Introduction to Methods of Applied Mathematics
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Lecture – 17
Laplace Transform Applied to Differential Equations and Convolution

Hello viewers welcome back to this course, so today we are going to start with the lecture 17 and we are going to deal with some other differential equations and solving them with the help of the Laplace transformation So the most important the differential equation will want to solve with the help of Dirac Delta function.

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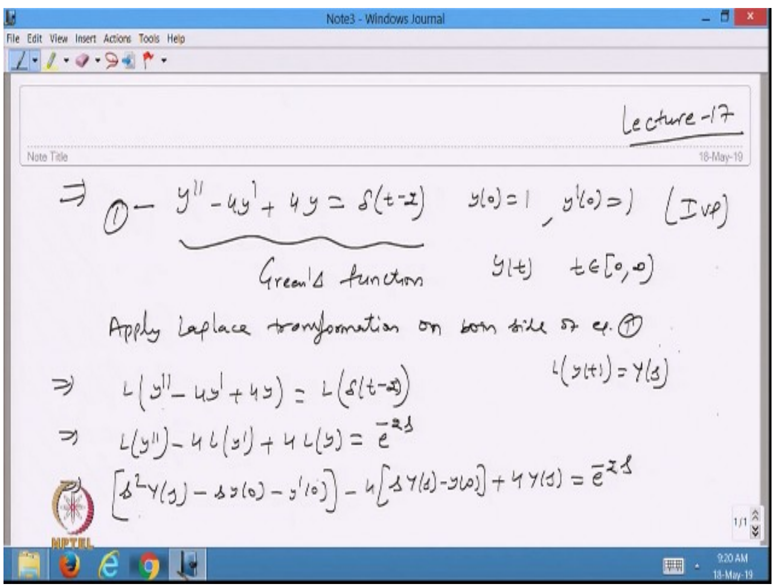

And if you know that suppose in this case suppose I want to solve differential equation of this type y double dash - 4 y dash + 4 y is equal to the right hand side of the Dirac Delta function t - 2 and may initial condition is given to me that y 0 is 1 and y dash 0 is equal to 1, now if you see that this equation has directed delta function the right hand side and if you remember then that whenever the right hand side function is a Dirac Delta function. And if you want to solve the differential equation the solution will be the green functions.

And in this case the solution for this will be the green functions. And the green functions you already know that how to find the green functions we have done lot of examples in the previous classes for the different type of differential equations, so in this case now and you also know that the finding the green function is very hectic process because you in that case you have to find out the for the second order equation you have to find four coefficient and then you have to apply the properties of the green function like the continuity, jump.

So in that only then we are able to find the green function but you will see that one that if I want to solve this one with the help of Laplace transformation then it is quite easy as compute the previous one. So this is my green functions so this is the equation number 1 and this is my initial value problem and if you also remember that if I want to apply my previous methods as we have done to solve the second order differential equation then I am unable to find out because we in that case we are unable to find out that.

What will happen when I want to find the particular solution? Whenever the right hand side function is a Dirac Delta function, so this is the benefit of dealing with the Laplace transformation because we already know the Laplace transformation Dirac Delta function, So let say in this case I am that my function yt is well defined it is piecewise continuous function for the whole t from 0 to infinity and it exponential order so I am using this one that t belongs to and yt is also exponential order then apply Laplace transformation on both side of equation one.

So we can take the Laplace transformation on both sides so from here I will take the Laplace transformation on the left side 4y and the Laplace transformation the Dirac Delta function with my initial condition. So if I apply this one this will be a linear property of the Laplace transformation - 4 Laplace transformation y dash + 4 laplace transformation y and the Laplace transformation the Dirac data function just now in the previous class we have found out so it will be - 2s this is my Laplace transformation of the Dirac delta function.

This will be s square y s because I am assuming the laplace transformation of yt is y s - s y 0 - y dash 0 - 4 the Laplace transformation is y s - y 0 + 4 y s is equal to e raise to power - 2 s.

