Introduction to Methods of Applied Mathematics
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Lecture - 18
Fourier series

Hello viewers welcome back to this course today we are going to start with the next lecture that is lecture 18. So today I am going to start with the new topic and that is called Fourier series.

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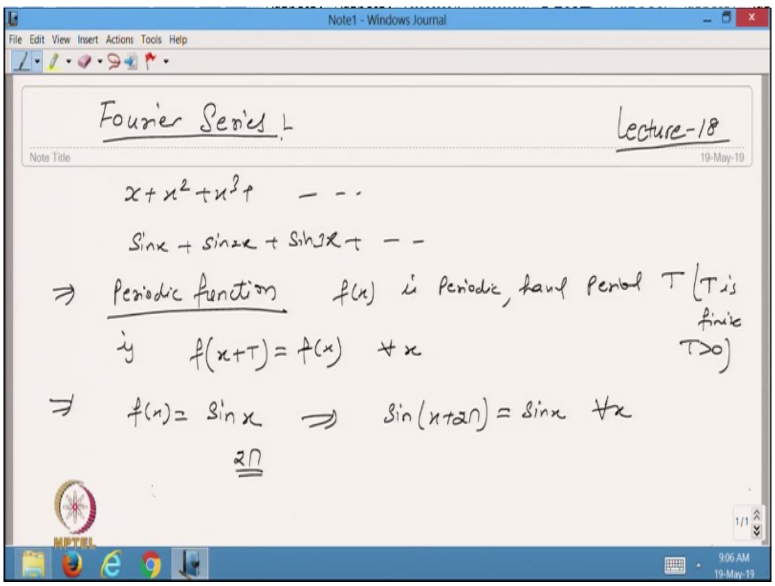

So this is the series has been given by the well famous French mathematician Fourier. So in this so to start with the Fourier Series so first of all I would like to tell that we have the knowledge about the series that what is the series so it is a summation of the terms we are taking that for example if I have a function you must have seen the series of this type x + x square + x cube like this one so this is a series made up of polynomial.

Similarly we can have we can have a series of trigonometric function like sin x + sin 2x + sin 3x and so on. So this is the series called the trigonometric series. So in this case so before that we start with the basics of the Fourier series I want like to develop or discuss one concept that is called the periodic function. So I have a function fx and I say that this function is periodic and having period capital T where T is finite so you can say that T is finite and greater than 0.

If my function fx + T is = fx. This is true for all x. Then we say that this function f is a periodic function and it having the period capital T. Now for example I want to take a function so let us take a function fx is equal to the well known functional just take that is sin x. So in this case I know that the sin x + 2 pi is again the sin x and this is true for all x. So in this case I can say that the sin has the period is a function periodic function and having the period 2pi.

Similarly I can define the function another function I take another function fx is = sin 2x or sin nx. So in this case if you see from here that from this one the period will becomes so 2 pi by 2.

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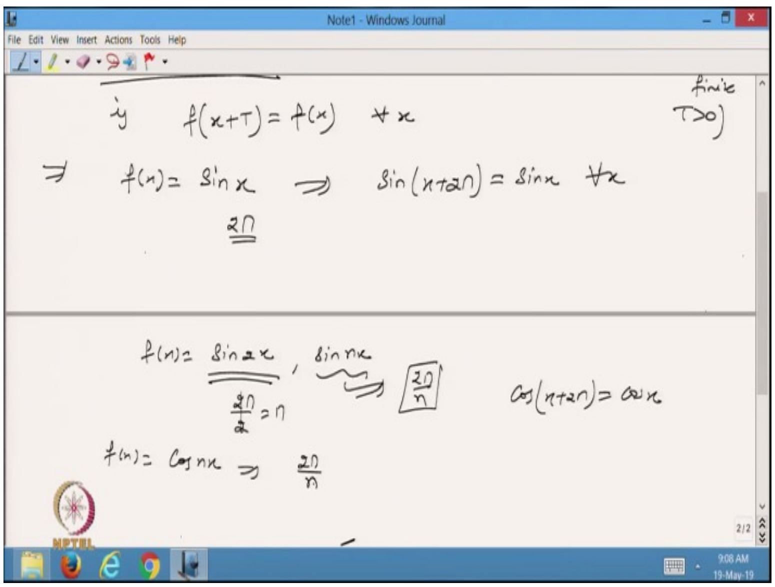

So in this case the period will be pi. So in this case the period will be 2 pi by n. Now 2 pi by n is the period of this function sin nx. Similarly I have another function f x is = cos nx so I also know that the cos x + 2 pi also cos x. So in this case we also know that this is a periodic function of period 2pi and from here also if I want to find out the period of this function so it will be 2 pi by n. So this is a well known function and we know that what is the period of this function is. Now I want to find out what is the period of function if I take f x is = some constant function 5.

