Introduction to Methods of Applied Mathematics
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Lecture-16
Laplace transform method for solving ordinary differential equations

So, hello viewers, welcome back to this course. So today we are going to start with the lecture number 16. So, and we are going to extend the Laplace transformation, because in the last class we have started with the Laplace transformation and then we found that how the Laplace transformation is useful for transferring the differential equation into the algebraic equation. So we will go further. And today I will discuss some another properties of Laplace transformation. So we know that Laplace transformation.

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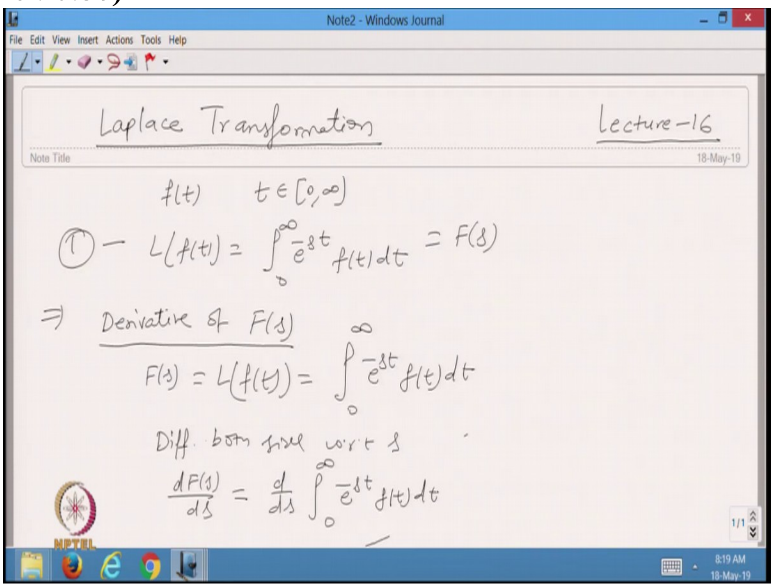

Now I have that my function that I have a function f t. And that is piecewise continuous and is of exponential order defined from 0 to infinity, then I know that so my t belongs to then I can take the Laplace transformation of f t that is from f t dt and we say that the Laplace transformation exists, just provided that this integral exist. So, now, in the last class we have discussed few properties of the Laplace transformation. So, today we are going to introduce that, what about the derivative of F s, because this is you know that this is equal to F s.

So, we will get from here. So, let us start so, you, you can call it a property. Now, this is my equation number one. So, I have my Laplace transformation for f t 0 to infinity e raised to power – st f t dt. So, what I do is that so this is my equal to F s. Now I differentiate,

differentiate both sides with respect to s. So the left side I will get dF s over ds and this is equal to d by ds from 0 to infinity dt.

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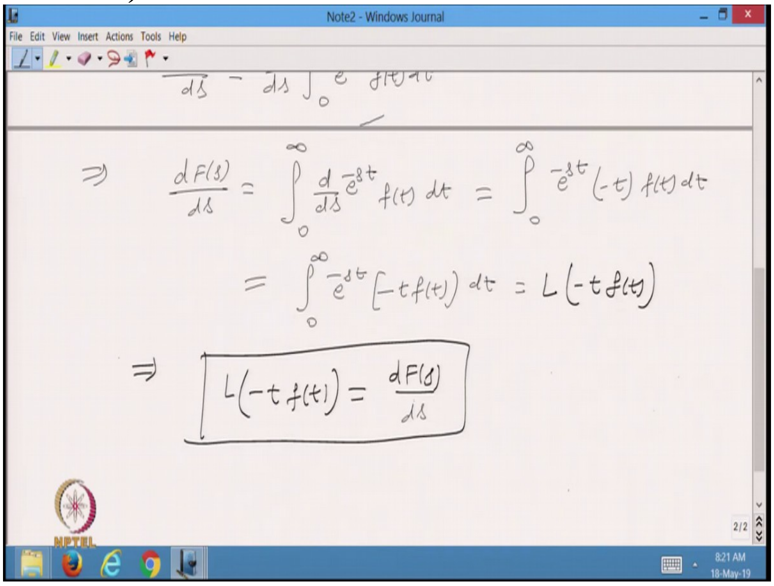

Now, so from here, so, by the Leibnitz rule I can take this differentiation inside and this will become d of d F s over d s. So, that will be integration. So, it will be the differentiation of e raised to power – st because we are taking that differentiation with with respect to s. So, f t will be same and then dt. So, from here, I if I take the derivative of this one. So, it is e raised to power – st and the derivative of this one with respect to s, so, it will be – t f t dt. So, from here, because I am taking the derivative of the s.

