Introduction to Methods of Applied Mathematics
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Lecture – 19
Fourier series (cond…)

So welcome back to this course. So today we will extend the concept of Fourier series. Today we will going to start with the lecture 19.

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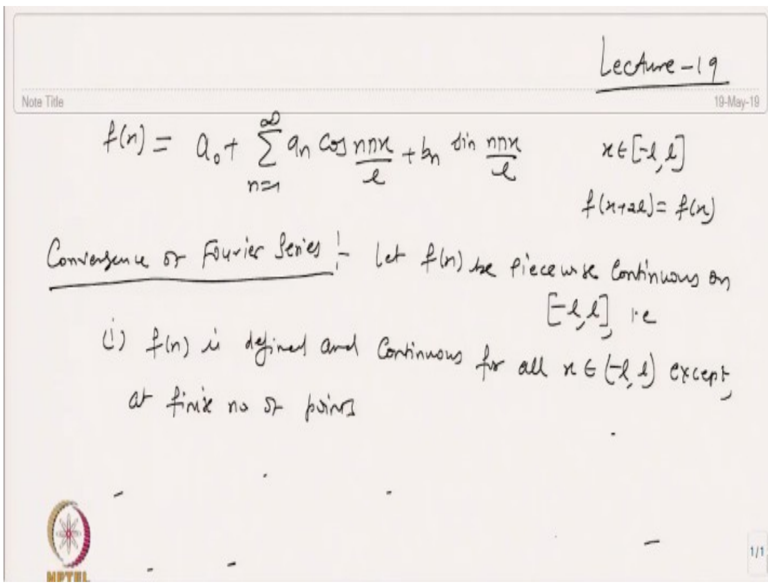

So, in the last class, I have discussed about that is when the function fx is a periodic function having a period from - l to l, then this can be written as a Fourier series a 0 + summation n from 1 to infinity a n cos n pi x by l + b n sin n pi x by l where x belongs to - l to l and this function is a periodic function with x + 2 l = fx. Now, will today I will discuss that what are the convergence of Fourier series so, what are the conditions so, in this case, so, let my f x be piece wise continuous on - l to l.

And so, what is the meaning of piece wise that is the f x is defined and continuous for all value of x belongs to open interval - l to l except at finite number of points. So, the function is continuous except at the finite number of points

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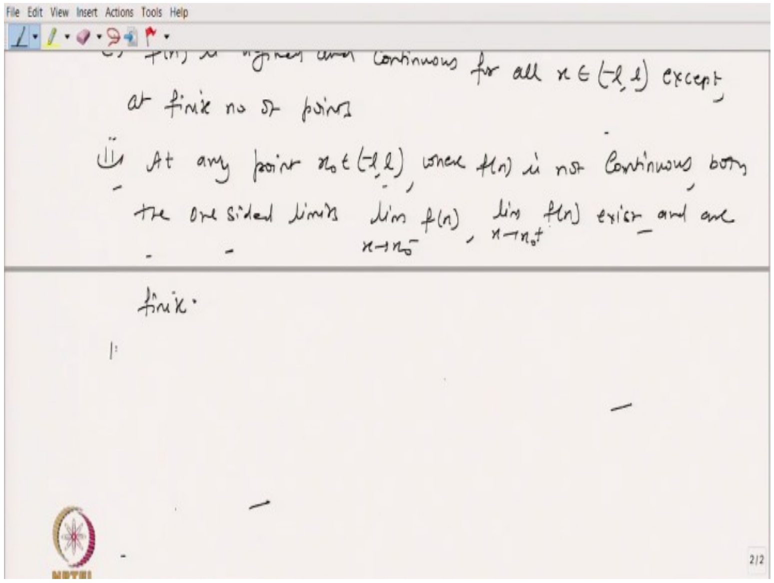

Then the second one is that at any point x 0 belongs to the interval - l to l because this is an interior point I am talking about where f x is not continuous both the one sided limit that is the limit x tends to x 0 negative f x and limit x x 0 positive fx should exist and are finite okay.

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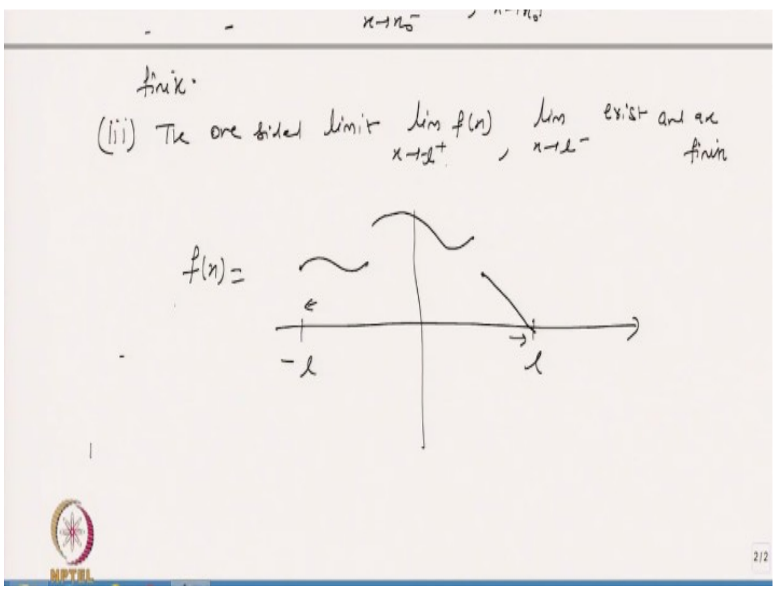

So, and the third one is that the one sided limit that is limit x tends to l positive fx and limit x tends to l negative exists and so from here because I my interval is from - l to l. So, it is - l and this is + l exists and are finite, it means that suppose I have any function so, let my take a function fx which is of this form. So, this is my x axis, this is my y axis and suppose my function is so this is - l and this is + l. So, suppose from here my function is like this to this and then from here my function is this one and then from here I have a function that is this one.