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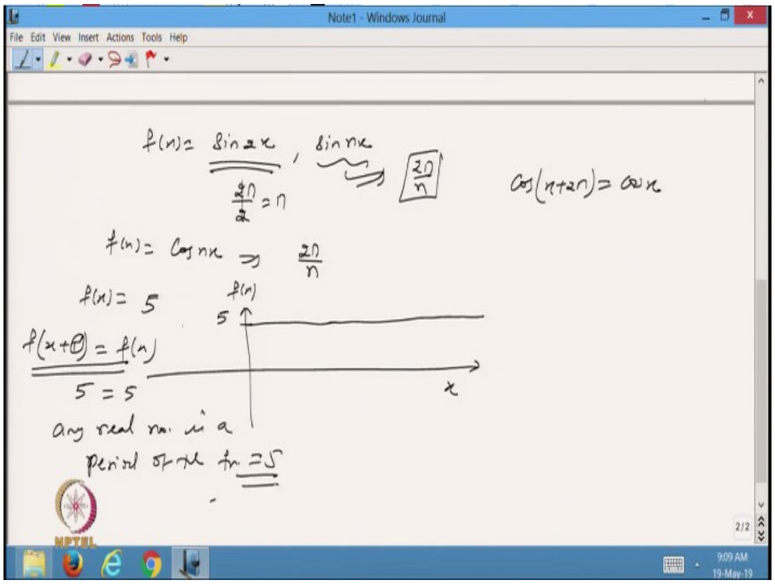

So this is the function I will taken and I want to find the period of this function. So if you see this one what is the function so this is a function and I define that x spot this is my 5 value and this is my fx this is x. So in this case if you see I take the value of x greater than equal to 0 then this function is always pi. Now from here I can say that function x + T is = fx. So what is the value of this T, but from here you can see that fx + T is = fx always true.

Because the left right and side is always 5 and left side is always 5, because this function is constant function say if somebody ask me that what is the period of this function then from here you can say that any real number if I take T to be 1 T to be 2 T to be point 1 T to be any value if I take a real number then I can say that this is the periodic function. So from here I can say that for the constant function any real number is a period of the function is equal to 5. So in this case say 0 is always there capital T can be 0 also, so that is true for everyone.

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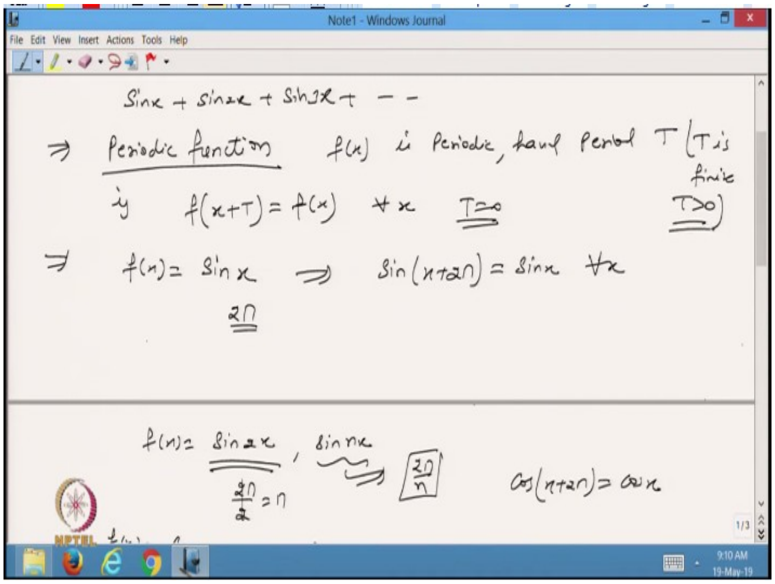

So in this case I am taking that including 0, so real number because the period t cannot be 0 so from here I can say that whenever we define the period T so in this case because you say that if I take T is = 0 then it is always true. So we always go for T greater than 0. So I am saying that the function constant function has the period of any so in this case T is = x I can say and x is greater than 0.So this one I can tell you so any positive real number.

So this is the period so this is the only period only function the constant function which has this type of property. Now you also know that like I have a function is = sin nx and my period capital T in this case is 2 pi by n. But if somebody ask me that it has a period 2 pi by n what about its period has it 2 pi. Then we can say that ya 2 pi also one of the period, so I have this period I have 2 pi period I have 4 pi all the periods are the period of this function so what is the period we generally take so we always take the minimum of all this one that is called the period of the function.

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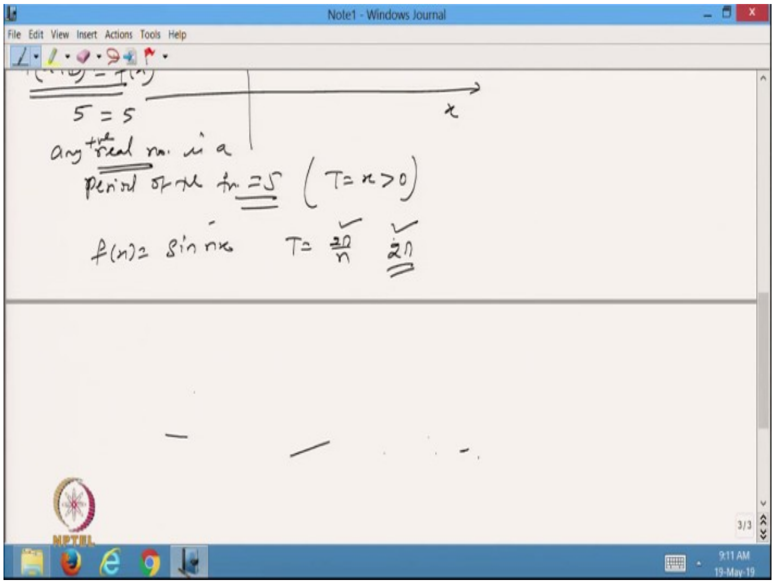

But if you see clearly that this is the minimum value so this is the period but 2 pi also one of the period. So for a one function we can have large number of period but for defining the fundamental period with always go for the minimum value that. So in this case I say that my period is this one for this function then so what the Fourier series says so will define with the Fourier series.

