Introduction to Methods of Applied Mathematics
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Lecture - 02
Introduction to First Order Differential Equations (contd…)

So welcome back viewers. So in the previous lectures, we are able to solve the linear differential equation, first order linear differential equation. So today lecture we will solve some important applications and that we start with the first nonlinear equation and that is Riccati equation, the very famous Riccati equation.

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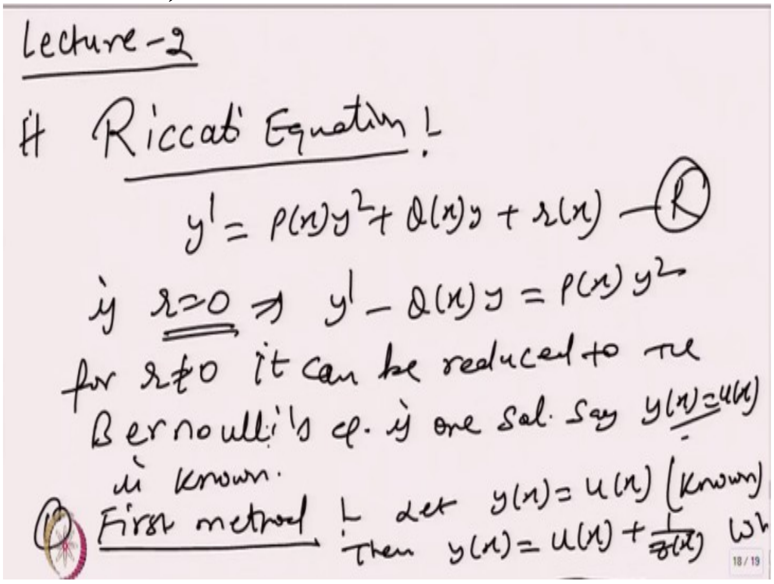

So like the Bernoulli equation, the Riccati equation also highly nonlinear. So the first order, this is the first order nonlinear differential equation of the type y dash is equal to some Px y square plus Qx y plus some rx. So where this function px, Q x and rx, they are continuous function of x. So in this case we have a y square. So that is this equation becomes a nonlinear equation.

Now if in this equation I put r = 0, then you can see from here then this equation becomes y dash – Q(x) y = P(x) y square. So that is the Bernoulli form with n = 2. So this is the Bernoulli equation for r = 0. Now for r is equal to not equal to zero it becomes nonlinear and we want to solve these equations. So however the Riccati equation can be reduced to the Bernoulli equation.

So for r is not equal to zero it can be reduced to the Bernoulli equation if one solution say yx = ux is known. So how to do that? So I will tell you the two methods to solve the Riccati equation. So the first method is, so this is the first method. So let yx = ux is known. So one of the solution is known to me. Then I take the transformation yx = ux + 1 over some zx.

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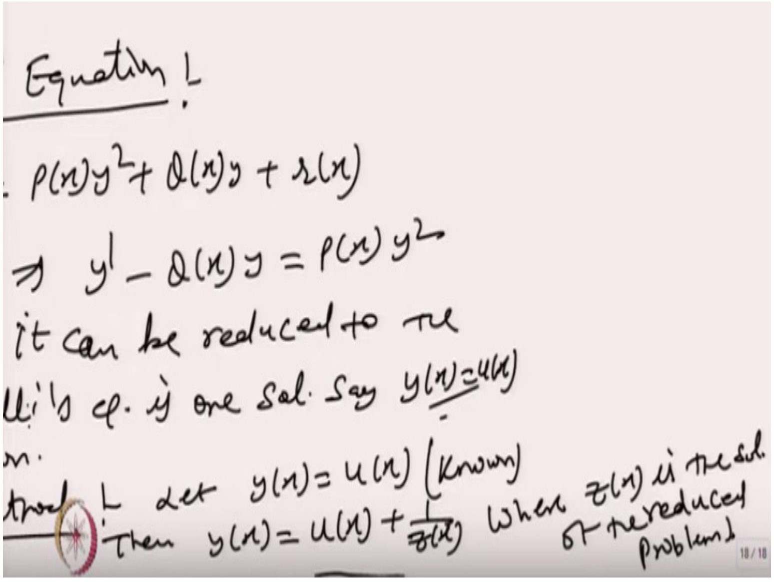

Where zx is the solution of the reduced problem. So now I will use this transformation to solve this equation.

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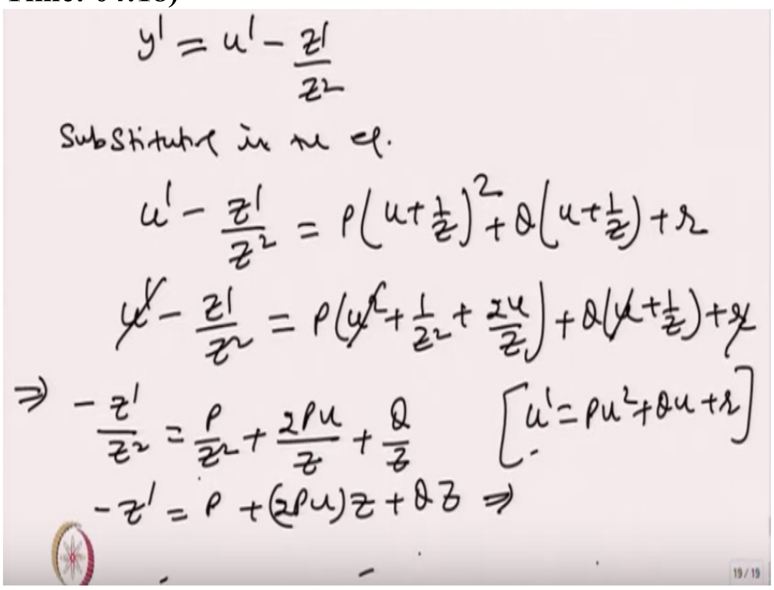

So now my y dash can be written as u dash minus, so 1/z becomes z dash over z square. So y dash become this one. Now substituting in the equation that is the Riccati equation. So I can give the name to the Riccati equation as R. So now substituting this in the equation R, I can write that as u dash – z dash over z square = P (u + 1/z square) + Q (u + 1/z) + r. So this is a equation I got.

So from here I can have z dash over z square is equal to P, I expand this one. So it becomes u square + 1/z square plus two times u/z this and this becomes Q (u + 1/z) + r. Now I know that the ux is the solution of, so I know that my u dash is equal to P u square + Qu + r because I know that this is one of the solution of the Riccati equation. So if I have this value, then if you see I can cancel out this term, this term, this term with this term.