So, what I say that from this condition that this function a piece wise continuous function it is continuous from here to here from here to here from here to here, and it said that at any point x 0 between these the function is not continuous both so, like this point. So, both the one sided limit so, from here the if I take the left hand limit for this function and right hand limit for this function left hand limit for this function at this point and right hand limit at this point should exist and should be finite.

So, from here you can see that this has a jump discontinuity, you have a jump discontinuity and also that the one sided limit of this function. So, here from right hand side I want to go and from here I should go from by the from the left hand side. So, this limit should exist only then will say that this is the condition it is called a piece wise continuous function. So now suppose this function there now.

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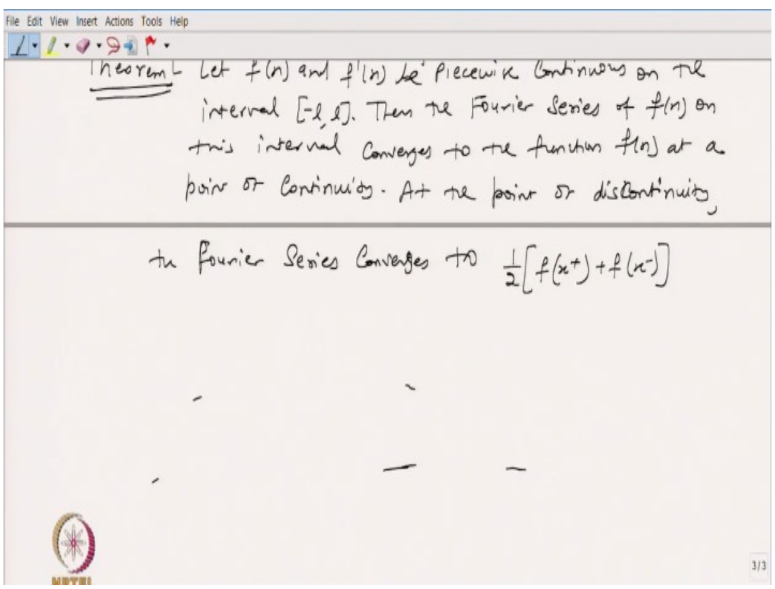

If so, this is a statement of the theorem, I am going to use now that let fx and f dash x be piece wise continuous on the interval - l to l then the Fourier series of the function fx on this interval converges to the function fx at a point of continuity. Now what about the point of discontinuity.

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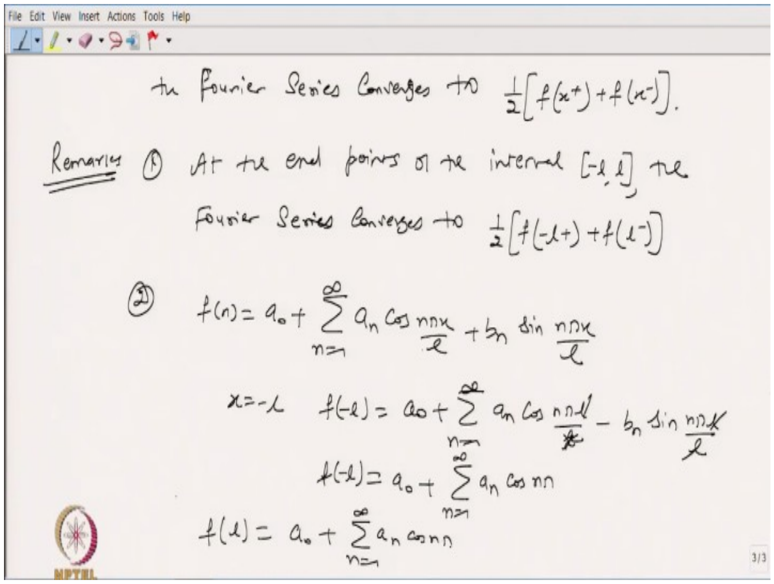

The Fourier series converges to the average. The function f at x +. So because this is a point of discontinuity at x. So what will do I will find out the left hand limit right hand limit and then taking the average, so in that case the function the Fourier series will converge to this value. So, this is the statement of the Fourier series or the convergence of the Fourier series. So, now, at this moment, we are not going to do the proof of this one, but this statement.

We just take as a statement for the converges of the Fourier series. Now we want to put some remarks. So remarks is that the first one is that at the end points, so at the end points in the interval the Fourier series converges to half f - l +. So, this will be the both the end points and the second one is that now I have the Fourier series. So my Fourier series is fx = a 0 + summation a n cos n pi x by l + b n sin n pi x by l. So, in this case, if you see that if I put x = - l, so what will get f equal to - l and from 1 to infinity a n cos - l will be cos l.

So it will be n pi l by x n pi l by l. So this will cancel out this will cancel out. So from here I will get a 0 + a n cos n pi so, because this l this l will cancel out and this is sin n pi and so for any value of n this value will be 0. So left with this only this value and similarly, if I want to find the value at l, then if you will see this one will get the same value a 0 + summation a n cos n pi.