So in this case the first thing is that let my fx is a function so it can be a piecewise continuous function in an interval where x belongs to from - l to l where l is a real number and fx + 2l is = fx, it means this function is a periodic function and it has the period 2n. Then and I can write this function as a combination of a 0 sin 0x + a 1 sin x so in this case I am taking a arbitrary period so I will define this series.

So I will just write the series as n is = from 0 to infinity a n cos and pi x by l + b n sin n pi x by l. So it a trigonometric series and we are for the time being we are saying that this series is convergent and its convergent to the function fx because when the series is convergent only then it can be written as summation can be written as a sum function.

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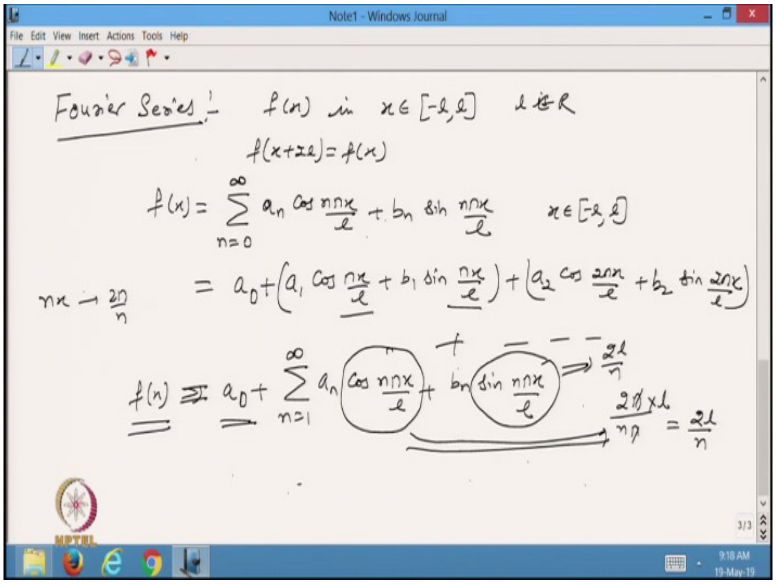

So we are assuming that will later on will do this one but at present here assuming that this series a convergent series and this series is convergent to the function fx, and where is my x belongs to this interval. So now from here I have this series if you expand the series so this if I put n = 0 here so this will be cos 0 is 1. So I will get the value a 1 and in this case if I put an = 0 this sin 0 is 0, so b 0 is it can be any value so from here I can say that it is a 1 if I put n = 0 so only a 1 will left + then I can have a 1 cos pi x by l + b 1 sin pi x by l.

So this is the first term then a 2 cos 2 pi x by l + b 2 sin 2 pi x by l and so on so similarly I can define all the terms of the series. So from here I can write this series as a 1 this is a series + then I can put the summation and from 1 to infinity a n cos n pi x by l + b n sin n pi x by l. So I can write the series as in this form from here it is starting from 1, because for n = 0 only we are left with the coefficient of the cos term that is a 1 a 0 sorry not a 1 a 0.

So my function is fx now in the if you see on the right hand side this is a cos function this is sin function and what is the period of this function. So we can write from here, so in this case I am considering my function having the period 2l so this function and this function so I can divide by 2 pi over it will be because when it will nx my period was 2 pi by n. So in this case I have n pi in to l, so this will be there.

So now if I so this will be 2 l by n, so in this case I can say that my cos function has a period that is 2l by n. Similarly from here I can say that it has a period that is 2l by n. So this is all the functions this function this function all the functions has a period 2l by n where depending on the n what is the value of n. So I can say that and I also told you that if 2 l bi n is a period of this function that is a fundamental period basically but 2l also period of this function and this is a constant function.

So I assume that it also has a period of 2l so from here if I see on the right and side I have the combination all the function and each function has a period 2l and the left hand side also I am assuming that the function is a period 2l. So from here I can say that the function on the left side and the right side and both are product function and having the period 2l.

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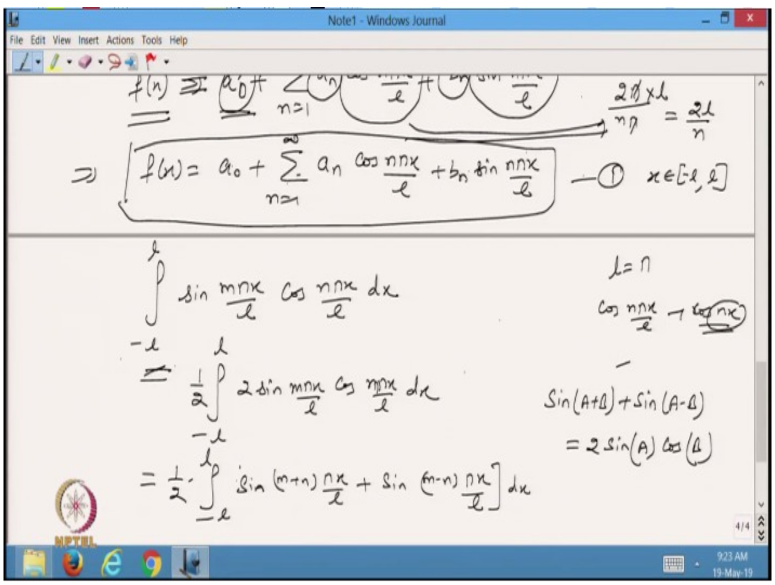

