Introduction to Methods of Applied Mathematics
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Lecture – 20
Gibbs Phenomenon and Parseval's Identity

Hello viewers welcome back to this course, so in the last classes on the last lecture we have started with the Fourier series and then we have done some examples finding the Fourier series of the function and then we also discussed that the complex Fourier series but for the Fourier series, we have considered that the function is a periodic function which has a period 2L. Now today I will define some other properties of the Fourier series and this is a very important phenomenon.

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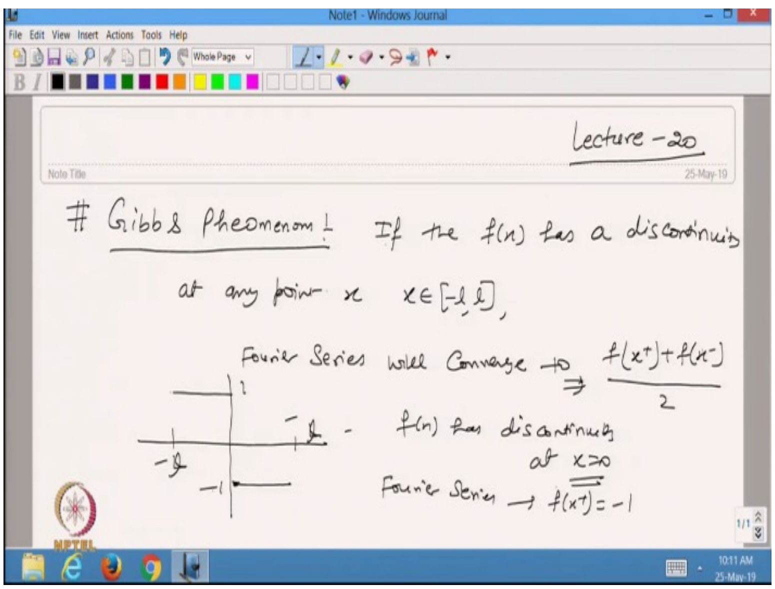

So today we will discuss with the Gibbs phenomenon so what is this Gibbs phenomenon Gibbs tells you that because when we deal with the Fourier series if the function is continuous then you will see that the Fourier series give you the nice approximation for that function but what about the function, so it say that if the function f x has a discontinuity at any point say x and x belongs to the interval - l to l because I am considering that my function f x is a periodic function of period 2 l defining from - l to l.

So in that case say that then the my Fourier series will converge to the function to this value so what is this value? It is divided by 2 so this is where the Fourier series will converge and if you see this one like suppose just define a function so let us take one function here suppose for the time being I defined a function like - 1 and it is 1 and it is – 1, so suppose it is - 1 and the value of the function is 1 and I am defining this one for - l and this is + l see is small l, so in that case if we see then the function and discontinuity at x equal to 0.

So in this case my function f x as discontinuity at x = 0, so if I in this case if I want to find out the Fourier series of this function from - l to l you will see that the Fourier series is convergent for this function at this value and this value but for the from value at x equal to 0 this will converge to the Fourier series will converge to f x + 1, so this is the right hand limit I am taking so I am taking the right hand limit is – 1, so in this case it is – 1.

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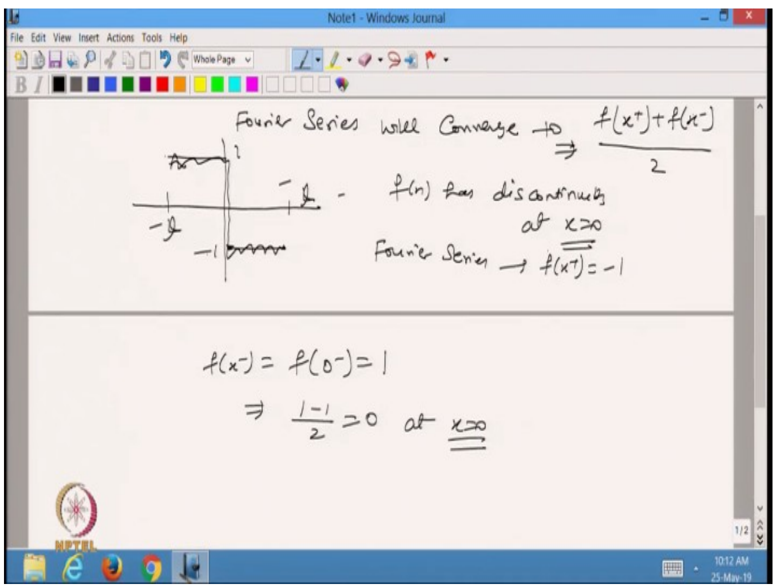

And f of x - 1 so this is I am basically I am considering this one so that value is 1 and from here if I see then 1 - 1 divided by 2 so that is value 0, so in this case if I find the Fourier series of this function then by the Gibbs phenomenon the series will converge to 0 at x = 0, so otherwise the series is convergent wherever the function is a continuous function so if I want to define the Fourier series of this function you will see that the Fourier series in this case will look like something and then this one and then like this type of.