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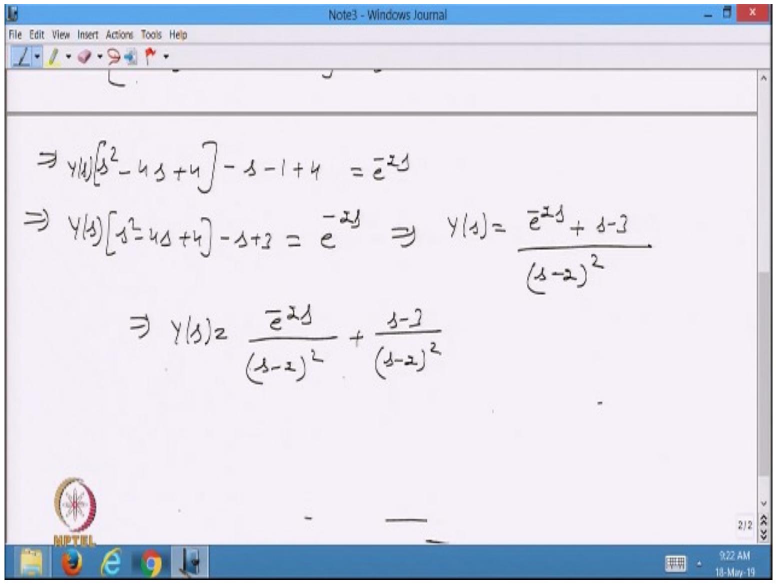

So from here and this value is given to me that is one and one value so from here I can write this is equal to s square i can take y s common so this will be s square will come from here - 4 s + 4 and the remaining things I will collect from corresponding 2 s, so this will be - s - 1 + 4 this will value is equal to e raise to power - 2 s and from here I can write from here then this value will be - s - 1 + 4.

So this will be y s s square - 4 s + 4 and this will be equal to - s + 3 so from here I have my y s is e raise to power - 2 s and taking the left hand side so it will be + s - 3 divided by this factor that is so this factor I can write as s - 2 whole square this should be s square + 4 - 4s so this equal to this, so if I further solving this one then so from here I can write my y s is e raise to power - 2 s over s - 2 whole square + s - 3 over s - 2 whole square.

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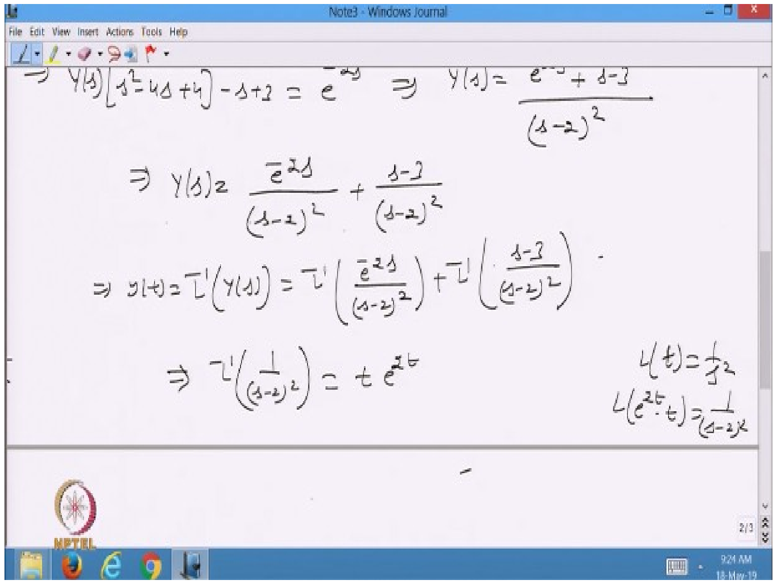

So in that case now I want to take the Laplace transformation inverse Laplace transformation so I want to find my yt and I know that my yt the solution will be the Laplace inverse of y s because this is the we are able to transfer the differential equation to the algebraic forms of this is the algebraic forms and this will be equal to the L inverse of s - 2 whole square + L inverse s - 3 over s - 2 whole square, so this one I want to find the solution.