Now the question comes here that this is my series and I know the value of cos and sin but what about this coefficients so this Coefficient how we can find out, because this series is not the exact series for the function fx x unless we know the value of the coefficient that what is the value of a 0 a n b n is for all value of n. So to start with this one so I have my effects 0 summation and from one to infinity a n cos n pi x bi l + b n sin n pi x bi l.

So this is a series I am considering now so this I will call it one where x belongs to -l to l. Now from here if you remember then earlier we have discussed that the orthogonal property of the functions so here we will assume we will take the help of the orthogonal properties of the function that is cos and sin. So from here if you see I just want to find out what is the value of this function when I take sin n pi x by l into cos n pi x by l dx from here if you see that when l is pi then this cos n pi x bi l becomes cos nx simply nx.

So whenever we are dealing with the period from -pi to pi then this expression becomes easier and becomes only nx. Otherwise it is n pi x, so because we are doing it for any arbitrary interval. Now for this one and in this case I am taking the weight function that weight function is assuming to be one value so if you solve this value then if you do the integration of this one and then I just multiply 2 and then the 2 sin n pi x bi l into cos n pi x bi l dx.

Then if I try to solve this integral and I am applying this inequality that is a trigonometrically inequality so I know that sin a + b + sin a - b that is = 2 times sin so this + this by so a + b + a - b by 2. So it will be in to cos this - this by two, so it will b. So from here you can see that this one can be written as half - l to l, so this can be written as sin so this is an I just take as m. So because one i just take as m and one another I take it n otherwise it will be always same.

Because what I am doing hear that n can be 123 so in this case I am taking sin is equal to so just I will write it in as a sin m pi x by l where n is also Integer and m is also integer.So this value is equal to m so from here I can write this as a sin m + n pi x by l from this one + sin m-n pi x by l and then dx and I am doing this integration from - l2l this function sin and if you do this one then if you do the integration then you will see that the value of this function.

If I put this after the limit then this become equal to 0 so in this case I can say that this is true is always equal to 0. For all m and n, even m is = n you take in that case also its value will be 0.

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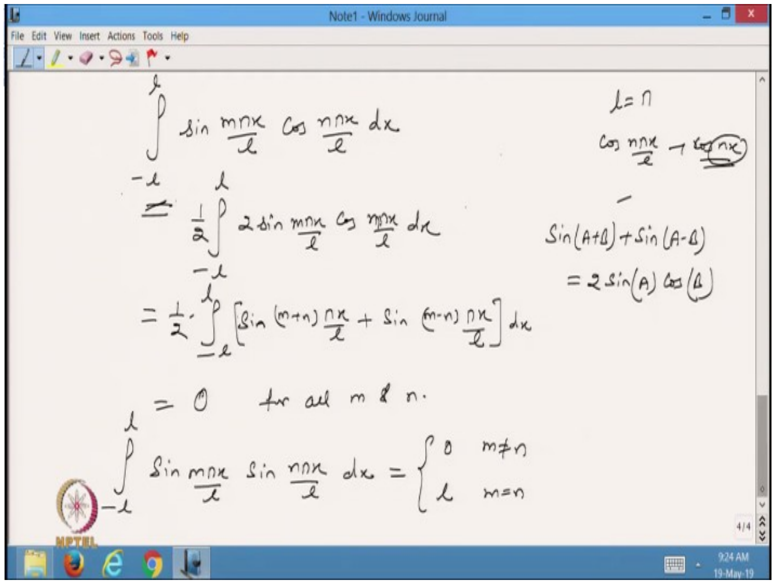

So from here I can say that the sin function and cos function with this argument with the weight function one taking the integral that integration from - l to l follows the orthogonal property so from here I can say that the sin function in the cos function they are orthogonal to each other. Also I just want to find another that orthogonal t, so in this case now what I do is I just want to define what this function, sin m pi x by l into sin n pi x by l dx. So this one we want to find out.

So in this case if you do the same way the integration and putting the trigonometric inequalities is here then you will see that this is = 0 when m is not = n and then when m is = n its value is l this one this integral you can do yourself similarly if I want to define the another value then you will see that – l to l cos m pi x by l cos n pi x by l dx again its value 0 when m is not equal to n and its values l when m is = l. So this orthogonal property is always satisfied because when m is = n.

Then you know that this become the square of that function and then in fact this is a magnitude of the function and that value is coming = l otherwise this so this properties is very important properties that is the orthogonal property because we are going to use this property to find out the coefficients in the Fourier Series. So now I start with this one so I have my function fx and I have written that function as a 0 + n from 1 to infinity a n cos n pi x by l + b n sin n pi x by 1.