So it depends upon that how many terms you are taking the Fourier series because the Fourier series you know that it contains the infinite number of terms but whenever we want to find out the approximation only in that case we know that we are considering only few terms of the series so it as soon we keep increasing the number of term in the series the our approximation will be better and better and I am saying that the once we are increasing the number of terms of the series the approximation will be look like this one. So this is because of the given by the Gibbs.

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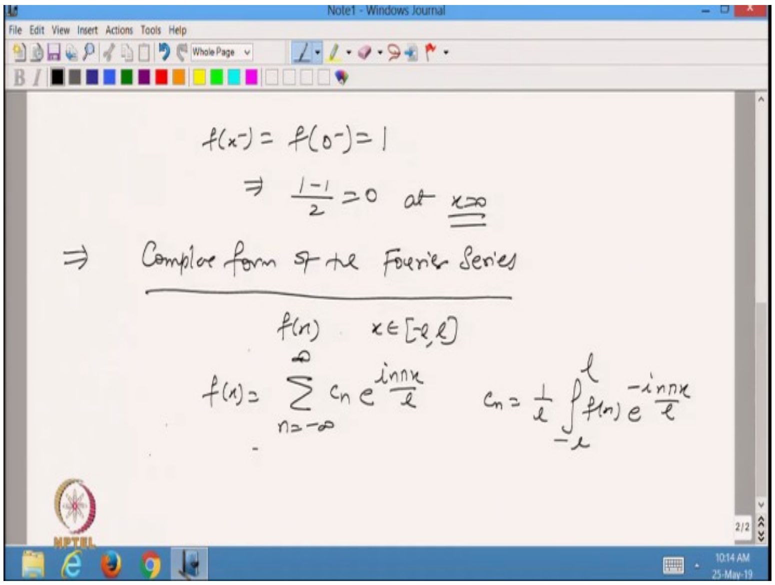

Now we define so after that another thing I want to define today is that in the last class we have discussed the complex form of the Fourier series so in that case we have defined the function so my function f x is continuous having a finite number of discontinuity and x is defined from - l to l then I know that in the complex form we have written the series as c n raise to power i n pi x by l where my c n is the coefficient and that was we have defined by 1 over l - l to l f x raise to power - i n pi x by l. So this one we have done in the last class.

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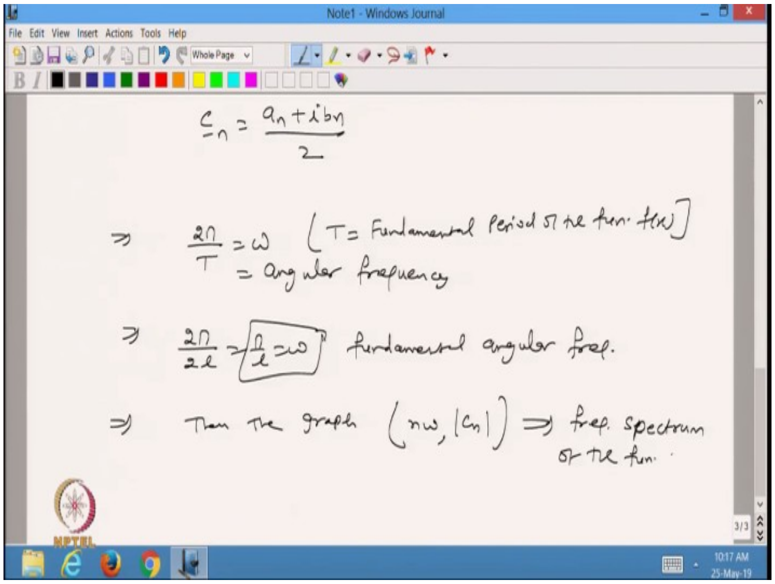

So now this to l and n is from 0 + - 1 + - 2 and so on now and I know that my c n in this case is a n - ibn by 2 and c - n is just the conjugate of this 1 by 2 now in this case what we do we define we take 2 pi by T and define this one as Omega where T I am just taking the T is the fundamental period of the function f x so this is a fundamental period I am taking fundamental period is that I told you at the function if the function has a period f then 2 pi also it is period 3 pi also period then the minimum of that one is the fundamental period.

So I am taking T here so this Omega in that case is called angular frequency and from here like suppose I have a in my function I have a period 2l so in that case if I have 2 pi by 2 l then it will be pi by l so this is called the angular frequency for we call it fundamental angular frequency for this function f and in that case so after defining things fundamental frequency my I can plot then the graph corresponding to an Omega because Omega is fixed here.