So after canceling this equation I left with –z dash over z square is equal to P over z square plus 2Pu over z plus Q over z. Now I multiply both side by the z square. Then I will get my z dash is equal to P + 2Pu z + Qz which can further be written as my z dash.

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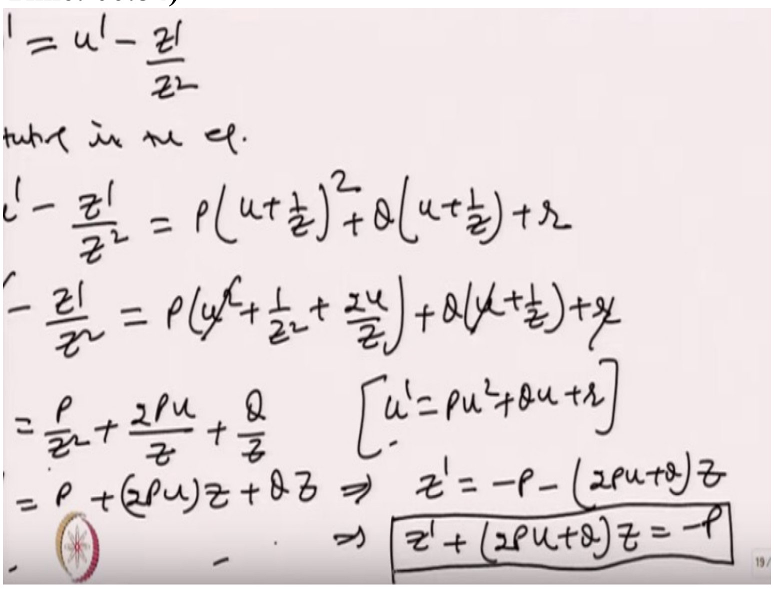

So this can be written as z dash = –P – (2Pu + Q) z and which further can be written as z dash + (2Pu + Q) z = -P. So this is the linear first order equation, which we know how to solve and once I solve this equation, I will get the value of z and then putting the value of z then I can define, I can solve. Because u is already known to me.

So I know the u, I know the z and then I can put back to the substitution and find the solution yx. So let us solve one example. So this is the example we are going to take.

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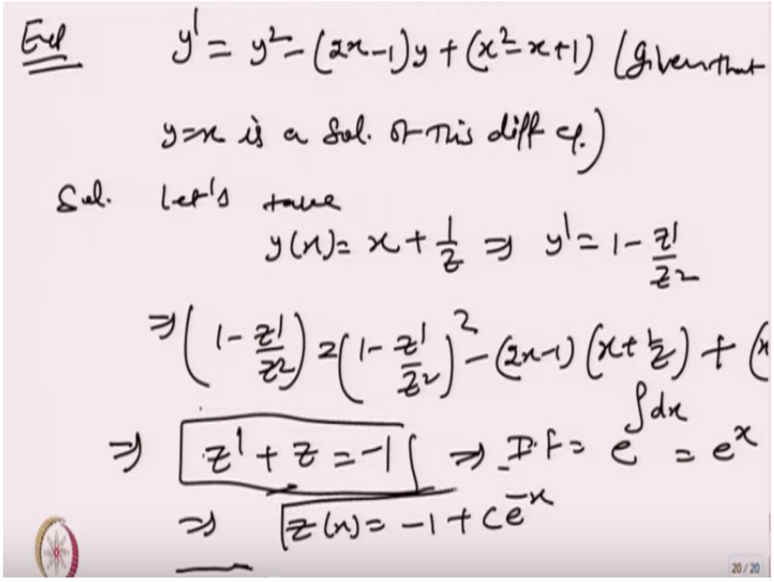

So let us solve y dash = y square – (2x – 1)y + x square – x + 1. So this is the Riccati equation we are taking. So in this case I want to solve this one. It is given that y = x is a solution of this differential equation. So it is given to me because I can start only with that if I know the one solution of the given equation. So for the solution, so let us take transformation yx = x that is known to me plus 1/z where z is the solution of the reduced problem.

From here I know that y dash will be 1 – z dash over z square. So if I doing calculation we will get after solving this equation here I will put I will get the 1 – z dash over z square. So I substitute in the equation. So it becomes 1 – z dash over z square the whole square minus 2x – 1 and my y is x + 1/z + x square – x + 1. So this is the equation we got.

So this can be further solved and we are left with after doing all the calculation and canceling the terms I will get this equation. So this is the first order linear differential equation which can be solved further. So from here my integrating factor is this. So I can in this case my integrating factor will be e dx. So that will be e x. So from here, my z, so z is the function of x that we know.

So it will become -1 + c e raised to power –x. So that is the solution of this equation. Now I want the general solution of the equation of the given equation.

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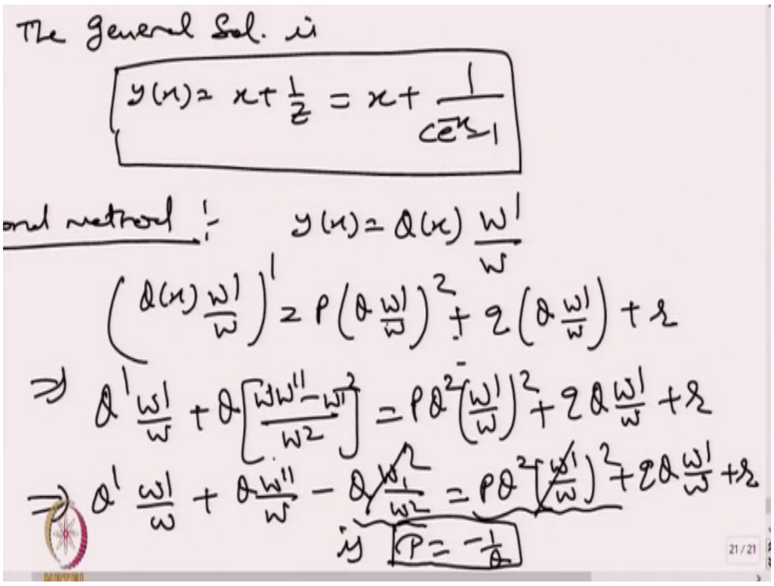

So the general is I have yx = x + 1/z and x is same and z instead of z, I can write this one value, so that become z raised to power – x - 1. So that is my general solution for the given Riccati equation. So in this case, I am able to solve the Riccati equation with the method one. So I have the another method, second method.

Now I take the transformation yx is equal to some function Qx and then w dash over w. So this is also a function of x where so I in this case I take a function, known function Qx multiplied by w dash over w. So this is my transformation and w is the solution of the reduced problem and Qx is some known function. So now in this case, I take the derivative.