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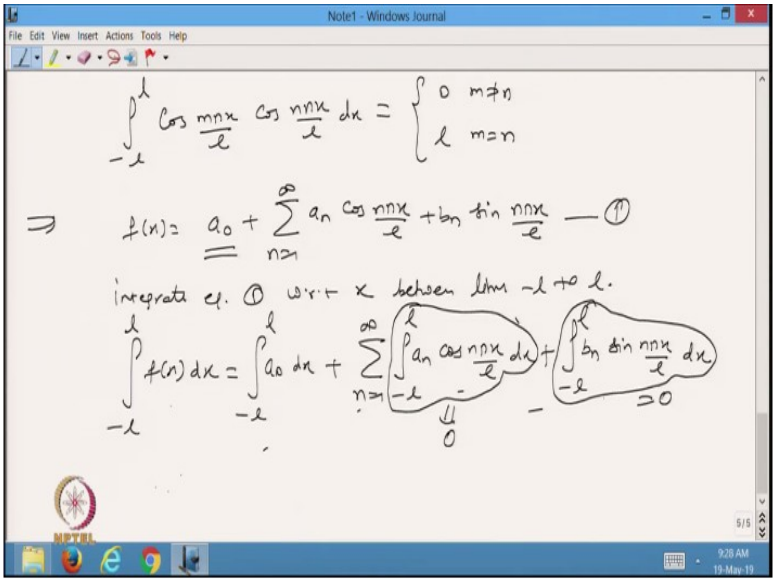

This is my series now first I want to find the value of a 0, that how I am able to find the value of a 0. So in this case what I do that so this is a first I have taken already I have given the name one how do you do that integrate equation 1 with respect to x between the limit –l to +l and I integrate this one so I have and this integration I am taking term by term integration. So – l to l I have fx dx and on the right inside I have a 0 + then the summation and from 1 to infinity.

So this is integration –l to l a n cos n pi x by l dx + integration from – l to l b n sin n pi x by l integration with respect to x now from here if you see this integration so if you see from here that cos and pi x by l i am taking and I am integrating from – l to l in which is this function is has the period 2l so if you see from here if I take the integer a n is a constant so integration of this cos will be sin and I am putting the sin of the value after putting the limit I am putting the value of l.

So l will cancel out I will get only n pi so sin n pi is always 0 similarly - l sin n pi is always zero, so if you see this one this will be always zero, similarly similar case here it is a odd function and I am doing the integration this odd function in the interval –l to l, so if you do the integration this is also coming to 0 so from here I can write from here that this will be = – l to l fx dx and then a 0 I can take common and this will be = 21, because I just do the integration x and putting the limit + everything is 0.

So from here I can find my value of a 0 so this is = 2l from – l to l fx dx, so this is the value I am able to find for the coefficient that is a 0 and if you see it clearly what I am doing is that I am taking the function integrating that function from – l to l and you know that the integration gives you the area so this is the integration and area and divided by the length so this is I can say that the average of the function. Function fx over the interval -1 to l, so it gives the average value of the function.

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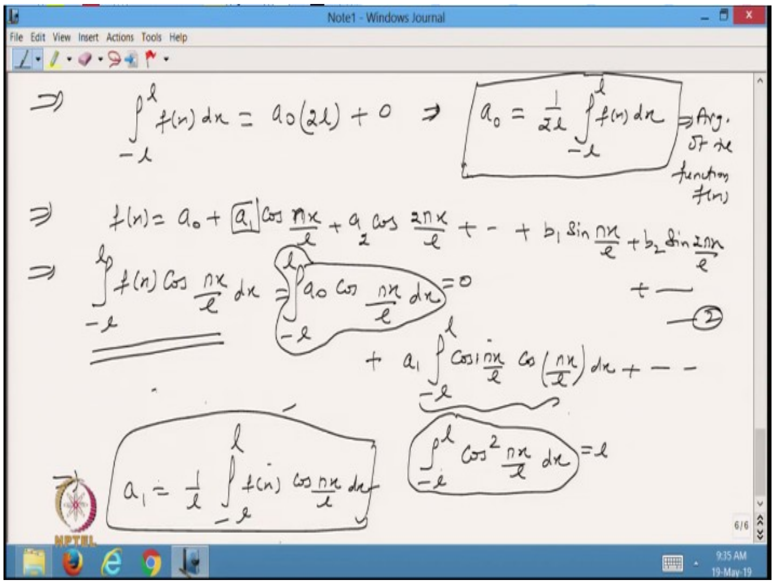

Because if you see that the summation become the integration in the form of the function so this is basically this I am taking the sum of the function over the values divided by the total length that value in this case become the average value of the function. So by this way I am able to find the value of a 0 now after finding this value I want to find the value of all this a n and b n, so first I want to find the value of a 1.

So let us I write the function fx now I write a 1 cos n is 1 and n pi x by l + a 2 cos 2 pi x by l and so on + b 1 sin pi x by l + b 2 sin 2 pi x by l and so on. So suppose I want to find the value of a 1 so what I do that I will multiply this I just give it name 2 so I do that in this case with the value of a 1 so I multiply this with the cos so this is a cos pi x by l so I multiply by the cos pi x by l or accept the cos pi x by l so I multiply this by Cos 2 bi x by l.