So n Omega means I am taking the value of n 1 2 3 whatever the value we are taking - 1 - 2 and then I take magnitude of c so magnitude of c n I would if I plot this one then this is called it is called frequency spectrum of the function f x.

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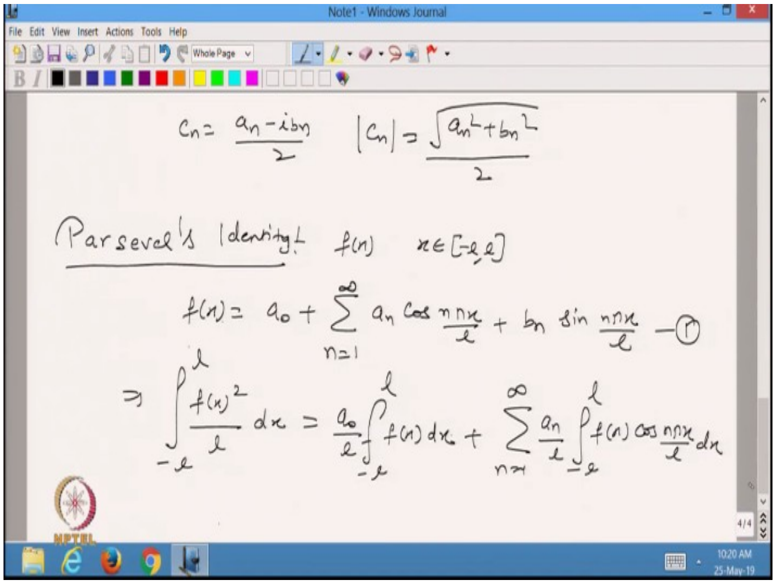

So then it is called a frequency spectrum and I know that my c n in this case is a n – ibn by 2, so from here I can define the magnitude of c n so this is a n square + b n square under the root by 2, so this one is the magnitude of c n, so I know that I can find out the magnitude of this one and then if I plot this one with respect to n Omega and then whatever the form we get is called the frequency spectrum the function, so this is called the frequency spectrum now I am also going to define another very important topic identity and that is called the Parseval's identity.

So this is the Parseval's identity, so it is say that my suppose I have a function f x and that is defined for the interval - l to l and in that case I have the Fourier series as a 0 + summation and from 1 to infinity a n cos and pi x by l + b n sin n pi x by l, so this in my Fourier series I have then what I do, so this is the Fourier series I am having and this Fourier series I know it is converging to the function f x for all value of x in the interval.

And wherever the function has discontinuity then by the Gibbs phenomenon it will converge to the average value of the left hand or right hand limit at that point so that we already know so now from here what I do is that I multiply the function f x both side of the equation 1 and then divide by l and then integrate with respect to x from - l to l so this is I am doing so the same thing will on the right hand side I am dividing by l integrating from - l to l f x dx so this will happen here plus now I take the summation and from 1 to infinity and integrating.

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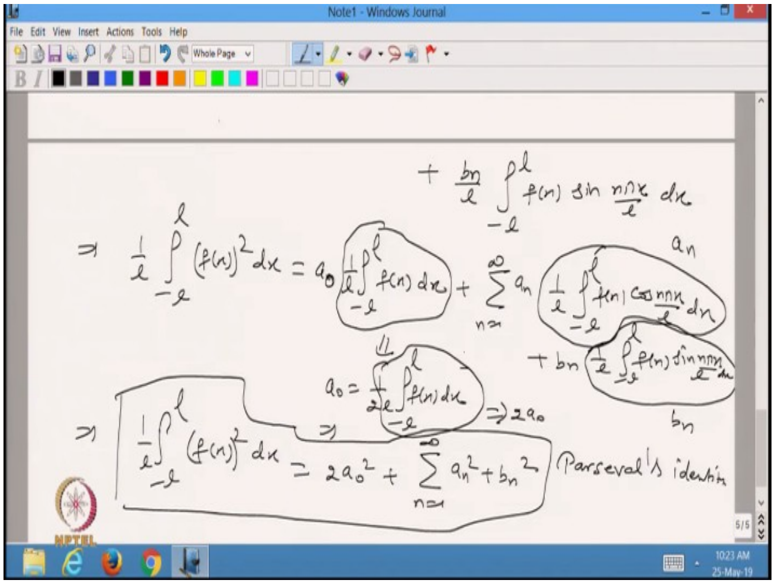

So this will be a n by l - l to l f x cos n pi x by l dx + b n by l - l to l f x sin n pi x by l dx so this is after multiplying by x or f x and then dividing by l and then integrating with respect to x from - l to l so from here the left hand side I can write l is 1 - l to l so f x whole square dx and on the right hand side I will get a and a 0 so - l to l f x dx over 1 by l + cos n pi x by l dx + similarly b n 1 over l - l to l f x sin n pi x by l dx so this is we have defined now if you see from here that what is the value this one this integral so this integral if you know then my a 0 is always equal to 1 over 2 l from – l to l f x dx.

