform if you see from this expression I remember that we have also done the Laplace transformation. In the earlier lectures we have done the Laplace transformation, so in that case also I have defined the Laplace transformation for the function f (t), so I call it laplace function of the function f (t).

If you remember it was from 0 to infinity f (t) e raise to power – st dt and we call it f of (s). So in that case I start with the function f (t) after doing the integration this convert to the s space. So in this case this is my Laplace transformation, so if you just compare this with the Fourier transform you will see that instead of s if I put i omega here in this expression. And outside it will be 1 over under root 2 pi then it become the Fourier transform.

So instead of s I just take the complex number i omega then the Laplace transformation can be converted into the Fourier. But only thing is that the Fourier transform is valid from - infinity to infinity and the Laplace transformation was from 0 to infinity. So just to remember that sometimes we want to remember the Fourier transform which is, so if you already know the Laplace transform then instead of s you just put i omega from - infinity to infinity and multiplied by 1 over under root 2 pi.

So this is called the Fourier transform of the function, so this is very important concept of the Fourier transform. That we have the function which is given to me in terms of my the physical space we call it the t or x phase and I convert that physical space into the frequency space. So this is my frequency space and once I am able to get the function in the frequency space and suppose I want to go back to the physical space, then we define the inverse Fourier transform.

Because this facility was not there in the Laplace transformation, in the Laplace transformation if you remember that we know the Laplace transformation but there was no mathematical expression for the inverse Laplace transformation. But in this in the Fourier case we have the Fourier transformation as well as the expression for the inverse Fourier transformation.

So now, so after doing this one, so like we had some sufficient condition for the Laplace transformation to be exist to be there for the function. Then the same way I can define the sufficient conditions for the existence of Fourier transform for a function f (x). So what are the sufficient condition for existence of the Fourier transform of the function, so the condition one is that, so here function f (x) is well defined from - infinity to infinity.

So first is that the same condition the f (x) is piecewise continuous on every interval from - l to l. So this is there and the second one is that f (x) is absolute integrable on the whole domain from - infinity to infinity. And the third condition is same as the that left and right hand derivative exist.

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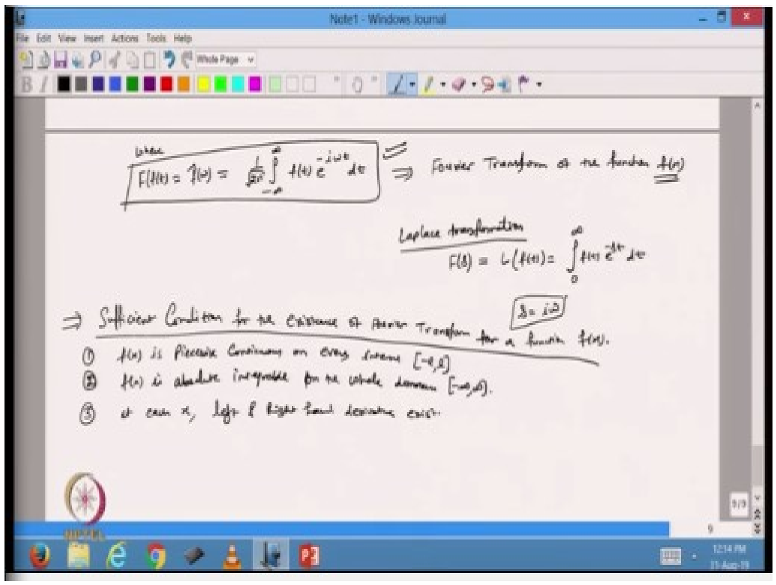

At each x left and right hand derivative exist, so if these 3 conditions are there, so in fact if the 3 conditions are satisfy then I can define my Fourier integral. And from there we have seen that we have found the expression for Fourier transform. So if this is the sufficient condition that we choose a function and satisfying all these 3 condition, then it is Fourier transform will always exist.