Now from here if you see it clearly then from here now Laplace inverse I can define for this factor I can so from here first of all I want to find out that the Laplace inverse of 1 over s - 2 whole square so whatever this 1 I know that the Laplace of t will be 1 by s so I am hearing - 2 so if you see this 1, so from here I can write this will be equal to t e raise to power of 2 t because I know the Laplace of t is 1 over s and the Laplace of e 2t into t is 1 over s - 2 whole square we already know, so from here L inverse (of) this one similarly.

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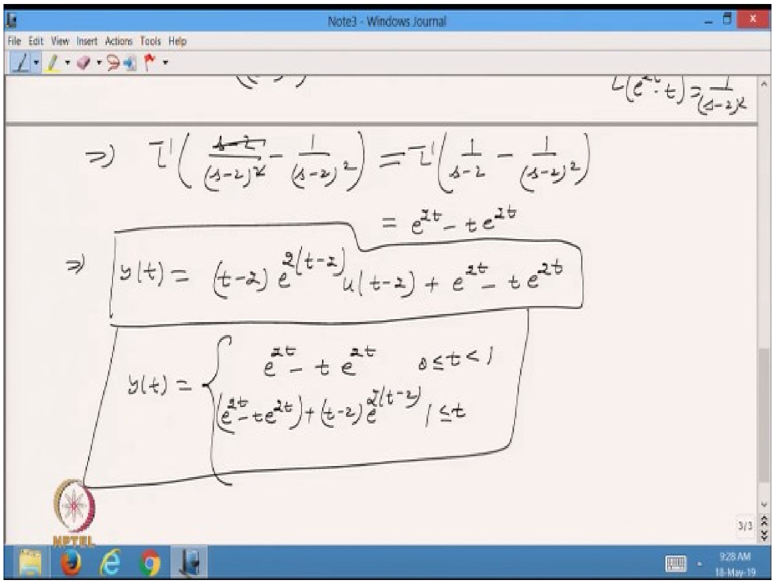

The Laplace inverse so this another factor I can write in this form s - 3, I can write s - 2 over s - 2 whole square - 1 will be there so it will - 1 s - 2 whole square and this 1 become so this term will cancel out so I have a Laplace inverse of the remaining part so this will be over s - 2 and - 1 over s - 2 whole square, so from here this will be equal to so 1 over s - 2 will be I can write as a e 2 t from here - and this is again the same t e 2t so now. Now we are I want to find my y t.

So Laplace inverse this is my t into e raise to 2 t so that the Laplace but I am here if you see that we are multiply by e raise to - 2 s so it means that whenever I want to find the Laplace inverse that will be involving with the unit step function so from here I can write the Laplace for Laplace inverse for this factor as t - 2 e raise to power 2 t - 2 this is I am writing u t - 2 + e 2 t - te 2t so this is the solution for the given differential equation now I told you that in this case it is a green function so I can just try to write the solution in the form a green function.

So from here I can write my yt s now if I put t less than to this part will be 0, so i can I write this function is e so I can write this one is e 2 t - te 2 t when t is less than 1 and when the t is is greater than 1 upto infinity then this value will be te + t - 2 e raise to power 2 t - 2 because that case this value be 1, so this is my green functions so this is the green functions or I can say the solution of the given differential equation so from here you can see that using the Laplace transformation very easily we are able to find the solution whenever the Dirac Delta function is given the right hand side. So that is another application of the Laplace transformation.