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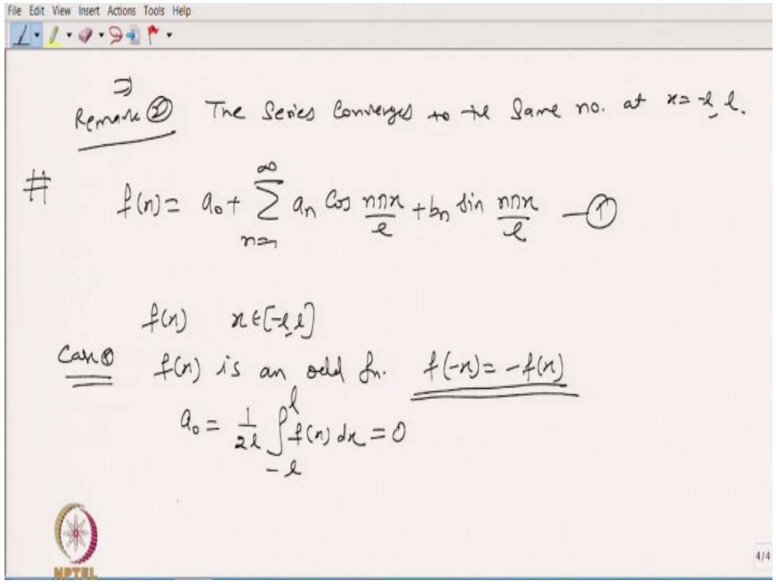

So in that case, I can say that so from here, I can say that. So this may remark number 2 that the series converges to the same number at x = - l and + l. Okay, so it means the series converges to the same number because I am putting l = - x equal - l and + l, I am getting the same value of the function. So this is the observation, now so once I am able to find the Fourier series, and then if the function is satisfying the conditions, then I can say that my series is convergent.

So let us go further and use some properties of the function to find out the Fourier series. So now I have my Fourier series a 0 + this one cos n pi x by l + b n sin n pi x by l. Now so this is I just take one now the condition, so now suppose I have a function fx for the x belongs to - l to l. So the case one is that, I found that my function fx is an odd function, odd function means I am getting that f of - x becomes so this is an odd function.

So in that case, if I somebody asked me to find out the Fourier series for this function, that function is odd function, then there is no need to find all the coefficients, and then we found that the this is a Fourier series, just before starting with the Fourier series we can say that if my function fx is an odd function. Then you know that my a 0 is a function of has a value from this to this. So this is an odd function and I am integrating the odd function from - l to l. So I know that this value will be 0.

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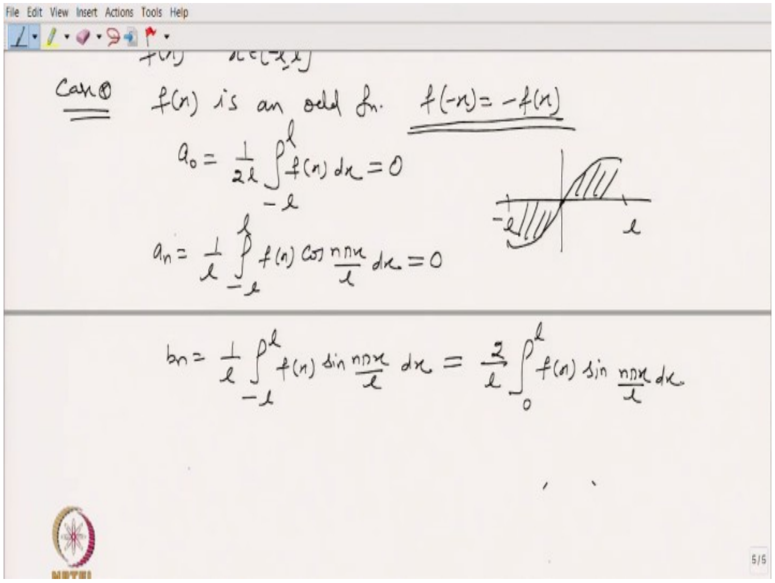

By the properties of the Integral that I am odd functions means this type function like whatever the function I have on the left hand side suppose this in my l and this is L. So, suppose I have a function of this type. So, on other side I also have the function on this type. So, this area and this area will cancel out and we get the value 0. My a n will be cos n pi x by l dx and from here also you can say that my function odd function cos function I know that is the even function.

So periodic will be the odd function and in that case also this value will be 0. So, from for the odd function my a 0 and a n will be 0 and my b n will be 1 over l from - l to l fx sin n pi x by l dx. Now, I have a odd function fx is odd function sin is an odd function and odd function to multiple of odd function is an even function. So, in this case, this is an even function and then I apply the properties of the even function that is 2 by l from 0 to l it will be fx sin n pi x by l dx. So, I am only left with b n which is not equal to 0.