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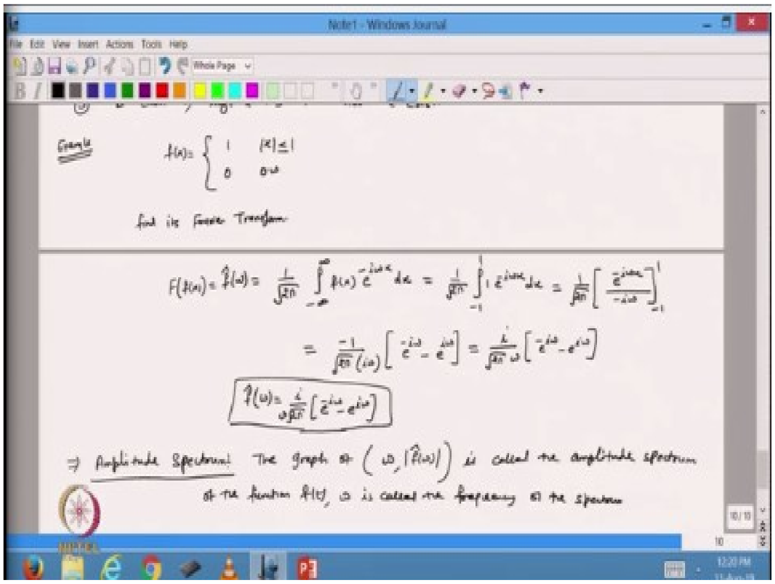

So let us do 1 example, so the same example I am taking because I know that this example is satisfying all the conditions, so I will take this one, so this is my expression. So I know that this function satisfy all these 3 condition, then I want to find it is Fourier transform. So in this case I want to write the Fourier transform, so I will write F of f (x) and I will call it f hat omega, so this will be equal to 1 over under root 2 pi from - infinity to infinity.

So this is my function I am applying f (x) into e raise to power - i omega x and that is dx, so this one I have write. So from here I can write my expression, so now I put the value of the function and here my function is defined from - 1 to 1, so and the value is 1. So it will be e raise to power - i omega x dx, so if I solve this, so it will be under root 2 pi, so this will be e to power - i omega x divided by - i omega and the limit - 1 to 1.

So from here I can write as 100 pi i omega and inside quantity will be e raise to power - i omega 1 - it is e raise to power - - + i omega. And this iota I can take on the i the numerator, so this will be I multiply and divide by i, so i square will be – 1. So from here I can write my expression will i over under root 2 pi omega into e - i omega - e i. So that is my Fourier transform for my function, so this is equal to i under root pi omega.

So this quanti is the Fourier transform of the function here, now from here I can define 1 quantity that is called the magnitude the amplitude spectrum. So from here I define another quantity amplitude spectrum, so what is the amplitude spectrum, the graph of omega because this function is in terms of omega and modulus value of f omega the corresponding Fourier transform. So if I plot this one is called the amplitude spectrum of the function f (t) and omega is called the frequency of the spectrum.

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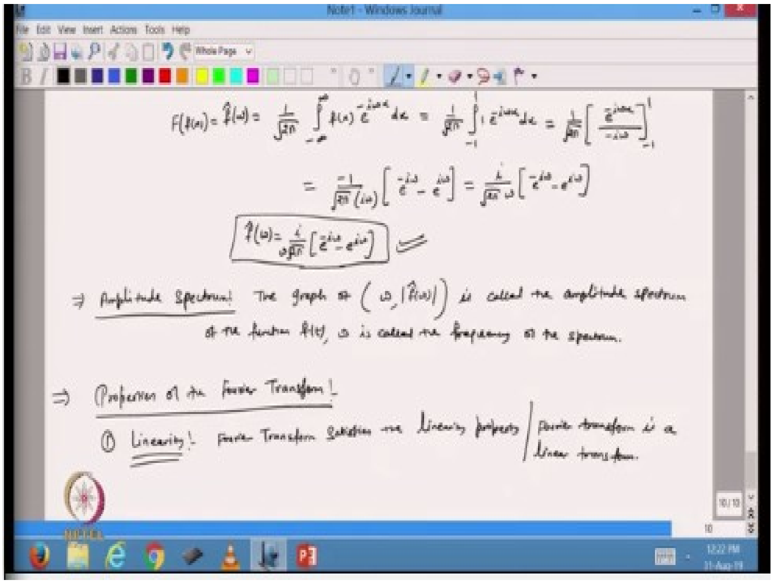

So based on this one once I get the expression for like here I am getting the expression for the f hat omega, then I can plot for a different value of omega and then whatever the expression I get that is called the frequency spectrum. Now once I know the Fourier transform then I will start with the properties, so properties of the Fourier transform. So if you remember that when we started with the Laplace transform after that we also discussed all the properties that how it behaves for different function what are the properties.

So the same way I can define the properties of the Fourier transform, so the first property is linearity. So for the linearity I can say that Fourier transform satisfies the linearity property or I can say that Fourier transform, then I can say the Fourier transform is a linear transform.