So putting this in the Riccati equation I get Qx w dash over w = P Q w dash over w square + q Q w dash over w + r. So this one I am getting. I am just substituting in the given Riccati equation. So from here, if I solve this one, I will get Q dash by taking the multiplication w dash over w plus Q and then it becomes w square by w double dash minus w dash square. So this is the derivative we have taken.

And on the right hand side I will get the same factor. So PQ square w dash over w whole square plus q capital Q w dash over w plus r. Now in this case, if I simplify further, so I am doing the simplification. So Q dash w dash over w plus so this can be written as Q w double dash over w because w will cancel out minus Q w dash square over w square.

So this becomes w w dash square plus qQ w dash over w plus r. So we are left with this factors. Now if you see clearly from here in this case, if you see that we are going toward the second order differential equation, but we want to reduce the factors because this is the problem because we have a square of the w and w dash.

So now from here you can see that this factor and these factors can be cancelled out if I can choose my P = -1/Q because, if I put P = -1/Q in this factor then this and this will cancel out. So from here I have relation between P and Q that if I choose my P = -1/Q then this factor will be cancelled out.

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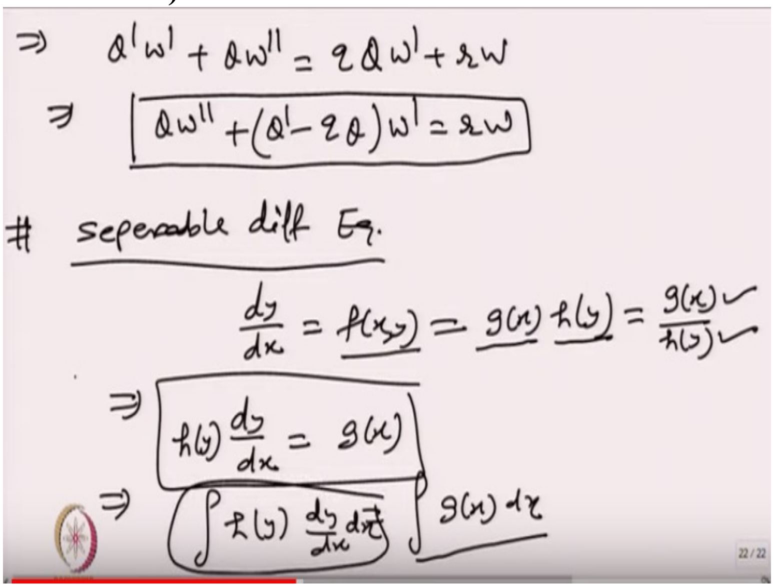

So after canceling this we left with Q dash w dash plus Q w double dash is equal to q capital Q w dash plus rw. And it can be written as this one further reducing. So I can write as w dash minus qQ. Then w dash is equal to rw. So this is the linear second order differential equation. So that we will going to do further in the next few lectures that how to solve the second order differential equation, but this is another way of solving the Riccati equation.

Now so after solving this Riccati question (equation), which has this special form, now we start with another type of first order differential equation, a general differential equation whether it is a linear differential equation or nonlinear differential equation. So let us take a separable form, separable differential equation. Because till now we have solved the linear differential equation and we have solved some special type of nonlinear equation that is the Bernoulli and the Riccati.

Now we take, we move further and we will take a differential equation, first order differential equation, which is called the separable differential equation. So let us take I will define a differential equation dy/dx = f(x, y). So this f(x, y) is a function which has a function of x and y. So let this function f(x, y) can be written as some gx into hy.

Now we can write this function as this one. So I can say that my function on the right hand side is separable in the form of a function which is function of x only and multiplied by some function in the y or I can write that I can have a function of this type, which can be written as this one. So in this case, also I am able to separate the function in the function of x or in the function of y.

So once I know this equation that this function is separable. And in this case, I am not saying that this function f(x, y) is a linear function. It can be linear and it can be nonlinear. So what I do is once I have this value, so I can solve, suppose I take this function, form of this function, then I can multiply by hy both side and I will get gx. So this is the factor I am taking.

Now, so this I want to solve which now I will try to solve this equation. So this equation I want to put in this form. Some form like some function of Fy and then so this one I want to solve. Now what I do is I will integrate this one both side with respect to x. I have my hy dy/dx = gx dx. So with respect to x I am integrating. Now what I do, I want to solve this differential equation.

So this differential equation can be solved because the problem here it is integrable but how we solve this one? So we further simplify this.

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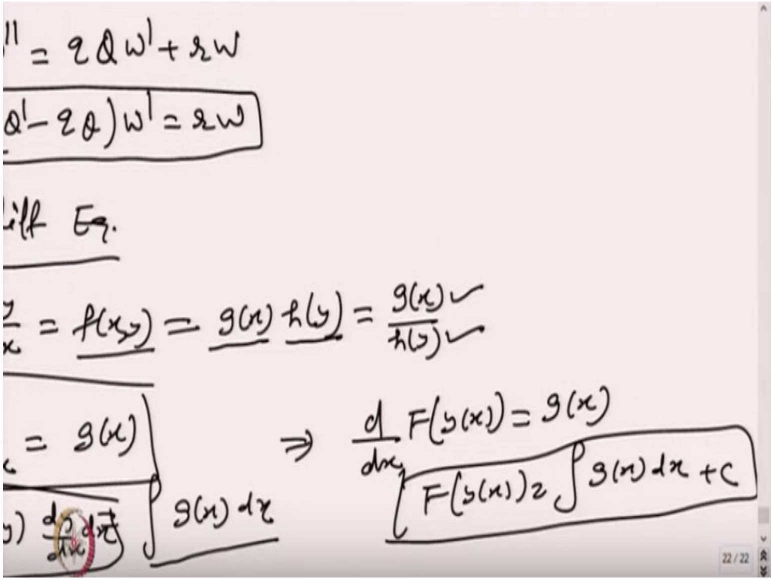

Now what I want, I want to write this in the form of some function d by dx of F yx = gx. So this one I want to do because in that case I am able to do that taking the derivative and then by taking the derivative the solution becomes gx dx and then I can solve this differential equation. So how to convert this one in that. So what you do is that I will define the function Fy.