So what I do that I multiply this function by let do this one and - l to l dx and so on .Now I just told u that cos 2 pi x by l if I integrate it will become sin and then putting this value so I using the integration I can verify that this value is = 0 this value depend upon the function what is the value the function so this integral we remain like this one what about this one so in this case I have cos so n and if you say this is a one and I know that just now we have done that this function cos m pi x and cos n pi x there satisfy the orthogonal property.

So this value will be 0 in all the cases except if I multiply this by the cos pi x suppose I multiply this whole series the 2 with the cos so with the cos 2 pi x we know that everything will be 0 except I will get here from a 2. So I am just finding the value in a 2 just so this is a 2 cos so I just want to find the value of a 1 what will do is I will pre multiply the function by the same factor as cos pi x by l. In this case so I will not multiply by 2 I will multiply the cos pi x by l and then here also.

So this is now I am multiplying by the same factor cos pi x by l. I just want to be show you that if I multiply some other factor only the coefficient of that factor will remain non 0 otherwise it will be 0 in this case this is 0 this is 0 and this is the remaining and now this value is not will be 0 because what is doing now it become the –l to l square pi x by l dx and this is what we know that it is the magnitude of the function so this value I just told you that this will be = l.

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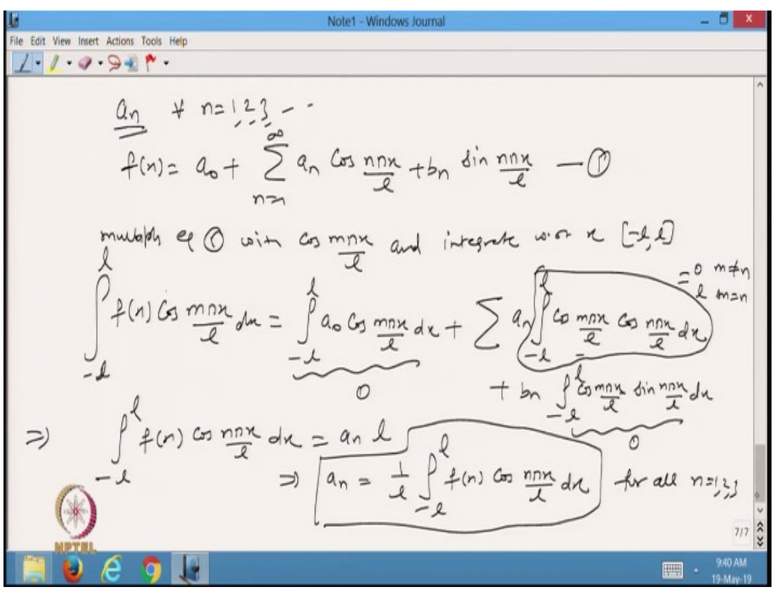

Then the other integral becomes 0 because only in this integral I have the cos pi x by l multiply by cos pi x by l otherwise next will be cos 2 pi x by l into cos pi x by l, so in that case this value integral value 0 sin pi x by l multiply by cos pi x by l and we know that the by the orthogonal property this integral will be 0. So all the integral becomes 0 only I will left with only the coefficient a 1 and this integral value will be l so from here I can say that my a 1 will be so this will be 0 and this is l.

So I just take the l here and – l to l I will write it as a fx cos pi x by l dx so I will come with only a 1 a l so now I doing the same property for other coefficient so now what I do that I have my series the Fourier Series whatever we have written so to find now I will try to find the value of the a n for all value find for n is = 0 is already there so 1234 all n I want to find the value of n so what I do now I multiply by multiply the equation number to 2 with the so I have my fx a 0 + sin n pi x by l ok.

So now I do that this is we have started with one so I just now it is it was my first equation now I multiply equation number 1 with function cos m pi z by l n integrate with respect to x between the limit – l to + l so in this interval so I want to integrate. So here I will get my fx cos m pi x by l – l to + l dx is = – l to + l a 0 + summation a and integration –l to l cos m pi x by l cos n pi x by l d x + b n -l to l cos m pi x by l sin n pi x by l dx.

So this integration is there now I know that this will be 0 this sin and cos this will be 0 for all value of n and m only thing is that this integration we know that this is = so this is = 0 when m is not = n and = 1 m is = l so from here I can write that –l to l fx cos n pi x by l dx because now I am considering m is = n. So instead of m I am putting n is = this 0 everything will be 0 when m is = n I left with only Coefficient that will be a n so it will be a n into l.

Other things will be 0 so from here my a n I can write as a one over l from – l to l fx cos n pi x by l dx this is my the coefficient a n and this is true for all n expect 0 because for the 0 we have found the different separately that for the a 0 but otherwise for n is = 1 2 3 so this is a formula for n 1. Similarly I can find the value b n multiplying by the sin n pi x by l so if you do the same way then you will see that my b n in this case will be fx sin n pi x by l dx so this is the coefficient I am able to find for the sine function.

So after doing this one so now from here I can say that I have the function fx which a periodic function in the and having the period 2l so this can be written as a series a 0 + summation n from 1 to infinity a n cos n pi x by l + b n sin n pi x by l where my a 0 is just the average value of the function doing the integration so here x belongs to -l to l and then if I want to apply this function for the whole period then the same function is vary so a 0 is this one dx a n is one over l my fx cos n pi x by l dx and my b n is 1 over l – l to l fx sin and pi x by l dx.