So I take the derivative of this exponential only. So for exponential this one and then so from here, I will get from 0 to infinity here e raised to power – st and then inside I will get –t f t. So, this factor is increased that is – t dt. So, from here, I can say that this is a Laplace transformation of – t f t. So this is Laplace transformation – t f t. So from here I can say, that the Laplace transformation of – t f t is equal to d F s by ds provided I already know the Laplace transformation of f t.

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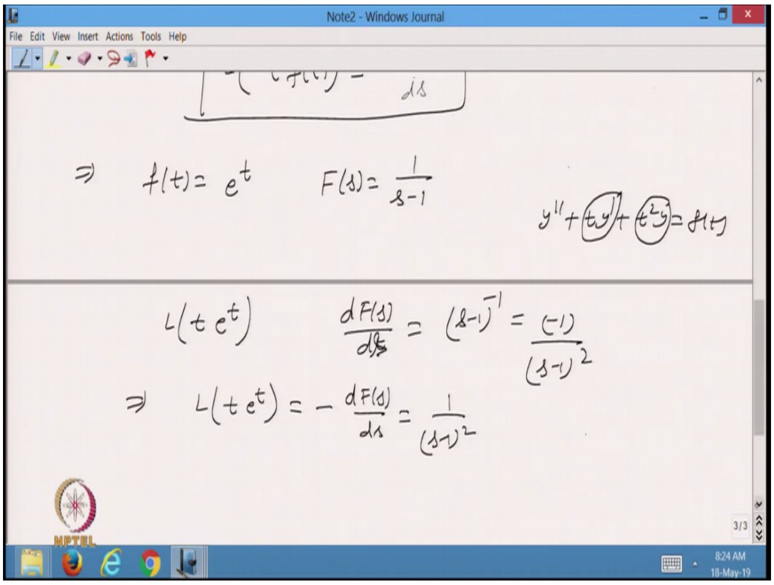

So, for example, let us take one example that suppose I want to take the f t, I just take et. Then I know that F of s will be the Laplace transformation. So, this will be as –1 to that is the Laplace transformation of et. Now, I want to find the Laplace transformation of the t e raised to power t. So, this one I want to find, and I do not know any formula related to any function, multiply by t, because why we are doing as you I told you that we are going to solve the differential equation and in the differential equation you know that the coefficient can be a constant and can be a function of f t.

So, whenever you have a constant function that is ok, but whenever you are multiply by some function of t, then you have to take the Laplace transformation of that factor also. For example, in the differential equations for the I have y'' + t y' + t square y is equal to sum f t and then I have to take the Laplace transformation. So, in that case I should be able to find the Laplace transformation when I multiply by t. Here also I multiply by t, here I multiply by t square. So, this way I should be able to find the Laplace transformation. So, from here I want to find the Laplace transformation, then I will apply this formula.

So, what I do? I will take the f s is given to me then I take the derivative of with respect to s, I will take the derivative. So, it will be if you see this will be equal to. So, I take this one and the numerator. So, s – 1 – 1. So, this will be equal to – 1 over s –1 square. So, from here by this property I know that the Laplace transformation of t e t will be equal to – of d F s by ds and this is – and –, it will be 1 over s – 1 whole square. So, from here we are able to find the Laplace transformation of a function t, it is.