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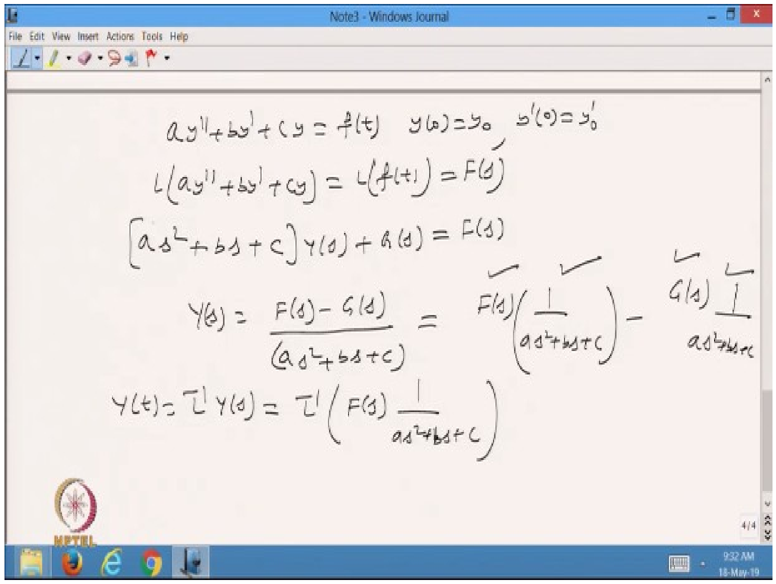

Now want to take the another very important concept that is being involved in the in the theory of mathematics that is called the convolution but before that one I want just want to discuss that why this is needed because if you see that I have like in this case we are solving the second order differential equation so I have a second order differential equation a some ay double dash + by dash + cy is equal to some function ft.

And I am solving this one with some initial condition y 0 is y 0 and y dash at 0 is y 0 dash then I take the Laplace and if you see that whenever we take the Laplace of this 1 I will take the Laplace and Laplace will be + cy is equal to the Laplace of ft and that is equal to fs and from here I will get the Laplace in the form of a so it will be as square will come from here + bs and then + c this will come + some function I can write in the form of s I just write the another function.

So that I can write as a function in Gs the remaining part will come to that is equal to Fs, Now if I find the solution of this one in the form of yt then I have to take the inverse of that Laplace so now from here I can write as ys is Fs - of Gs divided by this factor so if you from here you can see that this is equal to Fs multiply the some another function as square + bs + c - Gs multiply the another function.

So here what I am doing I have a one function of s multiply another function, I have one function of s multiply another function so I want to find out that if I want to take the inverse so I want find my yt so my yt in this case will be inverse of y s and this is equal to L I take this one so I will take from here 1 over s and another function as square + bs + c so I want to check that what will happen if I take the inverse the Laplace for the multiplication of the 2 Fs some function of s.

And some another function because whenever I want to solve differential equation I always come across this type of multiplication of 2 Fs and some function of s and then I want to take the inverse of that one so for this one I will take the concept to a start with the new concept and that is called the convolution, what is the meaning of convolution that I have 2 function let my function I have 2 function ft and g t so these are 2 function be defined on 0 to infinity then well defined.

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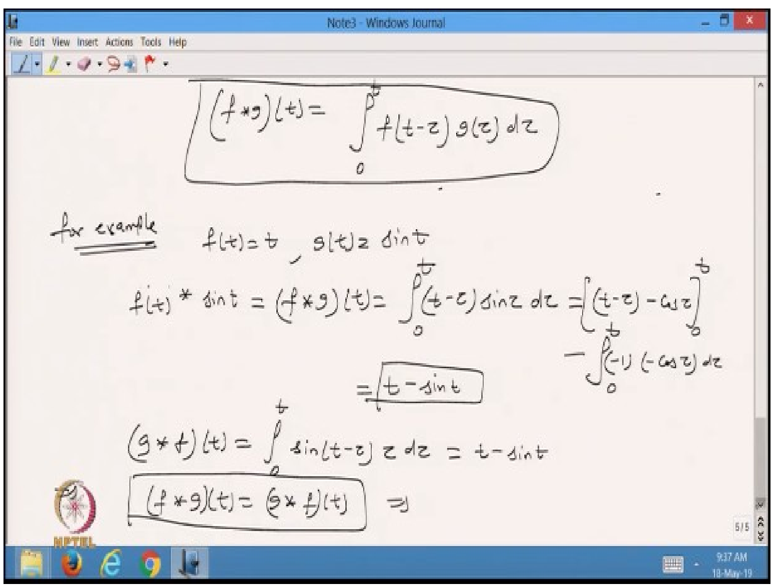