So this is given to me so from here I can say that this integral because this is the integral will be gives me 2 a naught similarly I know that what is the value this integral so this integral is n and if you see this one this integral is b n so from here I can write that 1 over l - l to l f x square dx that is equal to go from here it will be comes to a 0 square + summation and from 1 to infinity so this is given to me so now a n and this is also n so from here I can write this as a a n square + and this become the b n square so from here I get a 1 equality and that equality is this one.

So this equality is called the Parseval's identity so this identity basically gives you that the that on the right hand side we have only the summation corresponding to the coefficients and left side is the square of the function so this type of identity used for the large number of application so this identity I will use for that how this can be used

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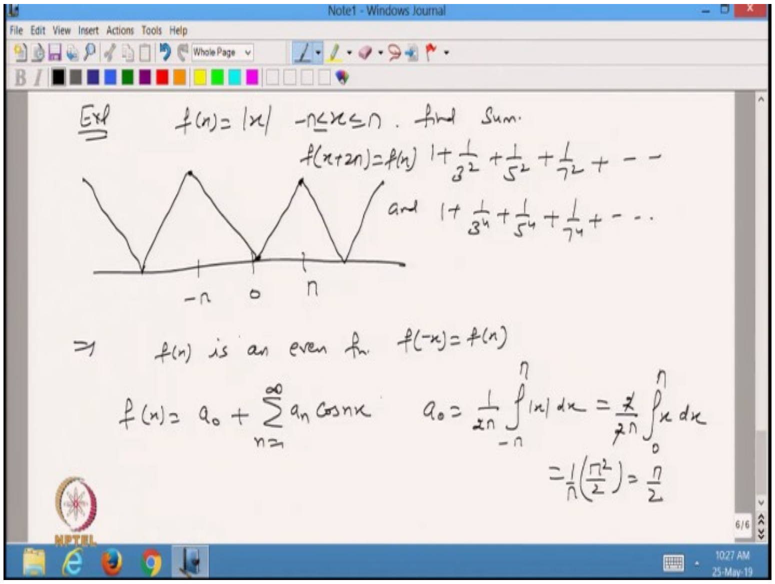

Let us start with the example so what I do I have an example f x = mod x I am just taking where x in this case I am taking from - pi to + pi and from here I want to find after calculate this one find summation 1 + 3 square 5 square 7 square and so on I want to I find the value of the summation and value of this summation 1 over 3 raise to power 4, 1 over 5 raise to power 4, 1 over 7 raise to power 4 and so on so let us see how the Fourier series is going to help us to find out the summation of this series.

So now in this case so what I do is that my function is well defined so if you see that this function is I am taking this in my 0 and suppose this is - pi and this is pi so this is my function so mod x and then this is this is well defined from - pi to pi and also it is given that the function x + 2 pi = f x so from here I can say that this function is a periodic function so if you see the periodic then this function will be can be extended for the whole real line and this will be the corresponding function.

So in this case you can see that this function is continuous function so in this case if I take the Fourier series that Fourier series will converge to the function at all the point value of x so from here and also I want to say that the function f x in this case is an even function because from here you can say that this is equal to this one because the mod x so it is an even function so with the help of the even function I know that if I write the Fourier series of this function then it will be only a 0 + summation and from 1 to infinity a n cos nx now my a 0 if I want to define the a 0.

So my a 0 will be 1 over 2 pi from - pi to pi and f x is mod x dx but it is a even function so I can write as a 2 pi from 0 to pi and from 0 to pi mod x will be x so it will be dx so this will cancel out and from here this will be pi square by 2 1 over pi so that gave me only pi by 2.