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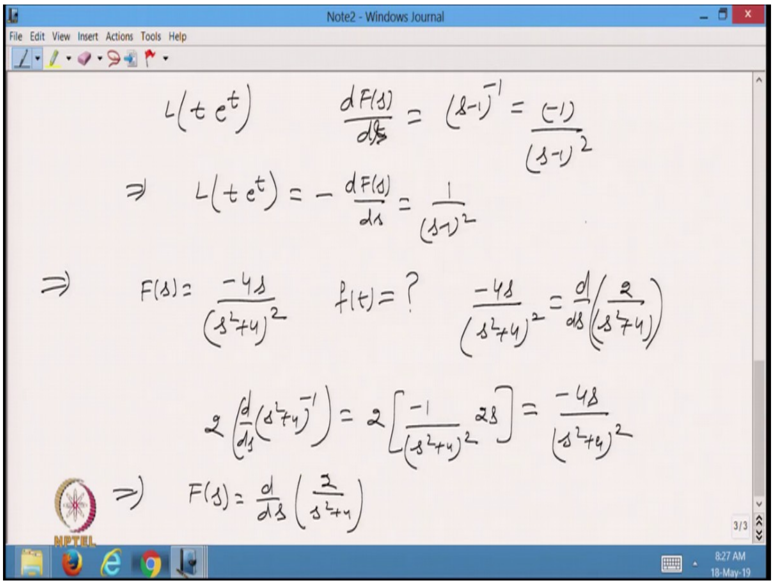

So, similarly I just want to use another way that I want to find the inverse of the Laplace transformation. For example, I have my F s and that is given to me – 4s divided by s square + 4 whole square. So, this is given to me and I want to find my f t such that the f s is a Laplace transformation on my f t. So my question is that what is my f t? And now from here, if you see clear carefully, I have factor s square + 4, but I have the power square, it means I can remember that okay, some where we have taken the derivative of 1 over s square + 4.

Because I take the derivative it become the square and the power will become square and if I take one more derivative, it becomes cubic. So like this one. So from here, what I do is that first of all I will find out that what is my, so this one I can tell from here I can see that – 4 s by s square + 4 square. Just check that if I take the s square + 4 and then I just try to find the derivative of this one. So, let us take the derivative of this one.

So, if I take the derivative of this one, so this will be 2 and then s square + 4 – 1 and then divide s, so this one I am taking. So, it will be 2 and then – 1 by s square + 4 whole square into derivative this one into 2s. So, from here I can say that this equal to – 4 s s square + 4 whole square. So, from here I can say that my F s is in this case, it is a derivative of 2 over s square + 4. So this is given to me. Okay.

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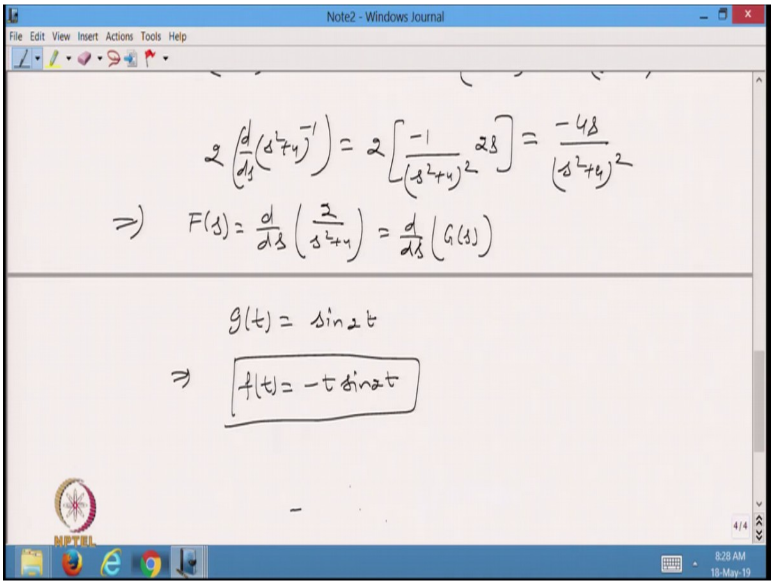

So from here I can say that this is equal to d by d s of sum G s. So what is my g t? So g t you know that this is equal to 2 times, so, it is cos 2t, so that is cos 2t. So, this is equal to the cos, not cos, it is sin basically sin 2t. So, from here I can say that my f t will be – t sin 2t. So, that is my answer. So f t in this case will be – t sin 2t, because if I multiply it by – t and then I take the Laplace than the Laplace I will get from there. So, this is a example just we have taken.