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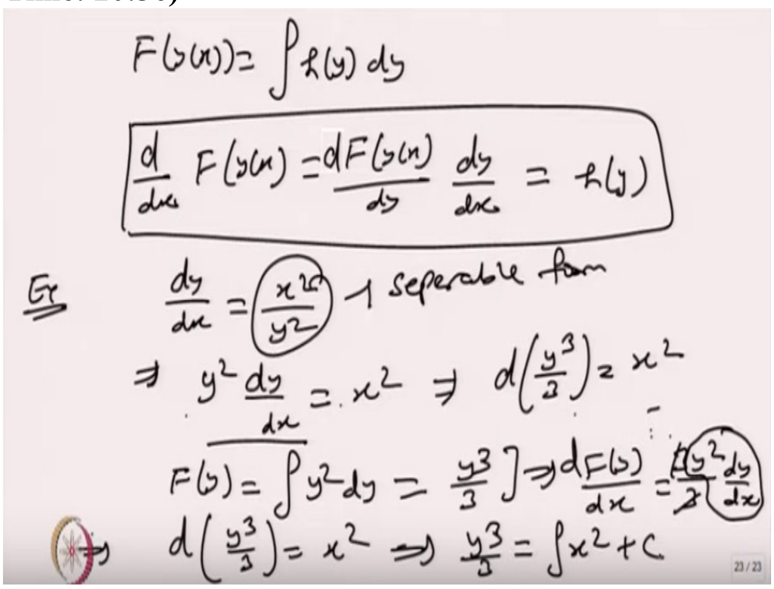

So this Fy the function I have taken is the integration of the function hy dy because if I choose this function as this one, then I know that if I take dy dx of yx then it will becomes, so I am taking with respect to x. So it will become F yx with respect to y and then d by dx. So and this will becomes, so I am taking with respect to x now. So that becomes a function of x, function of y sorry.

So now in this case I am going to do that. So let us to take one example, it will more clear how to solve. I have a differential equation dy/dx = x square + y square. So this is a nonlinear equations. And this is a separable form. So I can add that it is a separable form. Now I can take y square multiply it both side by the y square, so I get y square dy/dx is equal to x square. Now, what I want?

So this can be written as I can write this one as d by y cube by 3 is equal to x square. So what I am doing now? I am defining my function F(y) is equal to integration of y square with respect to y. So this is becomes y cube by 3. And if I differentiate this with respect to x, then I will get F(y) with respect to x. So this can be written as 3y square by 3 into dy by dx. So this will cancel out and you will get this factor.

So that is same as this factor. So from here I am able to find my capital Fy, that is the integration of this. So using this one, now I have converted my differential equation into the simpler form this one and which can be further solved by taking the integrating both side. So if I integrate both side with respect to x then I will get y3 is equal to 3 is equal to, so this is equal to integration of x square plus c.

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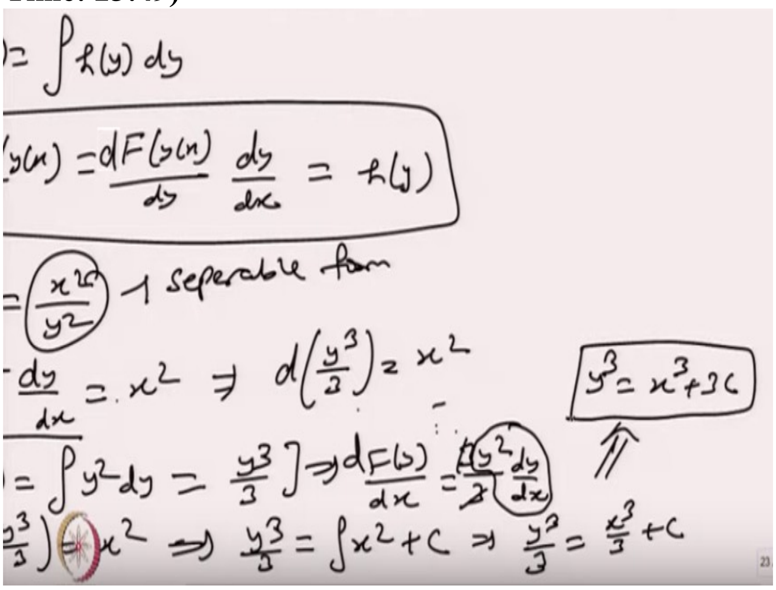

And which can be further solved. So y cube/3 = x cube/3 + c and the answer of, so from here, my solution is y cube = x cube + 3c. 3c is again, so that is my solution.

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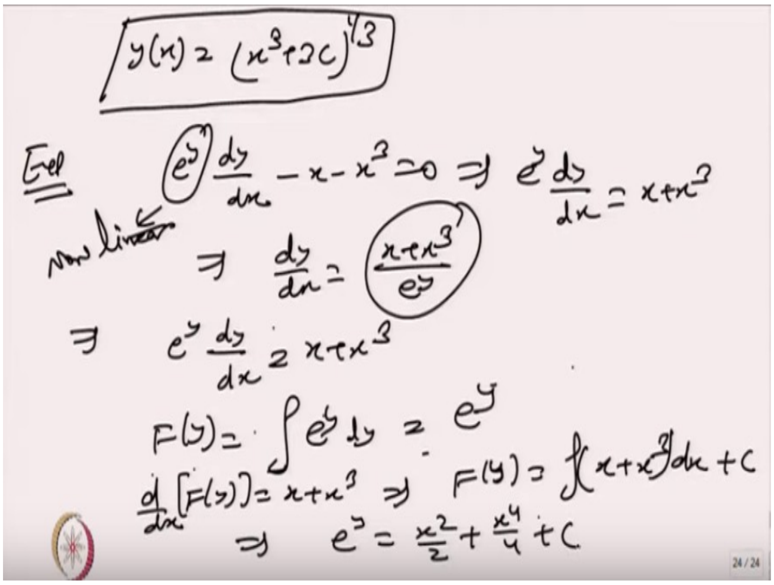

And if I want to write this solution in the explicit form, so I can write that yx will be in this case x cube plus 3c power 1 by 3. So that is the general solution of that separable differential equation. So we are able to solve the separable differential equation. So let us take another example. I take another example dy/dx multiply by exponential –x – x cube = 0. So in this case I have this factor, exponential.

So I know that this is highly nonlinear. So this is nonlinear differential equation. But so I want to solve this one, I do not know how to solve this one. But let us try to make it as a separable form. So from here, if I see this one I can transform or I can write this equation in the standard form. And this is a separable form. So from here, I can write dy/dx = x + x cube by e raised to power.

So in this case this is a separable form. So once I am able to separate my right hand side function, then I can solve this one. So I get this value. So in this case, I want to find my Fy first. So Fy will be the integration of e y with respect to y. So that will be E y. And from there, I can say that so I can write that d/dx F(y)] = x + x cube where Fy is this one. So from here, I can solve this one further.

So from here, I can take the integration both side and then my Fy will be integration of x + x cube with respect to x plus c constant. So from here my Fy I know, this is e y. So it can be written as x square/2 + x power 4/4 + c. So that is my solution.