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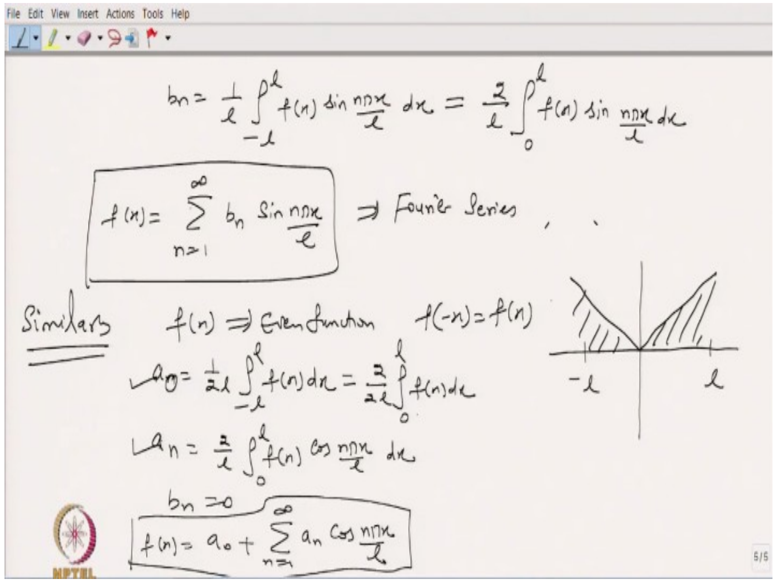

So, from here, if I want to write the Fourier series function which is the odd function, then you will see that I will come up with only and from 1 to infinity that is my b n sin n pi x by l. So, this is my Fourier series, because this is the odd function and on the right hand side also sine function in the odd function. So, I know that the equality the left and the right should we have same type so this is odd function then it should also be, for so this is the corresponding Fourier series for the odd function, similarly I can go with the function fx is a even function.

So, whenever the even function, this is my even function. And if you see the picture of the even function, that suppose this is - l this is + l suppose I have a function from here it is of this type then even means on the other side it is just the reflection about the y axis. So, in this case, you will see that this area and this area if I integrate this area and that cannot be 0. So this is my even function. So, now in that case, I want to write the Fourier series.

So, left hand side is the even function that means from my knowledge that I can find that on the right hand side which should be always even. So, you can see from here that in that case might b n will be 0, so a n will be remain like that one because a n is a 0 is 2 by l. So, this part will be always I will use a property of even function. So it is from 0 to l fx dx. So, this will be the value of a 0 similarly my a n will be dx fx into cos n pi x by l dx.

So, this is an even function that is an even function and then I use the property of the even function to find out the integration. So this will be able to find out this one and you will see that my b n in that case will be 0. So, in that case I left with only the values the coefficient of cos terms. So, from here I can find the Fourier series my fx will be a 0 + summation a n cos n pi x by l, where my a n and b n is like this one.

So this is that if I have some function, and I have knowledge about that function that whether this function is even function or odd function, then I can apply this formula directly.

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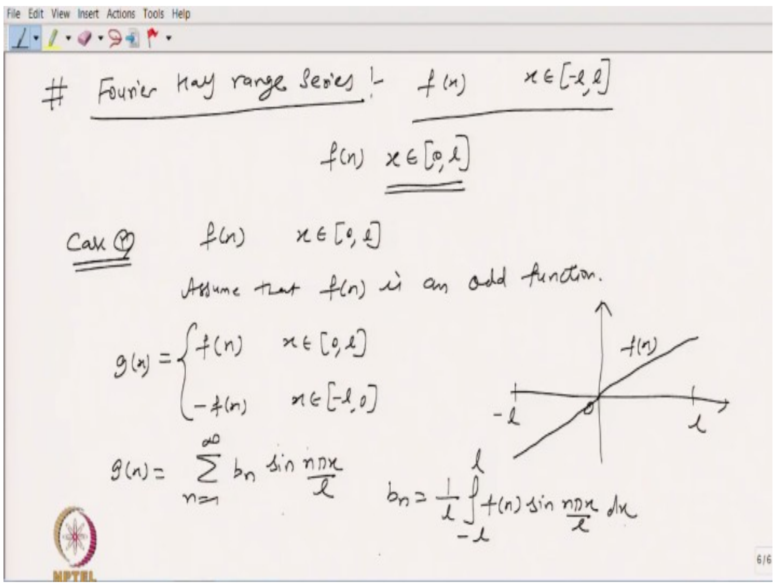

So from this one, I will going to start with the cosine series and the sine series. So let us start with the new way of doing the series is Fourier half range series. So what is this Fourier half range series? Actually, when started with the Fourier series, we know that I have a function fx and my x belongs from - l to l. The only thing is that suppose I do not know the function, I have a function whose value the x the value of the function is given to me only between 0 to l. So instead of this function of this form, now I have a function fx. So that x belongs to 0 to l.

Well, actually, if you see in general cases, always whenever we define a function, the function is given to us when the x lying from 0 to some finite value. So in that interval, we have the value of the function. So this in this case, I do not know what is the value of the function from - l to 0? So now somebody asked me to find out the Fourier series, then I know the value of the function only in the interval from 0 to l and I have no knowledge about the function from - l to 0.

So if I want to find the Fourier series, I am unable to do that because in the Fourier series we have started with the function is a periodic function, and it has a period from 2l and defining from - l to l. So what will do in this case, so in this case, we will take the help of that the case one I will take so, in this case I have the function fx this is given to me therefore the value of x between 0 to l. So this is given to me the function is well defined for the x belonging to 0 to l.

Now I take the case one and I assume that the function fx is an odd function so this is the assumption I am making that I am assuming that my function is the odd function and from here, I can define my function. So, suppose my function is given to me. So, this is my 0 and this is l. So, let my fx is given to me some function of this type. So, this function is given to me. Now, I assume that this function is an odd function.