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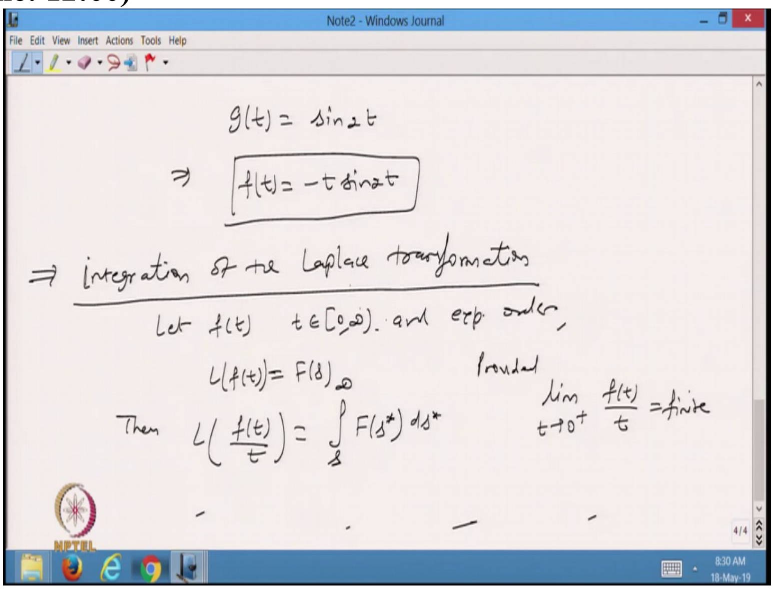

Now, I will try to find out another one and I will define what about the integration of the Laplace transformation. I just taken that differentiation, now take the integration, integration of the Laplace transformation. So, in this case I just want to define what will be the integration of the Laplace transformation. So let us do that one. So, let my f t is a piecewise continuous function as defined for the t belongs to 0 to infinity and be of exponential order. So that we already know, then, I know that let L of f t is equal to F s.

So I have the function F t is a piecewise continuous for t belong to this, and of exponential order, and so that Laplace over t f s then Laplace of f t by t. So f t by t, so this will be equal to s over infinity s star, ds star. So in this case, I am just integrating the function f s from s to infinity. So this is there, provided limit t turns to g o from the right hand side, f t over t is finite or exist. Because if this is this limit exists because this function as d tends to 0, 1 over t is going to be unbounded, it is going to be infinity.

So in that case, we say that if this both the function together, put the limit, and then that limit comes finite, then we can define this function and then we can take the Laplace transformation. So this is, in this case, I am taking the integration of the Laplace transformation. In the previous one I have taken the derivative the Laplace transformation, and in that case, we got – t into f t. But now I am taking the integration. So this is the proof we can do very easily.

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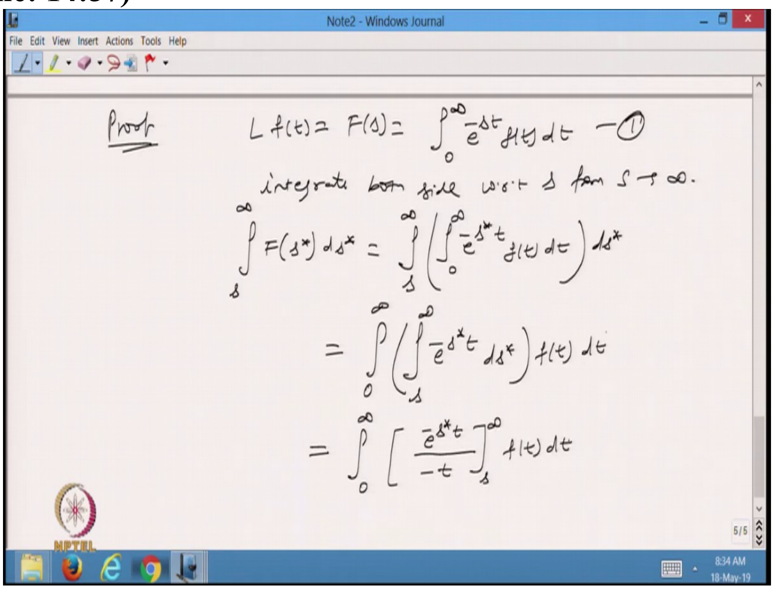

That let so I have my L of f t that in my F s is 0 to infinity e raised to power – st f t dt. Now, this is 1, now integrate both side with respect to s from limit s to infinity. So this is the limit we are taking. So on the left hand side I will get my s to infinity f of s star, I just put ds star and on the right hand side, I will get s to infinity and then 0 to infinity e raised to power – st f t dt ds star. So, this is the integration I have taken from both sides. Now, from this one, so, this is the limit we are taking and here I am taking the integration respect to us.