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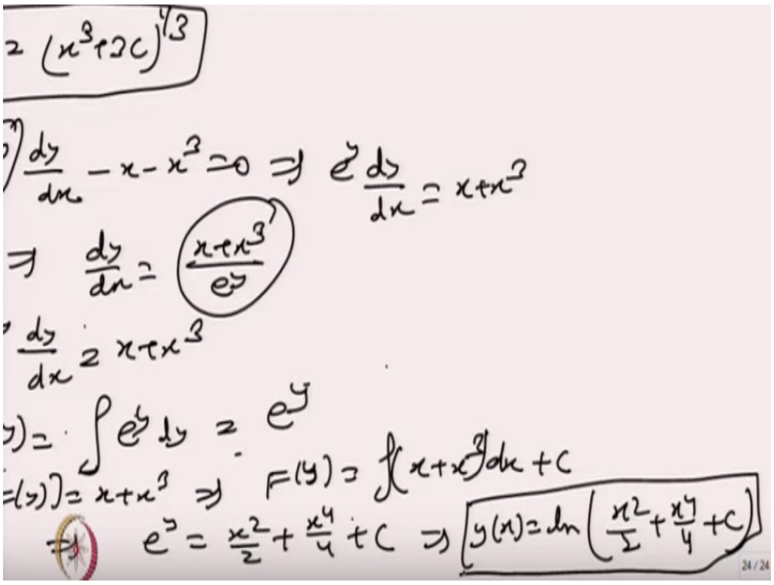

Which can further be written as, I am taking the logarithmic natural over both side. And from there, I can write my yx = ln over x square 2 x 4 + c. So this is my general solution for the given separable only. So using this one we are able to solve a nonlinear differential equation. But if the right hand side function is separable form. Now we have another important classification of the differential equation. The first order I am talking about and that is called the exact differential equation.

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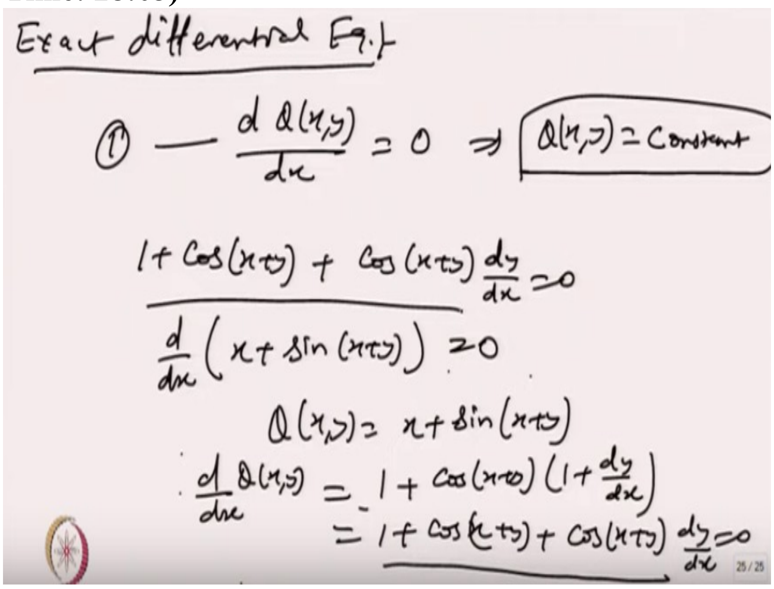

Exact differential equation. So in this case, what I want is that, that if I have any differential equation and I am able to transform that differential equation in this form, is equal to zero where q is a function of x and y. It can be linear, nonlinear anything. Then I know that from here I can integrate and I can show that this is equal to a constant. So if I am able to transform a differential equation into this form, so this form I can write as one then I can solve the differential equation.

So what is that, our purpose is that whether we are able to transform the differential equation into this form or not. So for example, I will take a equation like this one 1 + cos (x + y) + cos (x + y) dy/dx = 0. So this is the first order differential equation. And this is nonlinear because we have cos terms for x + y. If we expand that one then we have a power of x and y. So this is a nonlinear equation.

And I do not know how to solve this one. But if you little bit carefully look at this one then you will see that this equation can be written as it can be written as dy/dx of this factor, x + sin x + y = 0. Because if you take the derivative of this factor then, so where in this case I have my Q and xy this function as x + sin x + y okay. Now if I take the derivative of Q xy then this will becomes 1 + cos x + y into 1 + dy/dx.

So that will be my derivative and this can be written as 1 + cos x + y. I will be multiplying this one plus cos x + y dy/dx and that is equal to zero. So if you see this one that this becomes same as this one, differential equation.

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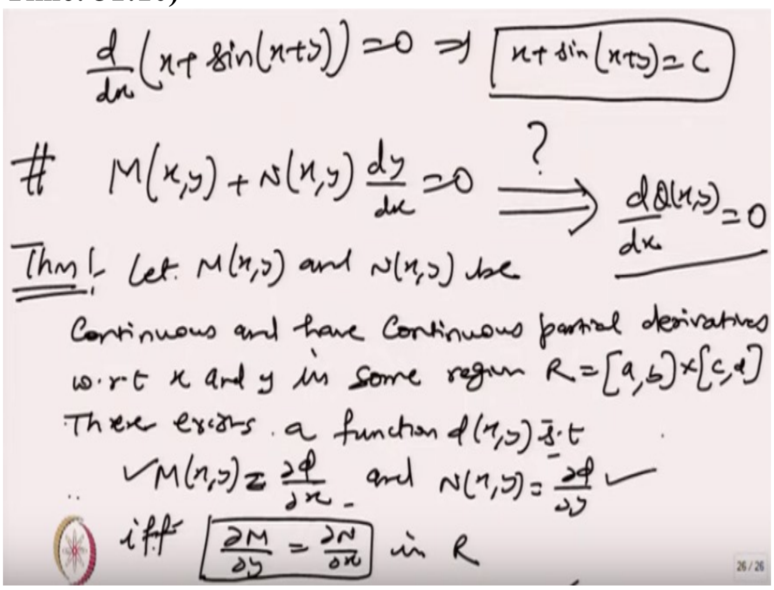

So it means I am from here, it means from here I can write that my differential equation becomes dy dx is equal to x + sin x + y = 0. And I can integrate both sides with respect to x and from here I can get the solution that my x + sin x + y equal to some constant c and that is my solution. So this is a implicit form. We in this case, we are unable to write the solution in the form of explicit form, but here we this is the implicit form.

So how is (means), the next question is that whether we are able to convert all the metric all the differential equation in this form or not. So there is a theory for that. So our what is our purpose now. Our main purpose is that suppose I have a differential equation, first order differential equation M xy + N xy dy/dx = 0.