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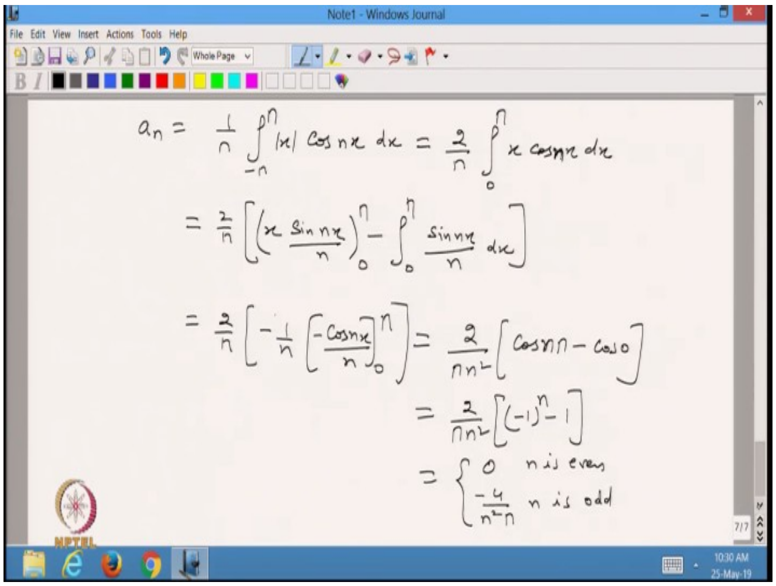

So that is the value of a 0 now I want to define the value of n so n is 1 over pi - pi to pi mod x cos nx dx so this is even function cos nx function from here I can write as 2 over pi 0 to pi it will be x cos n x dx so to find the integration of this integral I can apply the by parts rule and from here it will becomes 2 over pi so I can define as x + cos pi x the integration of cos pi x is sin pi x divided by pi note it is n so it is not a pi it is n putting the limit and then - 0 to pi and this will be again sin n x by n and dx now if I put the limit for this one.

So I know that the pi will put sin pi 0 sin 0 0, so from here it will be only 2 by pi and then inside I will get minus so this one end will come out and I have to integrate from 0 to pi sin n x so this one I can write as a - cos n x by n and then putting the limit because sin n x if I want to find out the integration of this one so it will be – cos n x divided by n and putting the limit 0 to pi so from here now I can define from here so this n where I can take common so this will give me only 2 over pi n square and inside I will get cos n pi - cos 0 I am putting the limit here.

So from here I will get 2 over pi n square now cos n pi I know that when n = 1 the cos pi is - 1 when n = 2 cos 2 pi is 1 so from here this will be always equal to - 1 raise to power n - 1 so now this 1 = 0 when n is even and this is equal to - 4 by n square pi when n is odd number so that is the value of the coefficient a n.

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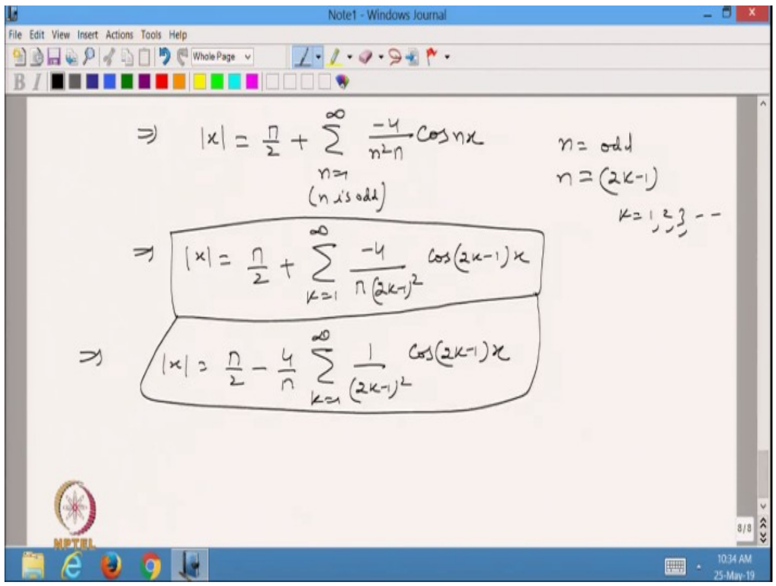

So if I use this coefficient then from here I can write my Fourier series for the mod x, so this will be a 0, a 0 was pi by 2 so it is pi by 2 + summation n from 1 to infinity and this is so here I am taking n as a even because n is not even odd where n is odd and then it will be - 4 over n square pi cos n x, so that is my series, so from here I can define this series now I can also write this series as because my n is a odd value so from here I can define my n as 2 K – 1.

So where K is 1 2 3 and so on so from here I can define my mod x = pi by 2 + K from 1 to infinity - 4 instead of n I can put so it is 2 K - 1 square and then it is cos so it is goes to K - 1 x, so that is the series so this series is that is the Fourier series for the function mode x, so further I can write as pi by 2. So this one I can take outside 1 over 2 K - 1 square and then it is cos 2 K - 1 x this one