So, this factor is there other factors I can take outside and I can interchange the order of the integration. So, the order of integration if you change, in this case it will be from s to infinity e raised to power – s star t ds star and then the remaining f t dt, this will be there. Okay. Because in this case the t is moving from 0 to infinity and s star is moving from s to infinity. So, if I change this in order of integration you can check, so, this s to infinity I can take inside and then this will be 0 to infinity.

Okay, so, from here then if I do this one, so, I am taking the integration with s, so if I take the integration. It will be e raised to power – s star t divided by – t. So, divided by – t, because I am taking the integration with respect to s star and then putting the limit s, putting the limit from s to infinity f t dt and if I further solve this one. So from here, I will get my integration from s to infinity.

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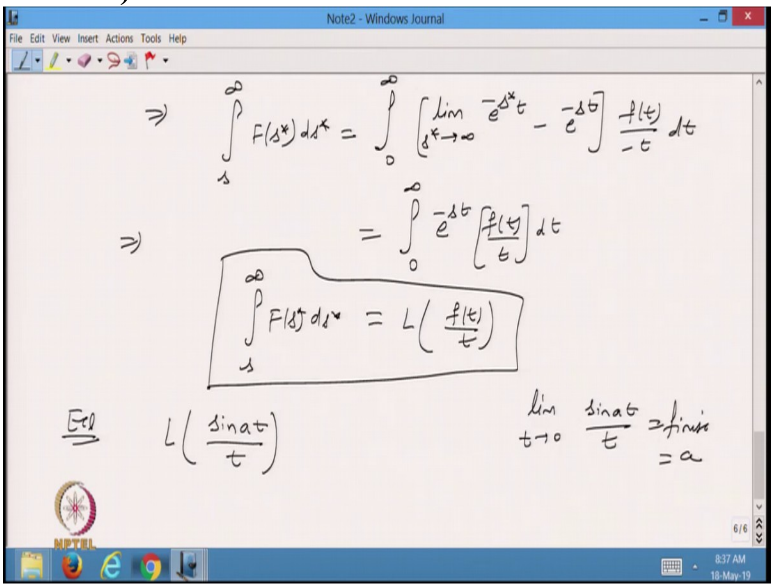

This one I can write as and now put the limit. So from 0 to infinity, I am putting the limit here. So it will be I can take the – sign outside. So, from here I can take limit s star tends to infinity e raised to power – s star t – e raised to power – st and this one I can write f t dy – t dt. So I am putting this the limit here and from here, we get this thing on same on the left hand side and on the right hand side. So this factor you know that we always taking this is greater than some number positive number, so this is going to be 0 here.

So, from here I will get 0 to infinity, – sign and – sign will cancel out. So I can write from here e raised to power – st and then I can write f t by t and then if you do it carefully, then this is the Laplace transformation of the function f t by t. So, from here I can write this is equal to Laplace transformation of f t by t. And so, it means that if I want to take the Laplace transformation the function f t divided by some t, then I to integrate the function, the Laplace transformation of the function f t. So, this is a very important properties.

Let us do one example. And so I want to find the Laplace transformation of a function. So I take sinat divide by t. So, this function I just define, I want to find out Laplace transformation this one. For this one, I to check that whether it is satisfying the properties or not and why I am taking because in this case, we know that the limit t tends to 0, sin at over t is finite. In fact, it is a if I multiply by a here and divide by a, so, this will be equal to a.

So, it is a finite value and it is satisfying the precondition that for the integration of the Laplace transformation. So, in this case what I do, if I do not remember the formula for the integration of Laplace transformation, then I can apply this Laplace transformation directly from here, but you will see that you will not be able to find the solution because it becomes cumbersome, very, very cumbersome. So, in that case we have to use this property.