Now the question is that whether we are able to convert this one into this form dy dx some function x y equal to 0. So that is my purpose that whether I am able to solve this differential equation or able to convert this differential equation into this form, because if I am able to solve, able to convert this one then it is also solvable, very easily we can solve and we can find the solution for that.

So there is a theory behind that. So let us define one theorem based on that. So let M x, y and N x, y be continuous and have continuous partial derivatives with respect to x and y in some domain region R. So R we define as some a, b multiplied by c, d. So this is the region, square region, a rectangular region. So this is a rectangular region.

Then there exist a function phi x, y such that my M x, y is equal to this is equal to del phi by del x and N x, y is equal to del phi by del y if and only if del M del y is equal to del N by del x in R. So if this is given that if I have del M by del y is equal to del N by del x then from these factors, these two equations, I can find the value of the phi and once I am able to find the value of the phi, then I am able to solve this differential equation.

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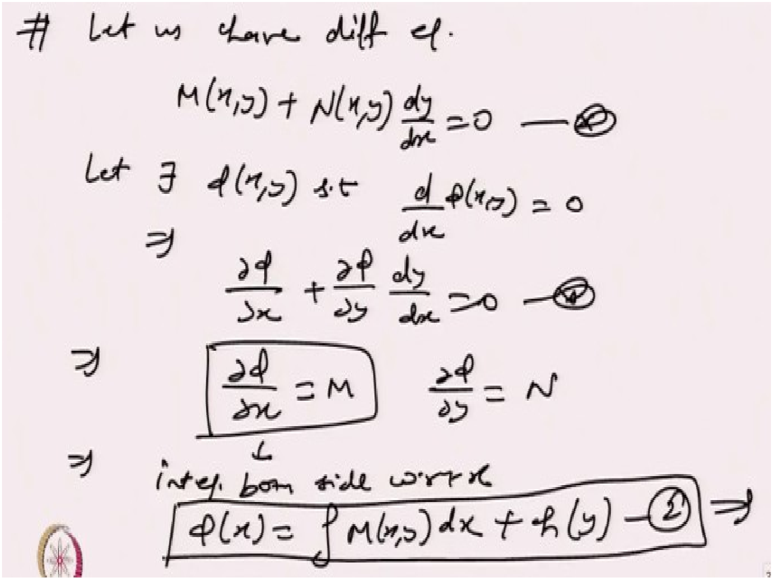

So let us do this one for the proof of this theorem. So lets we have let us have differential equation plus dy/dx = 0. Now let there exist function phi x, y such that d/dx phi x, y = 0. So from here I take the derivative of this one. So this is total derivative. So I know this derivative can be written as d phi/dx + d phi because it is a partial derivative so I should write a partial form.

So it will be total derivative del phi/del x + del phi/del y dy/dx = 0. So if I compare this with this, from here I get that if I have this phi then del phi/del x = M and del phi/ del y = N. e Now I want to find so from here, I just solve this differential equation, this one. So integrating both side with respect to x, so in this case integrating both side with respect to x I will get phi x, y = M.

So this is understood that it is a function of x and y dx plus constant of integration. So in this case we have the partial derivative. So the constant of integration will be a function of y because if I take the partial derivative with respect to x both side then this will be 0. So this is my, so I can write that this is my equation number 2, so I have the solution of this one. Now, so this equation.

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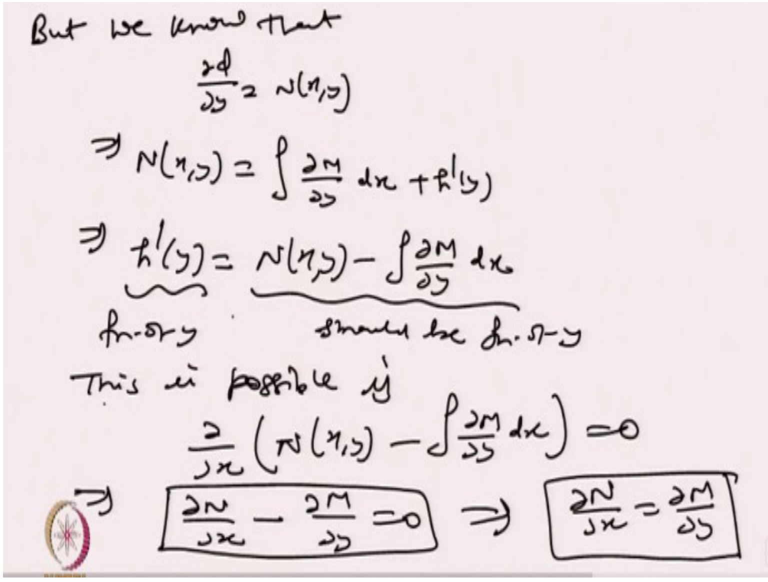

But from here I know that this so but we know that del phi/del y = N x, y. So from here I taking the again integrating both side with respect to y. I can write that, no but this is given to me. So N x, y is given to me. Phi is also given to me. So from here I can write del M/del y dx + h dash y from here. So because this is now I have this equation 2. So what I do is that I differentiate this one with respect to y.

So if I take the differentiation with respect to y what I will get? Del phi/del y = taking the derivative with respect to y and this will h y. So this is written and that is equal to N x. So I have this factor now. So from here I can write h dash y = N (x, y) – del M/del y dx. So this is a function, now what is this? It is a function of x, function of y sorry.

And on the right hand side it should be also, should be function of y because if we have a function of y on the left hand side, we also have a function of on the right hand side. So this is possible if partial derivative with respect to y of the right hand side and x, y – del M/del y dx = 0. So because if it is a function of y only and I am taking the partial derivative with respect to x, then it should get the value of equal to 0.

So from here what I will get is that del N/del x minus, so I am taking this one. So this will becomes del M/del y =0, because here I am taking the derivative, first I am taking the integration with respect to x and then I am taking the derivative with respect to x. So by the fundamental theorem of calculus, I will get only del M and del y so that become this one. So from here what I will get is that dell N/del x = del M/del y. So now I got this one.