Then I want to defined the convolution f star g so we represent by convolution by the star and it is again the function of the t so I will say that this is from 0 to t so that do I will right here function ft - tau into g tau d tau that is a convolution of the 2 function so in this case what are you doing if function of this time taken integration from 0 to t so what I do ft - tau so it is just the shifting of the function by tau multiple another function g tau and taking the integration with respect to tau.

So whatever this value is coming that is called the convolution of the function f and t, for example so for example I just want to you from take that let it take ft is some function t and g t I take as sin t so these are 2 function and I want to define the convolution so convolution of this 2 function will be, so it will be ft taking the convolution with sin t I can write similarly just we have returned so it is from 0 to t then t - tau, so this function is t so I will write t - tau here into sin tau d tau so that is we want to take.

And from here if you from here I just want to integrate then I apply the by parts rule so from here if applied by parts then I can take t - tau and the integration is - cos tau 0 to t - and the derivative of this one so it will be - 1 into - cos tau d tau 0 to t and if you solve this one then you will ultimately will get the solution as t - sin t so that is my convolution where the equation comes that in this case we have taken this function as ft and another function g t what about if I take this as sin t and this is g t interchanging the function.

Or maybe I can define instead of f and f star g I can defined this value as a g star f because it is up to me that whether I am consider this as a function ft or g t so I should be able to find the what is the convolution by interchange the function so in that case you if you take this as a sin t function is a first function so from here I can define my sin t - tau and then tau and d tau and if you do the integration this one the same way by parts then if you solve this one you will find that this is also equal to - t - sin t so from here one thing I can write from here that f star g is same as and this property you know that it is called the commutative property.

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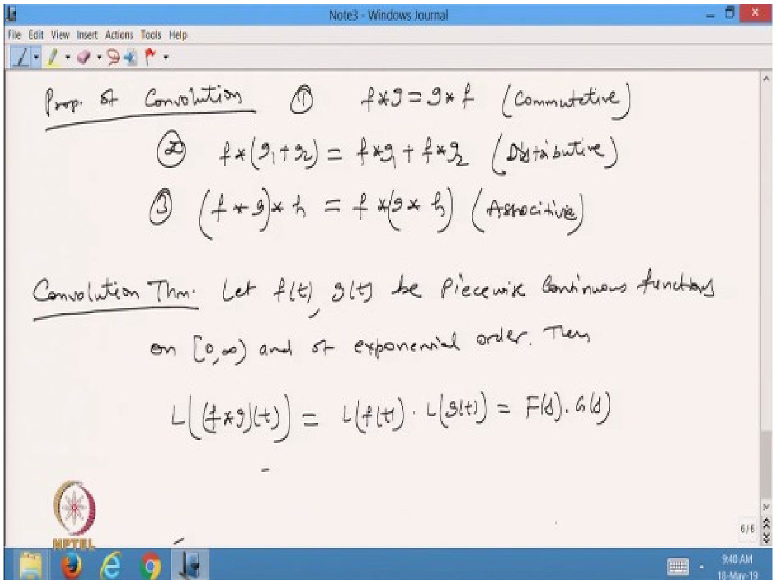

So will write down few properties of the convolution of the 2 function so the first one is that f star g is equal to g star f, so it is commutative, second property we can also verify that I f star and I have g 1 + g 2 so this is equal to f star g 1 + f star g 2 and it means this is called the distributive property distributive and the third one is the that I have 3 function f, g and h this is equal to f g h so does not matter I put the bracket here or I put the bracket here the as well is same.