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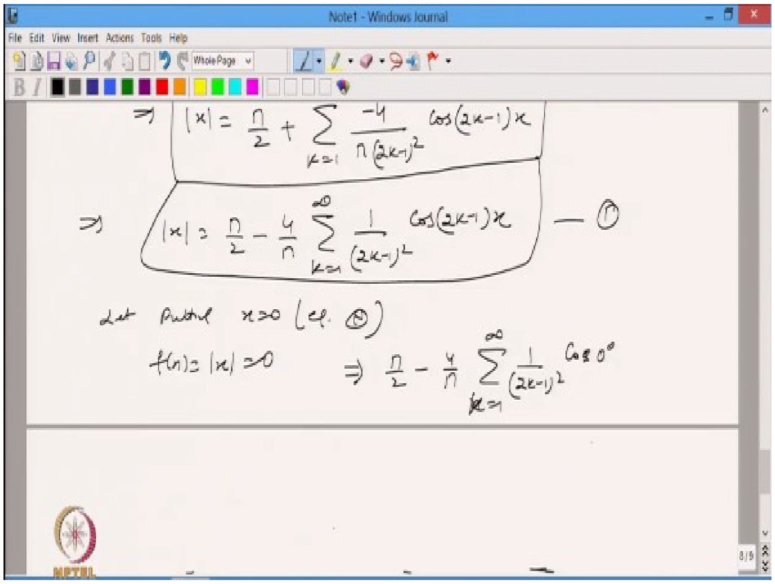

Now from here so I call it equation number one now what I do is I try to find the value of the function at so let putting x equal to 0 in the equation number 1, so on the left hand side my function f x is mod x and that is 0 and on the right hand side if I put the limit in this Fourier series so that gives me the series gives me pi by 2 - 4 by pi and this is the summation K from 1 to infinity 2k - 1 whole square and this will be cos 0 because x will be 0 so it will cos 0 and that is always 1.

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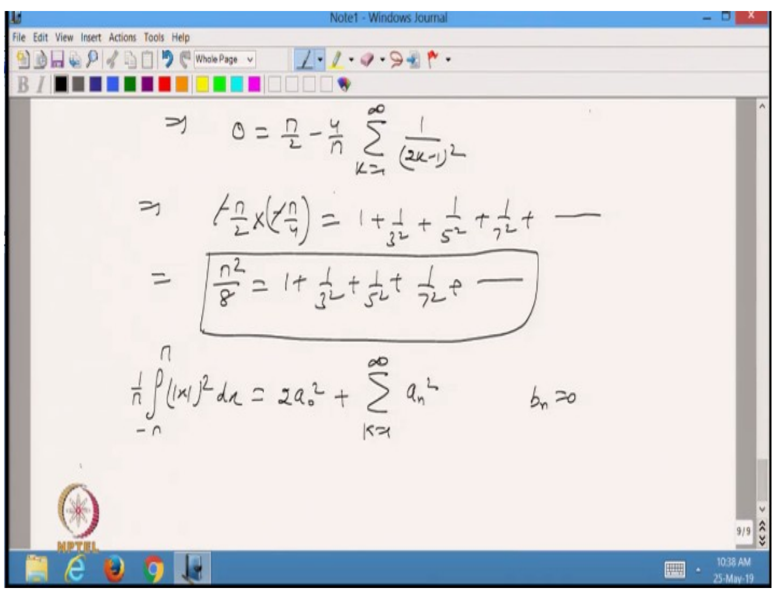

So from here I can say that from the equation number 1 I will get pi by 2 - 4 by pi and that is the summation K from 1 to infinity 1 over 2k - 1 whole square and what is this I can take this 1 on the left hand side, so it will be - pi by 2 so it will be - pi by 2 and then dividing by this one, so from here into - pi by 4 so this will come because I will take this one on the left side and then dividing by this one, so it will be pi by 4 other side I will get this value so K when K = 1 it is value will be 1 when K will be 2 it will be 3 square 1 over pi square 1 over 7 square.

And so on and this will be so I will cancel out this one so it will be pi square by 8 so that is 1 + 1 over 3 square + 1 over 5 square + 1 over 7 square and so on so from here I can find out the sum of this series if you see the series, series is the 1 + 1 over 3 square + 1 over 5 square and 1 over 7 squared so on so with the help of the Fourier series and by the property that it is giving it is converging for the function wherever the function is continuous by the help of this one I am able to find the summation of this series.

Luckily in this function in this case the function was continuous and I was able to find the value of the function directly from there now we have to find the another series so in that case I will use the Parseval identity so now I know that from the Parseval identity I know that 1 over pi - pi to pi mod x square dx should be equal to 2 a 0square + summation K from 1 to infinity and this will be a n square so that we already know because in this case my b n is 0 so by the Parseval's identity I should have this value now let us see what will happen.