So, from there to there – l I will extend this function from - l to 0 and I now I assume that my function, so this is my fx. So, now I assume that my function becomes fx when x belongs to 0 to l. And this is - of fx, when x belongs to - l to 0, this is I have extended this function and making this function of the odd function. Now, I know that if I have an odd function then in that case. Now I am able to find the Fourier series. So my Fourier series is in this case will be so, this is my function now I call it gx.

So, this function now, I am able to write the Fourier series. So, this is my gx and odd function we know that a 0 and a n all will be 0. So I will left with only and from 1 to infinity b n sin n pi x by l where my b n = 1 over l fx sin n pi x by l dx. So, this is I have now this function is given to me.

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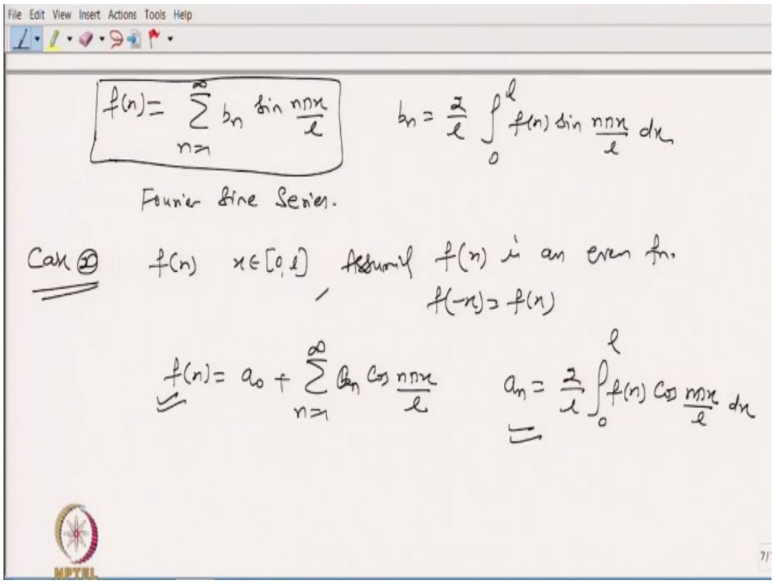

So, I know the property of the odd function and this integral become 0 to l fx sin n pi x by l dx. So now you can see that in this case, I have the value of the function from 0 to l only. And I know the value of the function from only from 0 to l. This is my assumption that I am making this odd function. So, now in this case, if I am taking this one then my if I take the x line between 0 to l in that case my function gx is the fx. So, in that case my Fourier series become this one.

So, this is called the Fourier sine series. So, now it becomes the Fourier sine series. So, for a Fourier sine series, when that series has contained the only sign terms and this is possible when my function is an odd function. So, this is the Fourier sine series. Similarly the case 2, I can take when my function fx is given to me for x belongs to 0 to l and I am considering assuming that that my function fx is an even function.

So, in that case I have my f of - x = fx and from here I can do the same way I can go and then I can write my function fx = a 0 + summation a n cos n pi x by l where my a n is 2 by l, 0 to l fx cos n pi x by l dx. So based on this one, I am able to write my a n and the function fx. So based on this one I am we are able to write the So, this is called the cosine series are the sine series and we also call it the Fourier half range series because in this case half range is given to us. So, let us do one example.

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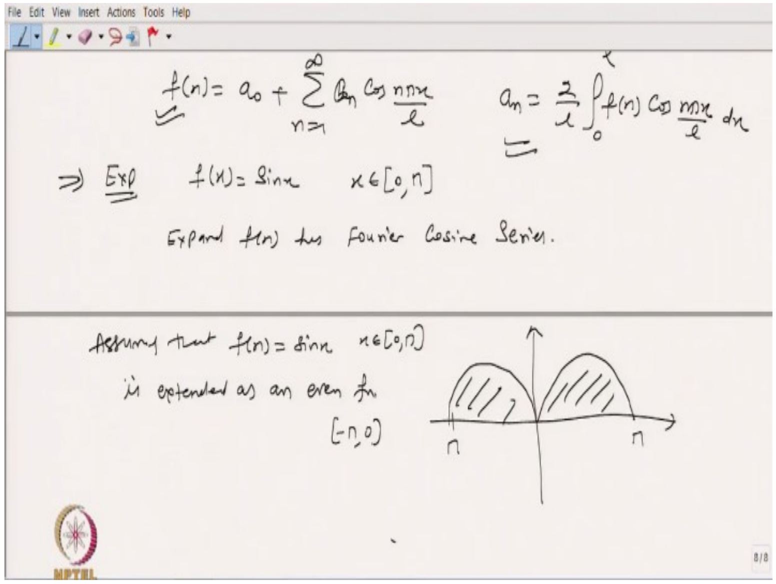

So, this is a interesting example I want to do suppose I have a function fx and this function is given to me as sin x and x belong to 0 to pi. So, this is a sine function is there now from here, but the question is that expand this one expand fx as Fourier cosine series. So, the function is a sin, but I want to expand this one as a Fourier cosine series it means now what I am need to do I have a function so, this function is given to me. So from this is my sine function, it is given to me pi and I want to find the Fourier cosine series means I have to assume that that my function fx is even function.

So, what is going to happen in this case that I am assuming that my function is look like this one? So, this is my only then this function is the even function. So, from here I can say that we are assuming that the function fx = sin x where x belongs to 0 to pi is extended as an even function from - pi to 0, only then I am able to find the Fourier cosine series.