So this property is called the associative property so the convolution satisfy all these 3 properties it is commutative distributive and associative now with the help of this one convolution so let us define convolution theorem, so this is the convolution theorem so convolution theorem say that let a function ft and g t be piecewise continuous functions defined on interval 0 to infinity and of exponential order then if I want to find the Laplace of convolution of the function.

So this is the convolution and I am find the Laplace of this one so that will be equal to product of the Laplace of the corresponding function or it will be equal to f of s multiply g of s so this is the simple multiplication we are taking and this is the convolution.

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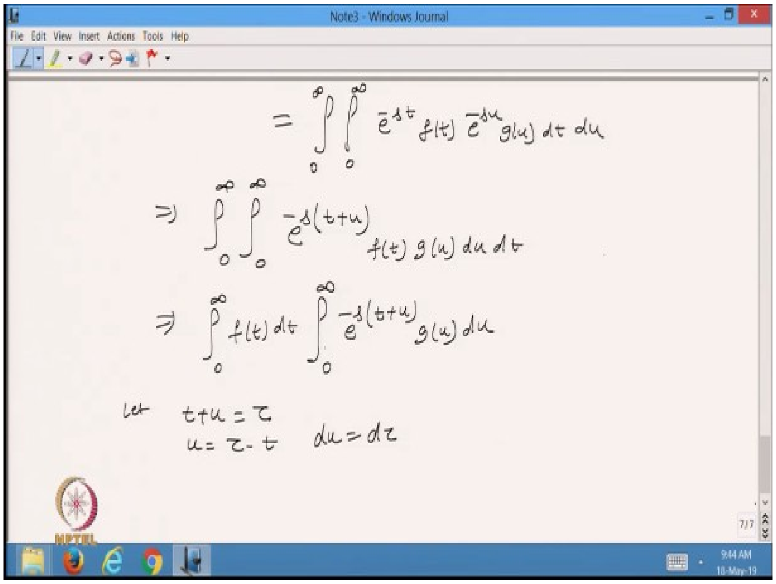

So that is the convolution theorem from here so I can prove this one it is very easy to prove here that now. I know that if I take the product of Fs into Gs so it is defined as 0 to infinity e raise to power - st ft dt multiply by 0 to infinity e raise to power - su I just take as a u because is there g of u du so this is given to me is that from here it is a function in the t integration is ft and this is integration with ft to you so from here I can take this one and mixing together this 2 integral.

So from here I can write from 0 to infinity I can I write for - st ft e raise to power - s u gu dt du just mix because it is I am taking integration with respect to t and u. And here with respect to u does not matter now from here I can write this 0 to infinity and this one as e s I take t + u and then I write ft. So this one I am taking with respect to t sp ft gu and then du d t this one, now I do is that from here I will write like try this one it is ft d t and then inside define 0 2 power - st + u gu du.

Now so I want to solve this one so let assume t + u is equal to some. I should choose that some tau and from here my u will be tau - t now t the variable u the variable another variable so u is equal to this one so from here I can say that my du is equal to d tau.