So base on this one so I am now able to write the Fourier Series a whole and we are able to find its coefficients so it depends on the function that which function you are taking will do this integration and then we are able to we should be able to get the Fourier Series. So let us do one example so will take the example that is a very simple example just want to take so take example number 1 and we start with example 1 as just now I just I take function my fx is = x and my x belongs to – pi to pi to this.

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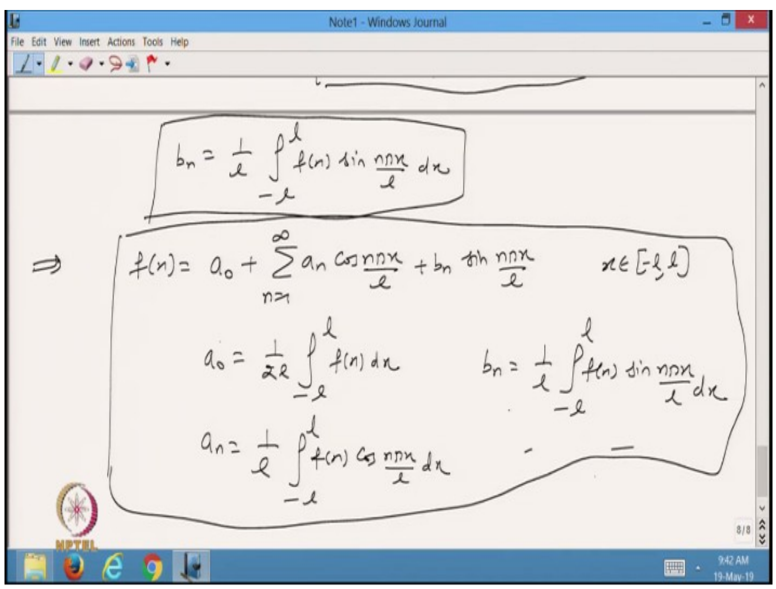

I am taking but if you see that my function and if I plot this function this function is this function this is my x axis and y axis and if you take this function pi and this is –pi. So this is a straight line so this is my function this is a function fx is = x and if you see that this function is well defined in between – pi to pi and from here looking at this graph we cannot see that this function periodic function because nothing it is defined only in this value from -pi to pi.

So to make it periodic I what I do is that I apply the condition that have x + 2 pi is = fx. So this one I have to put this condition based on this condition now only I can say that this function fx is a periodic function and having the period 2pi so from here if the function is of having the period 2pi then I want to find the Fourier Series for this function so basically I want to find my Fourier Series that is fx is = x is = a 0 a n now in this case my n is pi so I know that this becomes cos n x + b n sin n x.

Now I want to find the value my a 0 so a 0 already know then this is = 2 l so l is in this case l is = pi so this should be 2pi from – pi to pi x dx so this value will come now from here if you see this function this function is odd function so if I integrate from –pi to pi the area under this and the area under these are similar but of different sign so this value if you do the integration this will be 0.

Because this is odd function now I want to find the value of a n this one you can do very easily that because it will be one or 2 pi and this is a x square by 2 from – pi to pi and then it will be 1 over 4pi it will be pi square – pi square. So this will be 0. Now I want to find the a n so a n will be 1 over l so l is pi from – pi to pi now fx is x into cos n x dx. So using this one I want to find the value of a n now my function x is you know that x is a odd function here. So x I can say that this functions odd function and the cos function.

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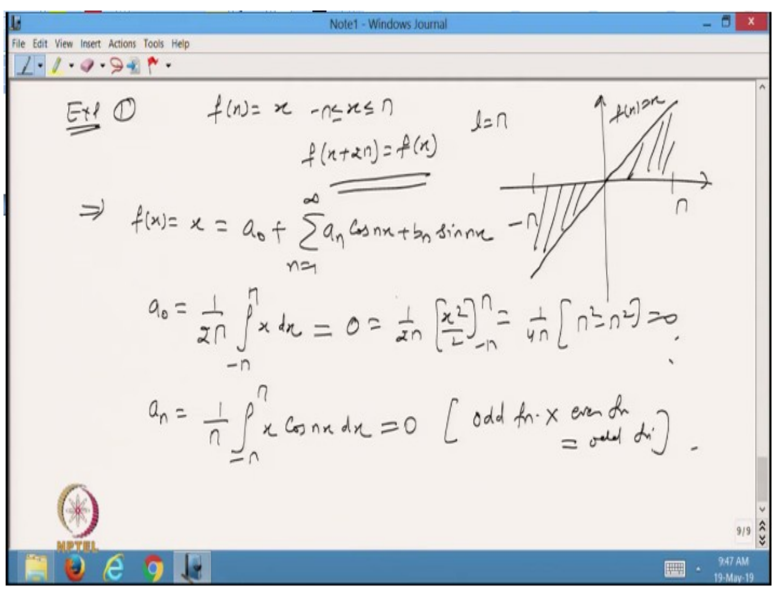

I know that is a even function so if I look at this one I am having a function and Odd function multiply by the even function so it is just you can say that if you remember that odd number multiply by the even number so result should be the odd and then its value should be = 0. This one also we can check because I am from here I am multiply odd function multiply by even function. So this is = odd function so if I do the integration its value should be = 0 so what but only we are using the property of the odd function and the even function.