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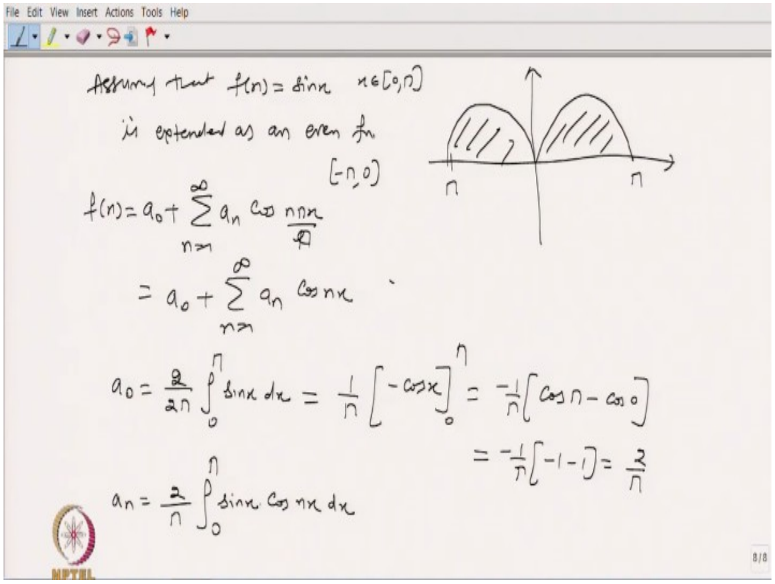

So, after doing this one, now I have my cosine series. So my fx will be a 0 + summation and from 1 to infinity a n cos n pi x by l. But in this case I am taking l = pi. So, this will be pi it will cancel out. And from here I can write my a 0 + summation and from 1 to infinity cos n x, this is already given to us. Now I want to find my a 0. So a 0 will be 1 over 2 pi and now what I am going to do is I it is a half range series so, I know that this written in 2 times 0 to pi and fx is given to me sin x dx. So, this is given to me.

So, from here I can write this is a pi and this is - cos x 0 to pi because the integration of sin x is - cos x, it would be cos pi - cos 0. So, what is the cos pi is a - 1 so - 1 -1. So, it will be 2 by pi, my a n it will be 2 over pi from 0 to pi fx is sin x and then this cos n x dx. So in that case now if it is from - pi to pi in that case I know that this function using the property of orthogonality this integral was going to be 0. But now in this case we are not taking the integration from - pi 2 pi we are taking the integration from 0 to pi.

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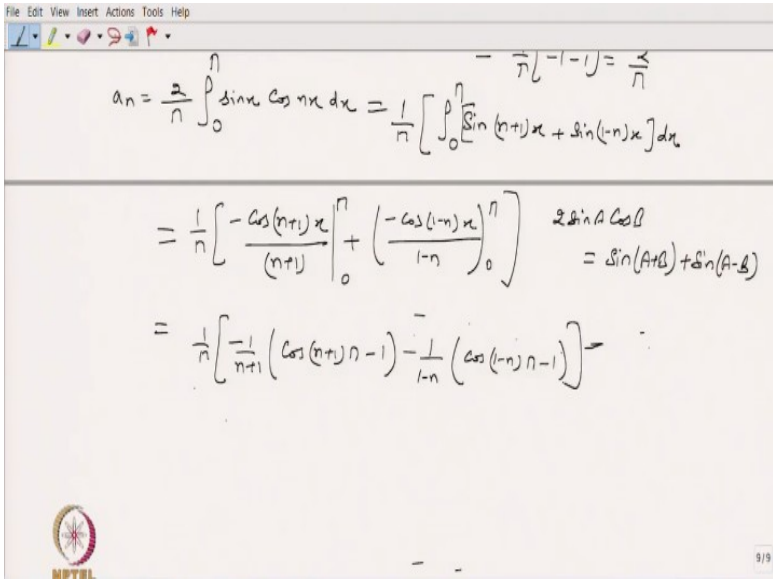

So, if I solve this one, so I can solve this integral from 0 to pi and then if I apply the formula here, so you can apply the formula or I can use as a first function of the second function. So I what I do is that I will use this formula 2 sin A cos B = sin a + b + sin A - B. So, from here I can write that this is 2 times this one. So I can write directly from here 0 to pi sin n + 1 x + sin so sin A sin B, A sin x and B sin x so A - B so x - 1 x and then dx. So this one I can do that integration.

So this 2 will cancel out because we have already used this formula and then it will be 1 over pi and then doing the integration. So it will be - cos n + 1 x divided by n + 1 doing the integration from putting the limit from 0 to pi + again it will be - cos 1 - n x divided by 1 - n disintegration. So from here I can write from that 1 over pi so this is a pi again put here. So minus 1 over n + 1 cos n + 1 pi - cos 0 0 1 it will be 0 and from here I can write again minus 1 over 1 - n cos 1 - n pi - 1. So this value I am able to find So, everything depends upon that what is the value of n.