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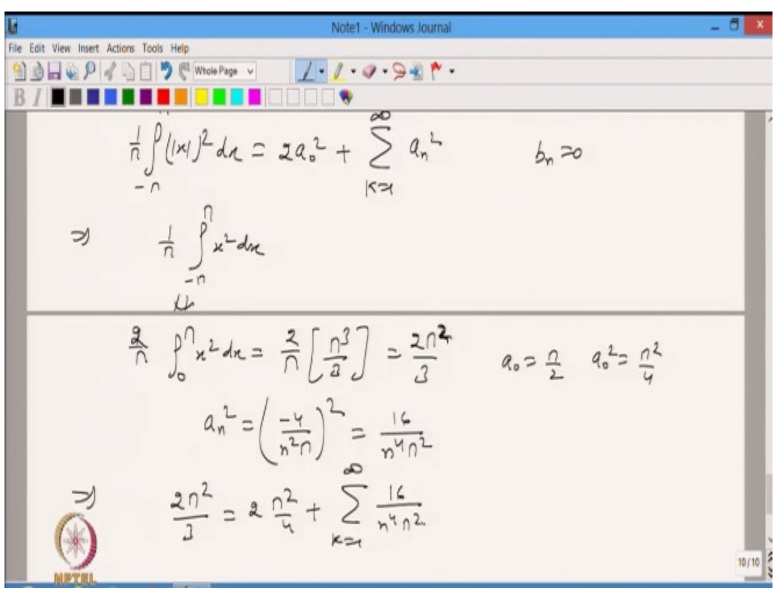

So from here just I want to find what is the value of this one so it is 1 over pi and from - pi to pi and mod x square is x square dx so it is an even function so from here I can write 1 over pi 2 times 0 to pi x square dx so it will give me 2 over pi and then it will be pi cube by 3 so this will be 2 pi square by 3 pi square by 2 so that is the value of on the left hand side of the Parseval identity now a 0 we already know was pi by 2 so in that case my a 0 square will be pi square by 4 and now I want to find what is the value of a n square.

So my a n square is if you see the a n was this value so this is my a n so in that case if I take the square it will be 4 square n raise to power 4 pi square only these things will change so it will be - 4 n square pi whole square so that will be 16 over n raise to power 4 pi square so by the property of the Parseval identity now I can write the Parseval identity so this will be 2 pi square by 3 = 2 times pi square by 4 + K from 1 to infinity and a n square is this one 16 over n raise to power 4 pi square.

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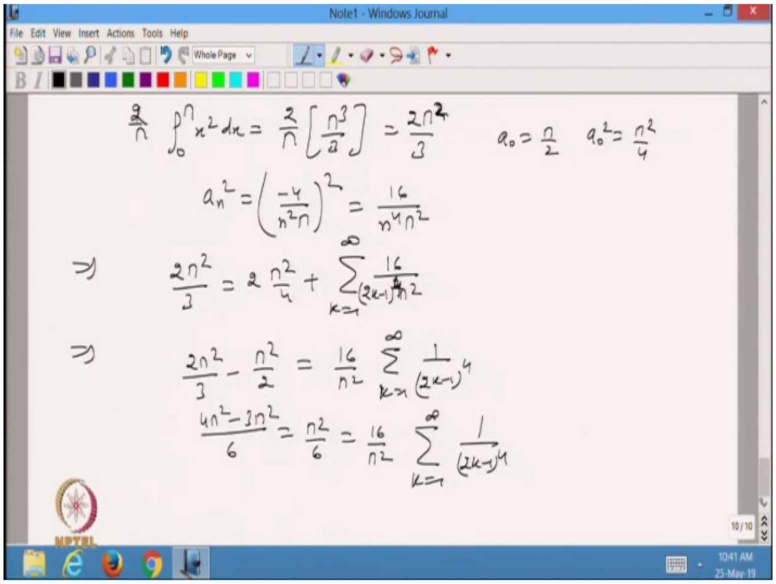

Now I will use this one this property so from here I just take this one on the left side so it will be 2 pi square over 3 - pi square by 2 and this will be 16 by pi square I can take common and inside you will get only K from 1 to infinity 1 over so n is basically 2 K - 1 so 2 K - 1 raise to power 4, so this will get so this also and instead of n I should write to K. So this is basically 2 K - 1 power 4 and then n square.

So from here now from if I do the calculation so this will be 6, so it will be 4 pi square - 3 pi square so that is basically pi square by 6 = 16 by pi square and summation K from 1 to infinity power 4.