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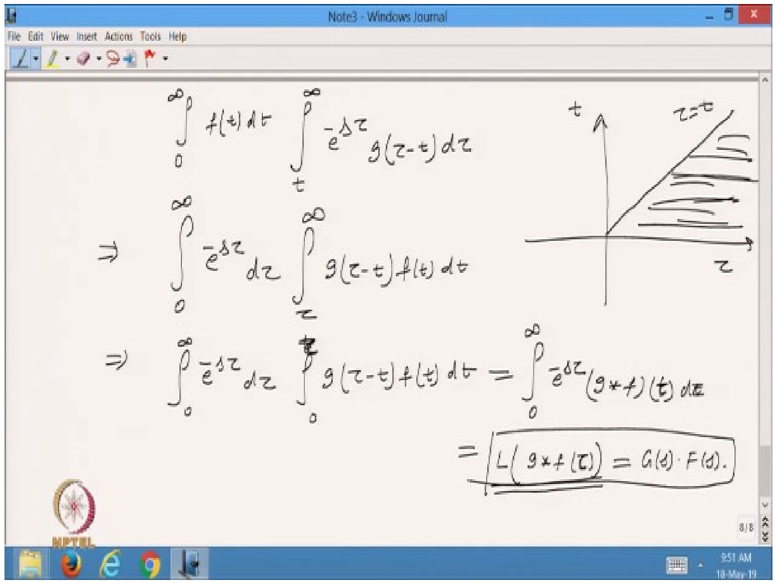

From here I can define that my integration becomes now, so ft dt 0 to infinity it become 0, when my u is 0 but tau is t it is starting from t to infinity I can write as e - s tau g of u is this one tau - t and this is equal to d tau so that it is now my tau is moving from t to infinity so this will become this so from here so I defining e raise to power - s tau date d tau so this one I am taking here and inside I am taking this one g of tau - t was already there ft dt this is given to me now I just want to check out that what will be the limit for the integration.

So if you just see so I have a tau and t now my tau is equal to t this is the line that is tau is equal to t so for this integration I have a tau from t to infinity so my tau is this is the area of integration and then my t is from 0 to infinite so that is t from 0 to infinity this area and if you see this integration 0 to 2 this area and then the common area will be this form only then what I do now I take my t inside so t is now starting from tau to infinity so I can write it tau to infinity this one and then my tau will be from 0 to infinity.

So this is the how we can interchange the integration and then I will solve this one, so it will be now. If I solve this will be 0 to infinity e raise to power - s tau d tau and this is now. I can define this one as 0 to t dt. So this one we have taken from top and then this is what it is a convolution g star f at t dt so this is convolution and from here. I can say that this is tau so from here I can say that this is equal to the Laplace transformation of g star f of tau.

And that is so there we are taking the Laplace transformation of the convolution the function and this is coming equal to the G of s into F of s. So it means that if I take that Laplace transformation of convolution of the 2 function s that is equal to the product with Laplace transformation so this is very important properties because this is being used whenever we are solving some differential equation we are to use this again and again so let us do one example so from here.

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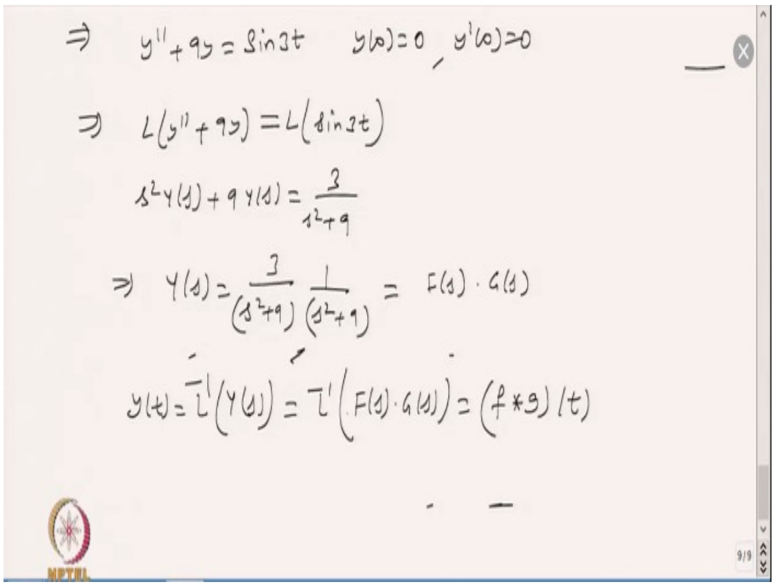

So from here now I have example so let us I want to solve this equation y double dash + 9 y is equal to sin3t because if you see in the previous one this type of things we are always so in this case we have started with the integration of 2 functions now do if you look carefully that how we are doing this one we just started with the integration and then we making this integration of the double integration take inside this one and then we combine this factors.