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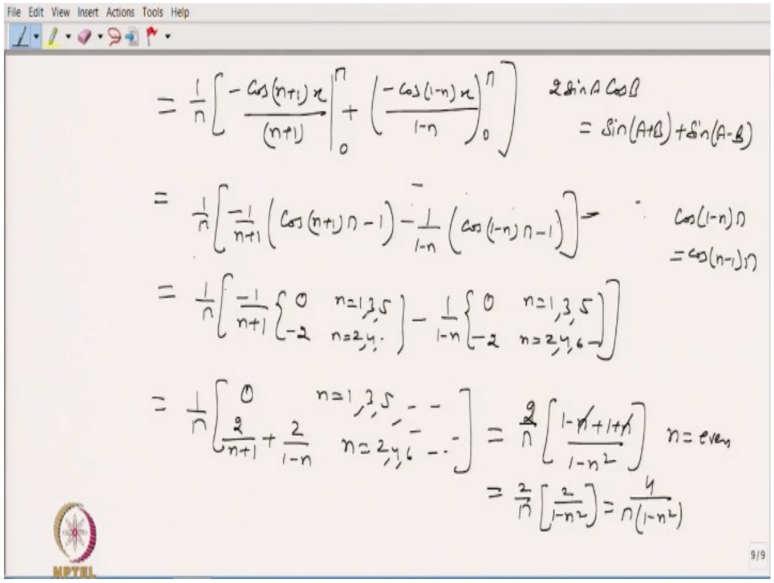

So, I can take my So, this will become 1 over pi - 1 over n + 1. Now, depending upon what is the value of n if I take n = 1 here, it will be cos 2 pi cos 2 pi is 1. If I take n = 2 it will cos 3 pi so its value will be minus 1. So depending on this value, if it is 1, then it will be 0. If it is minus 1 that will be minus 2. So depending upon that I can write here that its value will be 0 when n is 1, 3, 5 and so on. And otherwise value will be minus 2, when n is an even number 2, 4 and this - 1 by 1 - n and again this type of value.

So it if I put n = 1 here, so this will be cos 0, and cos 0 is 1, so it will be 0, when n = 2 here It will be cos 5 it will be minus 1. So again, I can write that this is 0, when n is 1, 3, 5 and so on. Because if I put cos 1 - n and same as cos n - pi, it does not matter, right. So when n = 1 its value is 0. When n = 3 it will be 2 pi 0 and otherwise it value will be minus 2 when n = 2, 4, 6 and so on. So based on this one, I can write the value 1 over pi. So this is so I am now going this one.

So its value will be 0 when n = 1, 3, 5 and so on otherwise value will be so minus minus. So it will be 2 by n + 1. And this minus will be plus 2 by 1 - n and when n is 2, 4, 6 and so. So, after finding this value. Now, from here, I just make it a little bit simpler, then when n = odd, but when n = even its value be this 1. So instead of n I can put the m here or I can write down. So this is equal to 1 over pi 1 - n square, this one I can take 2, I can take a common. So this is 1 - n + 1 + n, when n is even so it will be 2 by pi this will cancel out, certainly 2 pi 1 - n square. So it is 4 by pi 1 - n square.

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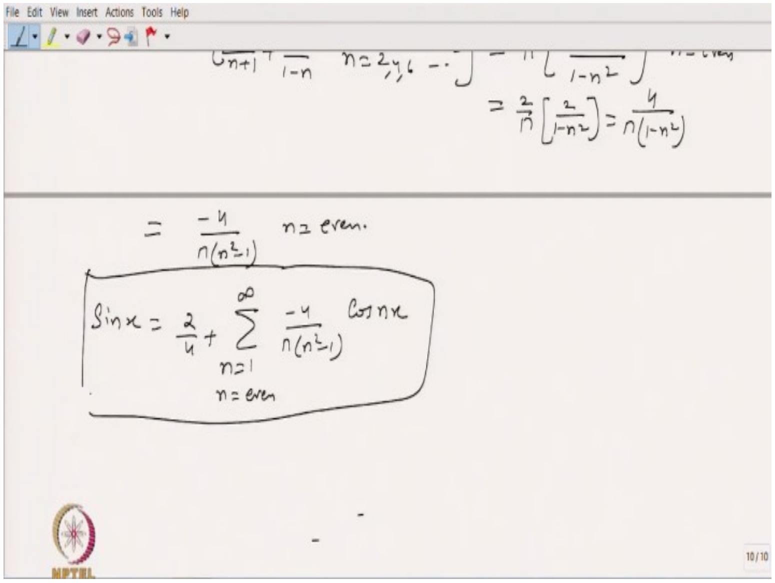

Or - 4 by pi n square - 1 when n is even. So from here, I am able to write the Fourier cosine series. So I can write sin x is equal to my a 0 was 2 by pi. So it is 2 by pi + summation n = 1 to infinity. Now I am taking n = even, so in that case, it would be - 1 by 4 - 4 by pi n square - 1 cos n x, but I am choosing only n = even. So n = 2, 4, 6 all this well we come up and this is we call it the Fourier cosine series for the function sin x.

So, that is a obviously a new thing that on the left hand side we have a sine function, but on the right hand side we have a cos function. So, this is the approximation of the function sin x in terms of the Fourier cosine series so, that is a way we can do assuming that the function on the left hand side extended to the function on the left hand side as an even function. So, after doing this one So, this is the way we have started that for a given function, any function well, when the function is well defined from 0 to l. Then we can write the Fourier half sine series or half cosine series. Now, we start with the next thing is that I have my Fourier series.