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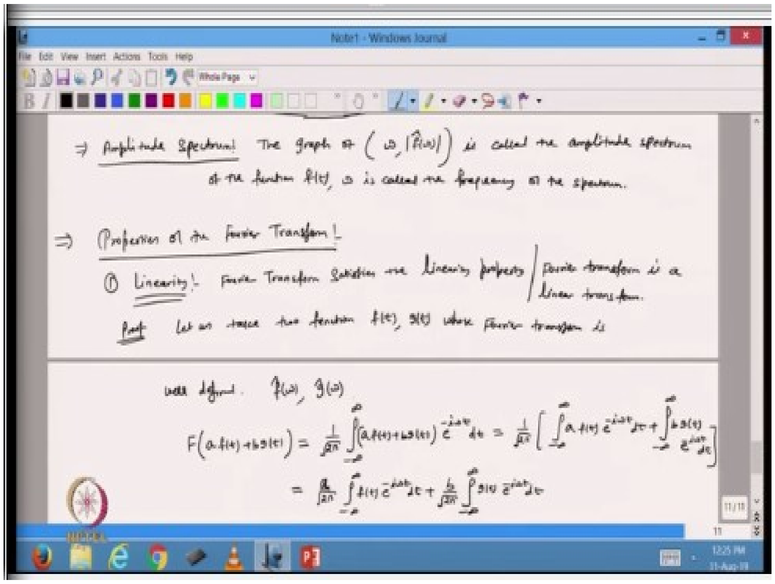

So let us do the proof for this, so proof is very simple, so I will take let us take 2 functions f of (t) and g of (t). And these both function f (t) and g (t) satisfying all the sufficient condition that both are piecewise continuous, absolute integrable and the third condition also satisfied. So then we can define, so let me take the 2 function whose Fourier transform is well defined, so well defined means I write it is Fourier transform of the function f (t) as omega.

And for another one we call it g omega, so these are the corresponding Fourier transform. Now I want to take Fourier transform of the function, some constant I take a of f (t) + b of g (t). So this one I want to define, so from here I can say that this is equal to 1 over under root 2 pi from - infinity to infinity, e raise to power - i omega t dt. And now I know that the integral is a linear function, so from here I can write a of f (t), a of - i omega t dt, so this is the expression we have taken.

And from here I can write that this is can be written as 1 over under 2 pi taking the constant a outside, so inside I will get f (t) e raise to a - i omega t dt + b over under 2 pi g (t) e raise to power - i omega t dt.

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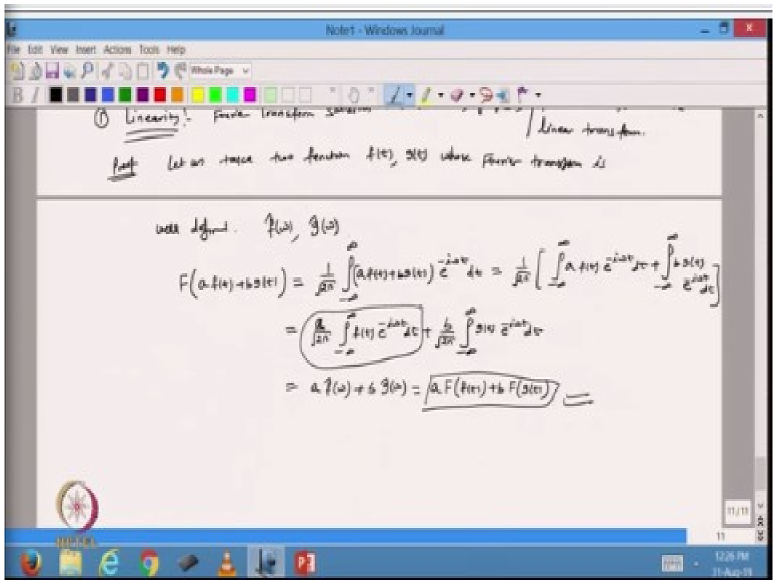

So this quantity can be written as and what about this one, so this is I can write as that a times into f of omega. So this is the corresponding Fourier transform of the function f (t) and this can be written as b times g hat omega. So from here I can write that this is equal to a Fourier transform of the function f (t) + b Fourier transform b into Fourier transform of the function g (t). So this is the expression we got, so from here we can say that it satisfy the linear property.

So today we will stop here, so in the today class we have started with the Fourier integral and then using the expression for Fourier integral. Then with the help of some mathematical expression we able to define the Fourier transform. In the next class we will go further and try to solve using the Fourier transform we try to solve some differential equation, so thank you very much for viewing this, thanks a lot.