And then I am taking the transformation, so this transformation become these from here my transformation become this one now it was tau here and t here so now interchanging the so this one we have written so it becomes t to infinity and then I am changing the factor from now I am changing this is my tau and this is t trying to changing the order of integration this is tau and t and from here you will see that if I do the integration so this becomes this factor becomes 0 to tau and it is from 0 to infinity because this area we have to choose.

Now it if I want to take here t from 0 to tau 0 to t and then taking the integration for them, So from here if you do this one so now I want to find out the Laplace transformation for this function this differential equation so I will take the Laplace 3t and then this is equal to s square y s other initial condition we have chosen to be 0 and then it becomes 9 y s and the sin3t is 3 over s square + 9 that we already know.

So from here my ys will be 3 over a square + 9 and divide by s square + now I have 2 functions so this is equal to I can say that is equal to some f s into some gs equal to some g of s so I have a product of 2 functions now and then I want to find out the value of yt that is the inverse of ys so if I do this one I will apply the convolution here so convolution for this one will be L inverse and this is I am taking F of s into G of s

And this will be equal f of a convolution will be equal to f convolution with g so that we know now from I have this Fs so we have choose the Fs is this one.

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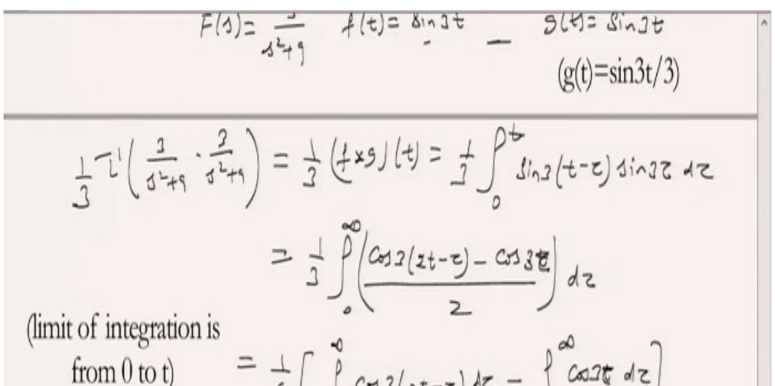

So my Fs is 3 by s square + 9 so in that case my ft will be sin3t so from here I can say that my f is this one and gt also sin3t so from here my L inverse I can take 1 by 3, so 3 by s square + 9 into 3 by q square + 9 so I want to play the convolution here so this will be equal to 1 over 3 so that will be equal to 1 over 3 0 to t and then sin3t - tau into sin3t tau d tau. So that will be the convolution of this function.

So from here if I want to find out the solution so this one we can solve and this with 0 to infinity then I have to apply that the trigonometric equalities for this one so this is equal to cos3 adding this one so it will cos2t - tau - cos3t divided by 2 and then d tau so that will be the because it is sin A sin B so this should be cos A + B - Cos A - B so if you solve for the it will be 1 by 6 and then I am taking the integration of this function cos 3 2t – tau d tau - 0 to infinity cos3t.

So this is tau d tau so from here I get the solution that is 1 by 18 sin3t– 3t cos3t so that is the solution for this equation so in this case we have applied the convolution we can solve it by the other methods also because here it is square so then we can look for the derivative that derivative the given Laplace transformation but here I want to use a convolution. So using the convolution we are able to solve this one if you go by the other method the same solution should come that you can verify yourself.

So from here so this is because Laplace transformation has numerous number of application that can be used to solve differential equation ODs and even PDs but in this course we are limited to the ODs only so that is all about the Laplace transformation and in the next class we going to start with a new topic and that is called the Fourier Series so thanks very much for watching this lecture thank you.