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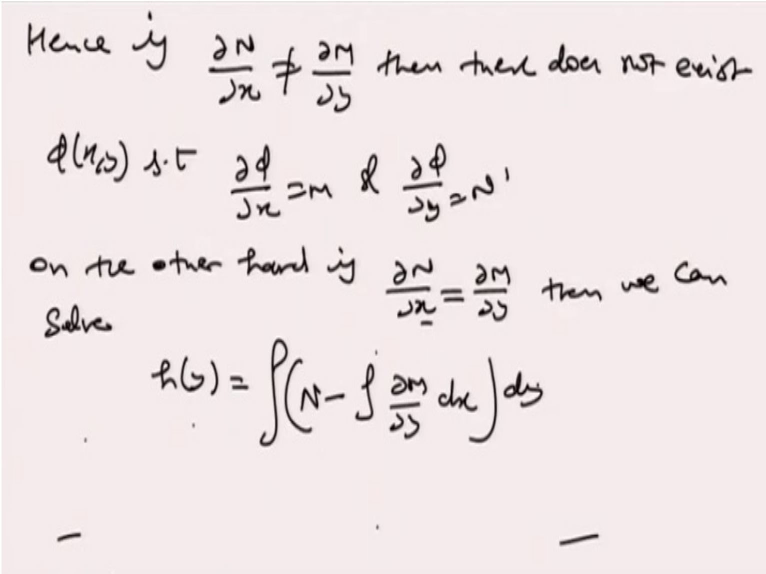

Hence, if del N/del x is not equal to del M/del y then there does not exist function phi x, y such that del phi/del x = M and del phi/del y = N if it is not equal, right. But if it is equal to that one, on the other hand if del N/del x = del M/del y then we can solve hy is equal to, that is the integration we have taken, del M/del y dx with respect to dy to find phi x, y.

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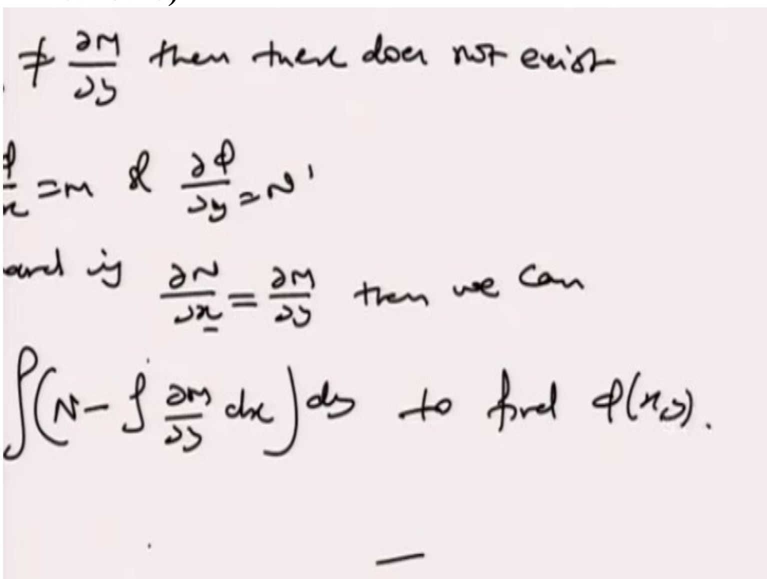

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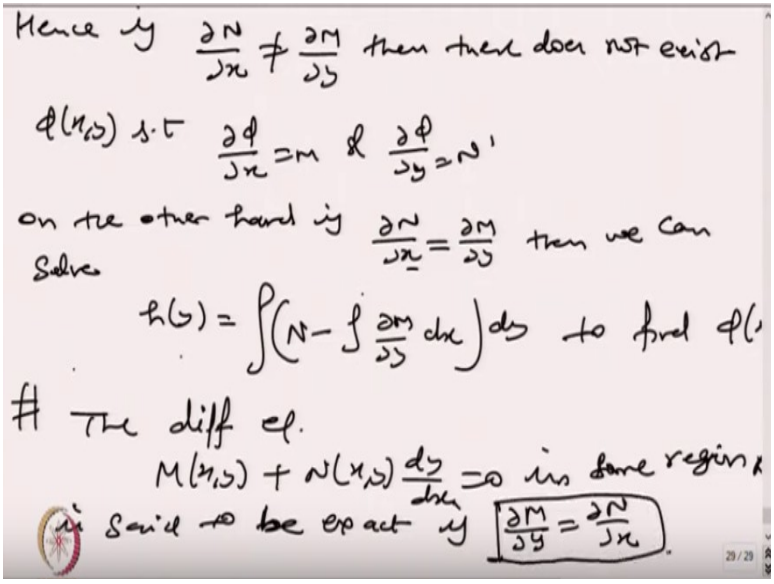

So from here I can write that, so this is the statement I can say that differential equation M x, y + N x, y dy/dx = 0 in some region R is said to be exact if del M/del y = del N/del x. So this is the end of the proof. So let us do one example.

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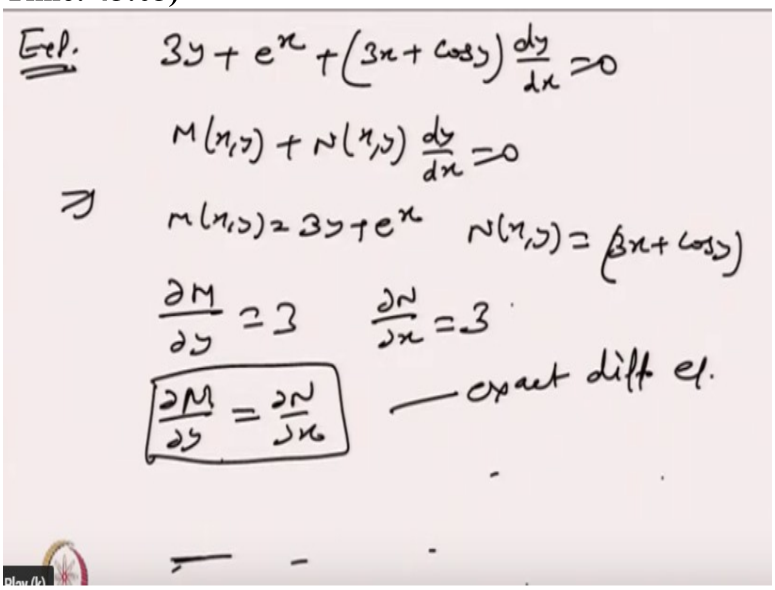

So I want to solve a differential equation 3y + e raised to power x + 3x + cos y dy/dx = 0. So this is a differential equation I want to solve and this differential equation has a factor cos y. So it is a nonlinear equation. Now if I compare this differential equation with the standard form that standard form is M (x, y) + N (x, y) dy/dx = 0 then from here I know that M (x, y) = 3y + e raised to power x and N(x, y) = 3x + cos y.

Now I just want to check that whether, so what is del M/del y? So it is 3 and del N/del it is also 3. So in this case my del M/del y = del N/del x. So if this is then I can say that this differential equation is a exact, is a exact differential equation. So if it is a exact differential equation I can solve this differential equation. So now from here, so now I want to find a factor.