Then I want to find the value of b n so if you see the b n this is one over pi sin nx dx so this is the odd function this is a sin also we know that the odd function so multiply by odd in to odd become the even then I want to solve this one integration. So this will be = 1 over pi and then I will apply the by parts rule.So this will be x so sin integration - cos nx by n this is the limit from - pi to pi - minus pi to pi and x the derivative excess one so it will - cos nx by n dx so we will get from here and then from here.

So I will just write here using from here I will get 2 over pi 0 to pi and from there if I go by this one so incoming 0 and this is 0 and this is two. Now this value will be not be coming 0 so if you do the integration and putting the limit now so this value comes here 2 by pi and then my 2 by pi - pi cos n pi divided by n. So this value will come and after doing this one I will get this pi will cancel out with this so I will get 2 by n and then cos - 1 this will be - and plus, because cos n pi when n is = 1 cos pi is -1.

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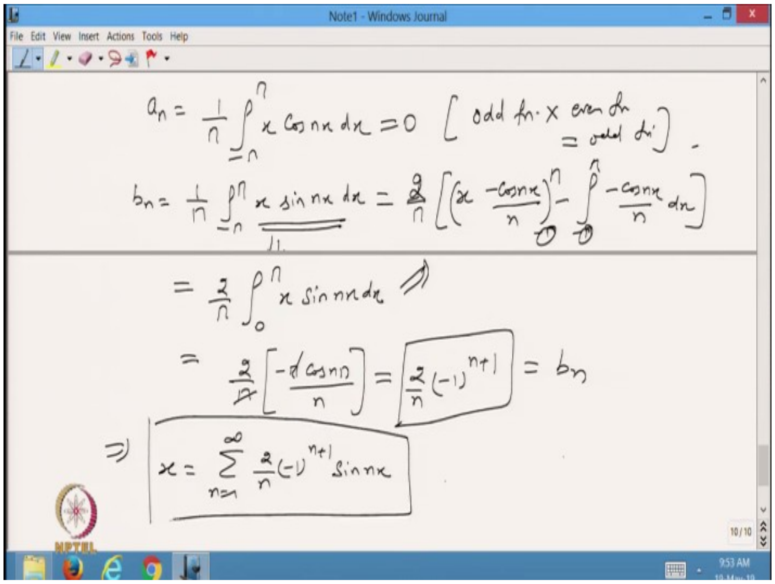

When n is equal 2 the cos 2 pi is 1 so depending on the value of n the value will be either – 1 or 1 and -1 is already there so I am taking - 1 is 2 power n + 1 so this is the value of my b n be careful that here we have use the property of the even function and then we have changed the integral into this form 2 by pi from 0 to pi and then we do the integration so from here I can write my function now show my function fx is = x it can be written as so a 0 0 a n is 0 so I will get only b n.

So b n is this one so from here I can write and from 1 to infinity 2 by n -1 power and + 1 into sin nx. So this is my Fourier series for the function x which is we assume that periodic function from –pi to pi and having the period 2pi. So this is the Fourier series if you expand this one I can write this series as and when n is = 1 it becomes positive so it will be 2 sin x then when n is = 2 so in that case is - so - 2 by 2 sin 2x then 3 it will be 2 by 3 sin 3 x - 2 by 4 sin 4x and so on.

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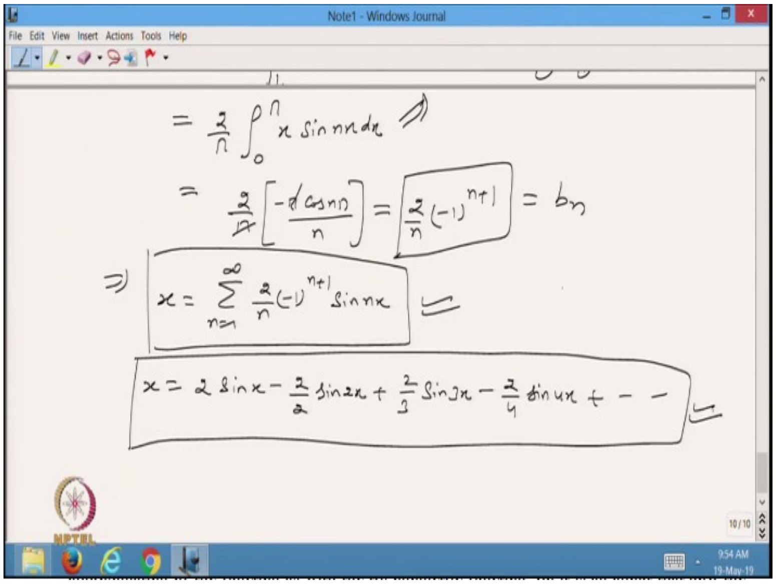

So from here you can see that that on the left hand side a function as a so this function and algebraic function that is = x so this x is being approximated by the summation of the trigonometric function. So on the right inside we have trigonometric function and the left side I have a polynomial all algebraic form that is x so that is we call it the Fourier Series or the approximation of the function as with the trigonometric function.

So it is all about that how the Fourier Series we can define for the function when the function is a product function and having a period either 2l or 2 pi. So in the next class will go further thanks for watching.