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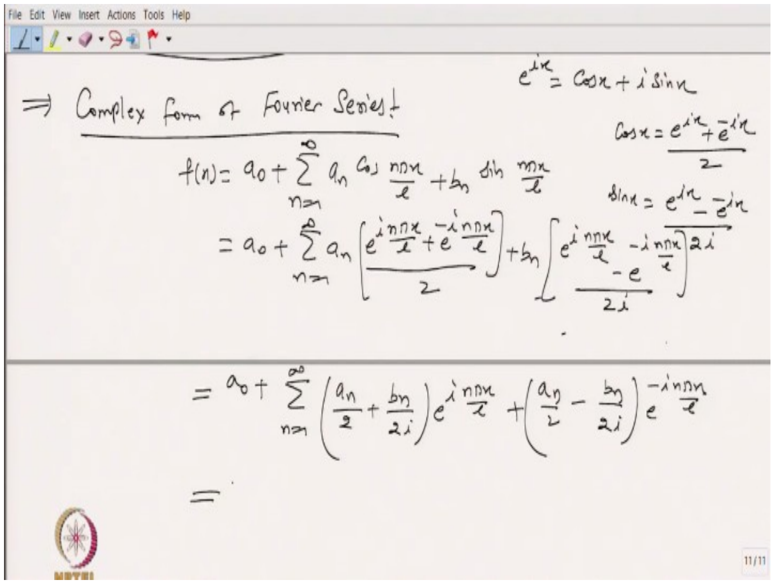

So, I just want to write the complex form of Fourier series, so, now I have my series fx as a 0 + summation 1 to infinity now, I have a n n pi x by l it is given to me. I will use this formula ix so this is I know that this is equal to cos x + i sin x now from here and also I know that the cos x can be written as e to the power ix + e - ix by 2 because if I take the - ix - i will come here and also I know that the sin x can be written as e to the power ix - of e to the power - ix by 2i. So, in this case I have 2i. So, I will use this property here.

So, it is denote that + summation a n and this one I can write as e to the power i so instead of x. I have n pi x by l for this is there, cos x + e to the power - i n pi x by l by 2 + b n now will write it as e to the power i n pi x by l - of - i by 2i, so from here I can collect the terms so, I would collect the terms correspond to this one. So, it becomes a n by 2 + b n by 2 i e to the power i n pi x by l + a n by 2 - b n by 2 i e to the power - i n pi x by l and then this is the constants we can take this constant. So, from here what I do that if you see clearly that this is just the conjugate of this one.

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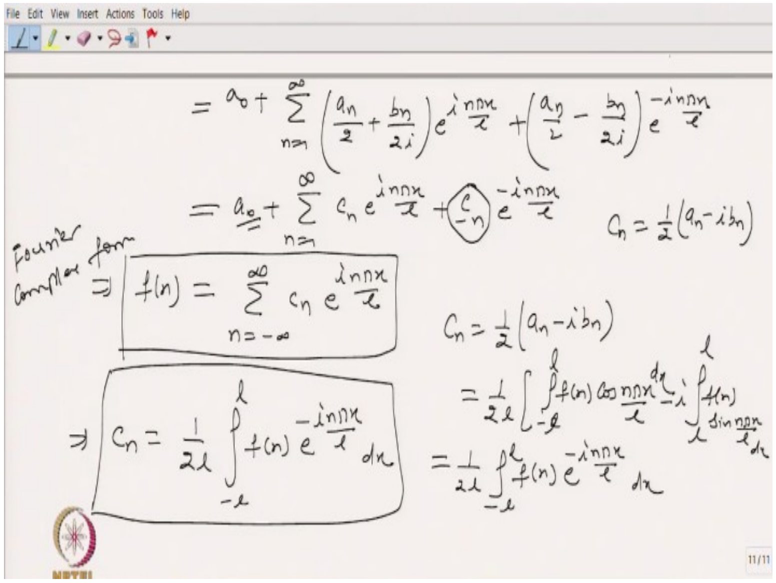

So, from here I will write this function as a 0 summation and from 1 to infinity. So, I call it as c n e to the power i n pi x by l + I will call it c - n because instead of n I am putting - n I will get the same value. So, this is n pi x by l, where I am choosing my c n is equal to So, c n we are choosing half a n - i b n. So, this I am choosing I will multiply by i i here and then this will be again this will be i i here. So, this will be - and this will be + so, I am taking c n will be like this one and c - n will be like this one. So, from here if you see from here I can write this out.

So, this a 0 also call it a c. So, I can write this one as a new one summation n pi x by l because from here if you see that when I take n = 1 it will be – 1 and my 2 it will be c - 2. So, it means the coefficient when I take n from 1 to infinity this coefficient is going from - 1 to - infinity. So, it is - 1 to - infinity this will be there at 0 this a 0 is there and after n equal to infinity this is there. So, I can write this series as in the one form that is called their complex form in this way.

So, this function becomes a now, summation and from - infinity to infinity c n this one and where my c n is half a n - i b n and that will be 1 over to 2 l. So, a n I already know. So, a n is there fx cos n pi x by l, So, I can write - i integration. So, it is from l to l fx sin n pi x by l dx. So, this is also dx. So, if I combine these together then from here will you see that it becomes - l to l and this becomes fx e raise to power - i n pi x by l dx.

So, from here I can write my coefficient c n will be 1 over 2 l because 2 is coming from here - l to l it will be fx e raise to the power - i n pi x by l dx. So, this is the coefficient of the complex Fourier series. So, this expression is called a Fourier cosine Fourier complex form and this is the corresponding coefficient. So, the advantage in this case is now, that whenever the function there, I can write the function in this form and only the all the coefficients because earlier we used to find a 0 a n and b n separately.

But now, I can find the value of c n together in just one go and based on this one we are able to write the Fourier series. So, in this way we can write the Fourier series in the complex form. So, this is a we will stop here. So, this is all about that how we can extend Fourier series into the half range series and then we are able to write the Fourier series in the complex form. So, in the next class will you go further and study more about the Fourier series. So, thanks for watching thank you.