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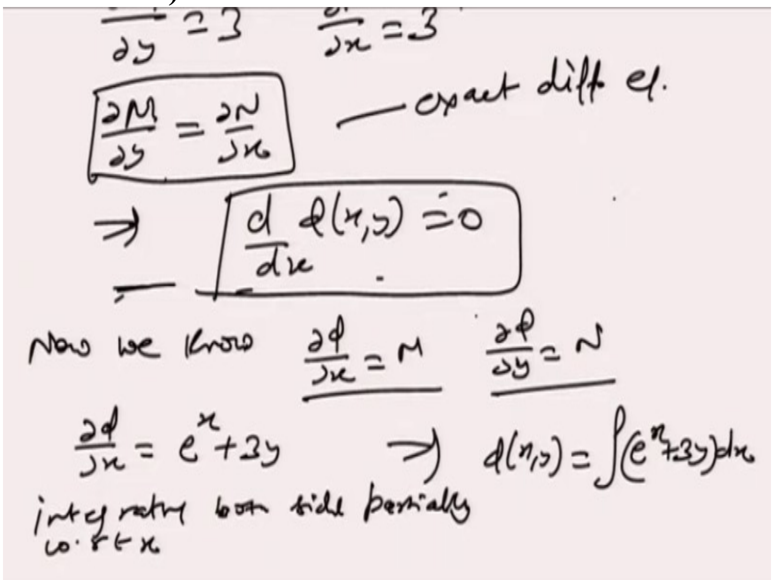

So from here one thing is true that now I can write this differential equation to this form, d/dx phi (x, y) = 0. So this form I can find out. So now I want to find what is my phi. So to find the phi, so phi can be found. So now we know that del phi/del x =M and del phi/del y = N, so that we know. Because if I differentiate this one taking the total derivative then and comparing with the differential equation, I can find that this should be equal to this and this should be equal to this.

So from here so now I have del phi/del x and my M is e x + 3y. So that is given to me. Now I integrate both side, so integrate both side with respect to x. So I integrate both side with respect to x. So from here I will get phi (x, y) = (e x + 3y) dx. And now integrating both side with respect to x partially. So I should write integrate both side partially with respect to x.

So this is a I should write clearly that integrating both side partially with respect to x. So if I do this one I will get this value.

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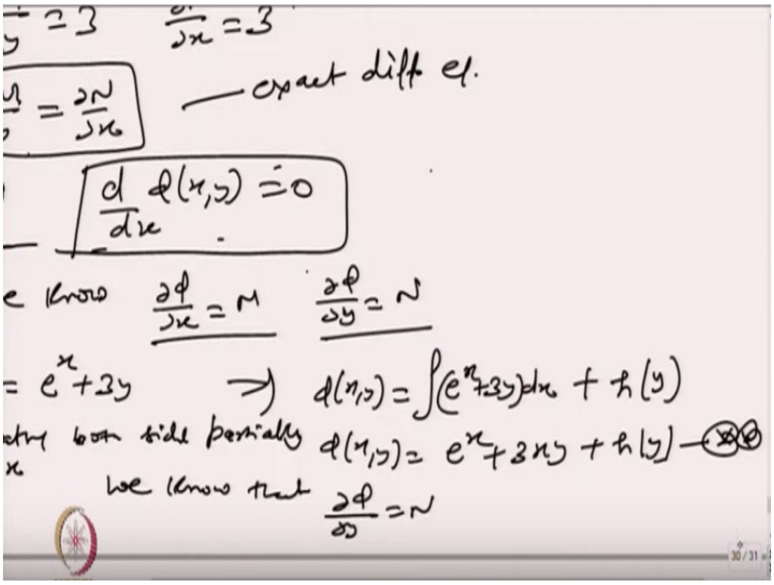

Plus some function of y because I am doing this one partially with respect to x. If I differentiate this one partially with respect to x this will cancel out. So from here I will get my phi x, y and now doing the integration partially with respect to x. So it will be e x + 3xy + hy. So that is given to me now. Now, so from here we know that del phi is given, so del phi/del y = N. So this is equation I am taking with a double star.

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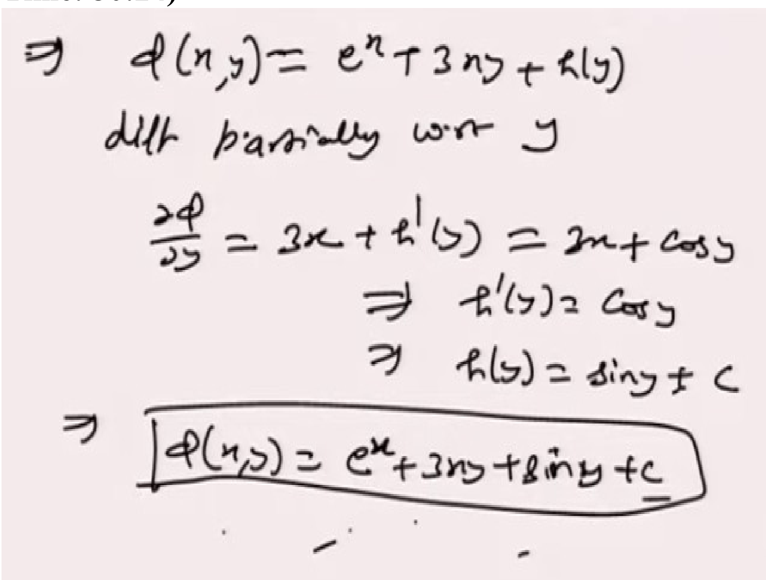

Now we are taking, so from here from the previous one I have that e x + 3xy + phi. So I have phi (x, y) = e x + 3xy and plus h y. Now what I do is that, so I differentiate this one, differentiate partially with respect to y. So I will get del phi/del y is equal to so this will be a function of x equal to 0. It will be 3x + h dash y. And this is given to me So and this is equal to 3x plus, so this equal to 3x + cos y. So that is given to me.

So from here I can compare and I will get h dash y = cos y. So from here I can get my h y = sin y plus some constant of integration, okay. So from the equation I can define my function phi (x, y) as e x + 3xy and h y is sin y and plus c. So this is my given function. So we are able to solve the differential equation of Riccati type, which is highly nonlinear.

Then we are also able to do the differential equation which is has a separable form and the differential equation which has the exact form. Now and it is independent that the whether our differential equation is linear or nonlinear. So in the next class we will discuss that how we can solve the differential equation of second order and the higher order. Because in today class and the previous class we have solved all the cases of the first order differential equation. So thanks for watching.