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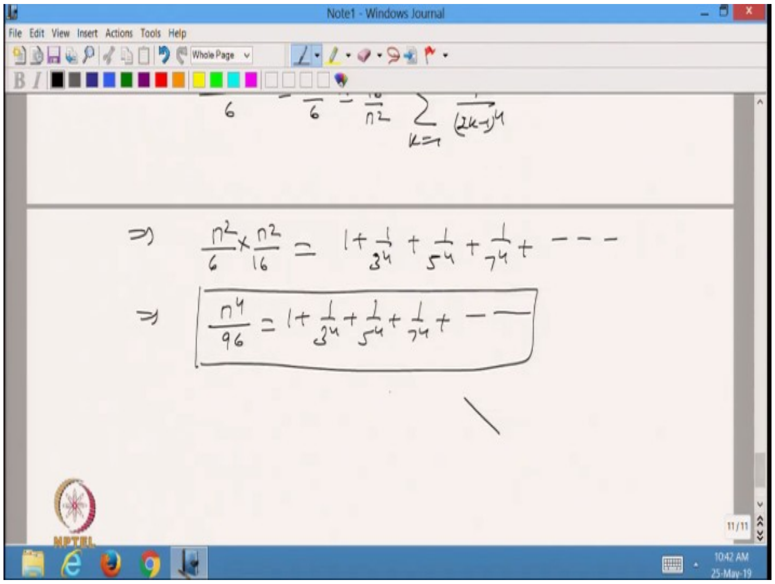

Now I will take this term on the left hand side so it will be pi square by 6 multiplied by pi square by 16 and on the right hand side if you expand this one you will see that I can write this when K = 1 it is 1 when K = 2 it is 2 4 - so it is 3 raise to power 4 then it is 5 raise to power 4 then it is 7 raise to power 4 and so on and on the left hand side you will get pi raise to power 4 by 96 so that is the summation of this series so this is the question we have asked that how the summation of this series will look.

So the summation for this series pi raise to power 4 by 96 so these are we are able to find from this one only when we are applying the Parseval identity so in that case the Parseval identity is very important to find out the various relation between the Fourier series and that generally is used to find out the summation of many series now so after this one so using this Parseval identity.

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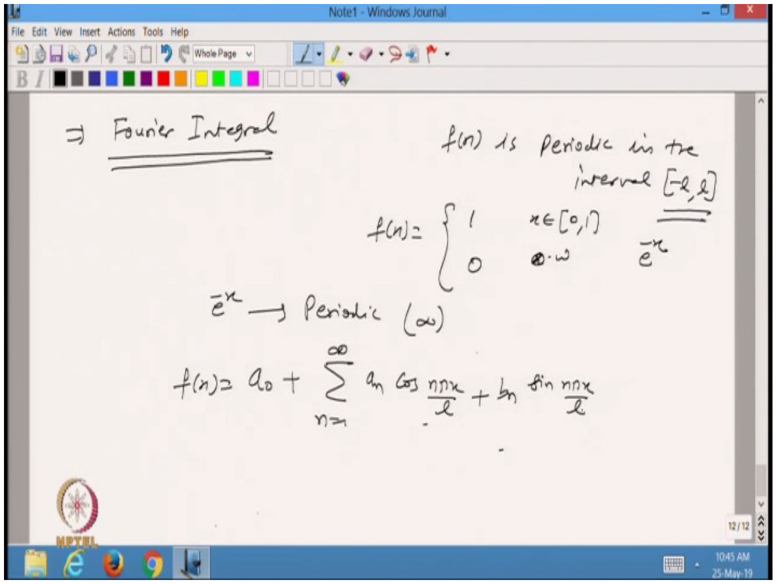

Now we going to the next topic and that is called Fourier integral now we know that the Fourier series is possible only when the function f x is periodic function periodic in the interval - l to l so in this case my function is periodic but what will happen when the function is not periodic because in generally we do not have my function all the functions are periodic function so in that case what will do like I have a function suppose I define a function like this one which is 1 when x belongs to some interval.

Suppose I say 0 to 1 otherwise 0 when so this type of function so this is not a periodic function or I have an exponential function e raise to power - x that is also not the periodic function in the finite interval for - l to l so if I have a function which is not a periodic function in the finite interval then how I can find the Fourier series so in that case obviously we are unable to find the Fourier series so if I have a function like this 1 - x then I can say that this function is periodic and having period infinity.

So that I can say that this function is a periodic function but it is interval of the period is can be treated as a infinite infinity so in that case whenever the function is not periodic in the finite domain then generally we treat that function as a periodic function and it has a period infinity so what will happen whenever I have a because for the finite interval I note that the Fourier's I can write the Fourier series or the function as this one cos n pi x by l + bn sin n pi x by l but what will happen in this case now my l is going to infinity.

And if the l is going to infinity then how this will treat and then in that case this summation will, what is the meaning of the summation in that case whenever l is going to be infinity, so that will, that how we can find out the Fourier series or we can say that the Fourier integral when we function is not a periodic function and it is period is stated as a infinity so these things we will do in the next class because it is going to take some time to define that how the Fourier series can be converted into the Fourier integral. So this we will do in the next class. Thanks watching thank you very much.