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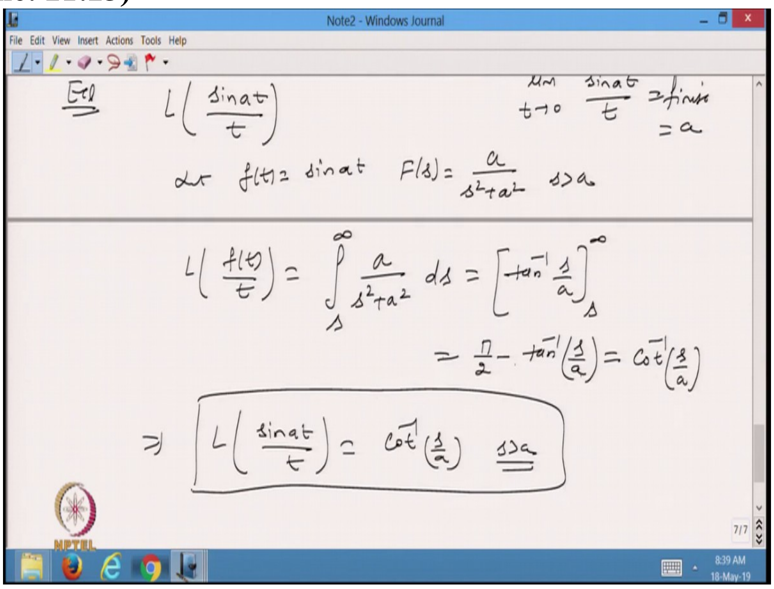

Now, so in this case what I do, I take choose that let my f t is sin at. So, I choose this one and I know that if I choose this one then in that case my Laplace transformation sin at will be a over s square + a square. So, this is my Laplace transformation. Now, I want to find the Laplace transformation of my f t over t. So, in that case I will do the integration from s to infinity and then my F s will be this one. So, it will be a, s square + a square ds. I can add also like this one or otherwise I put the star here, because the indexing.

So, in that case if I do the integration on this one, so this will be equal to tan inverse s over a, from s to infinity and if I put the infinity here, as tan infinity is pi by 2 – tan inverse s over a. So, this one if I solve further than from here I can write this equal to cot inverse s over a. So, that is the Laplace transformation. So, from here the Laplace transformation of sin at over t is cot inverse s over a, provided s is > a. So, that is there already. So, this is my Laplace transformation.

So, that is why it is very useful property that whenever we know the Laplace transformation of some function f t and the new function comes as a dividing by the value t, then definitely always we have to keep in mind that we have to use this integration formula. So, this all things will be useful for whenever we are going to solve the differential equations. Now, one more property I want to use for this one, what about if I want to take the Laplace, I in the, last class we have discussed the Laplace transformation of a derivative of the function.

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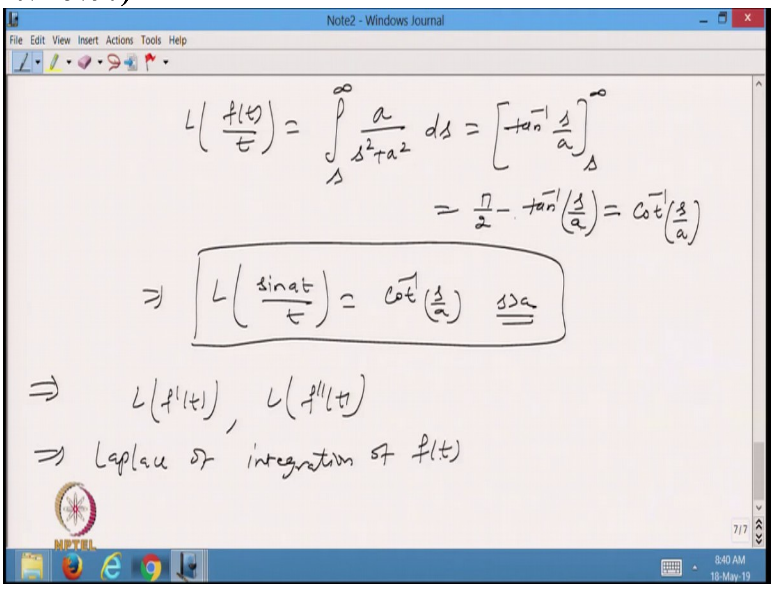

If you remember that I have taken the Laplace of f' t Laplace f'' t , in the last class. What happened if I take the Laplace of integration of f t?

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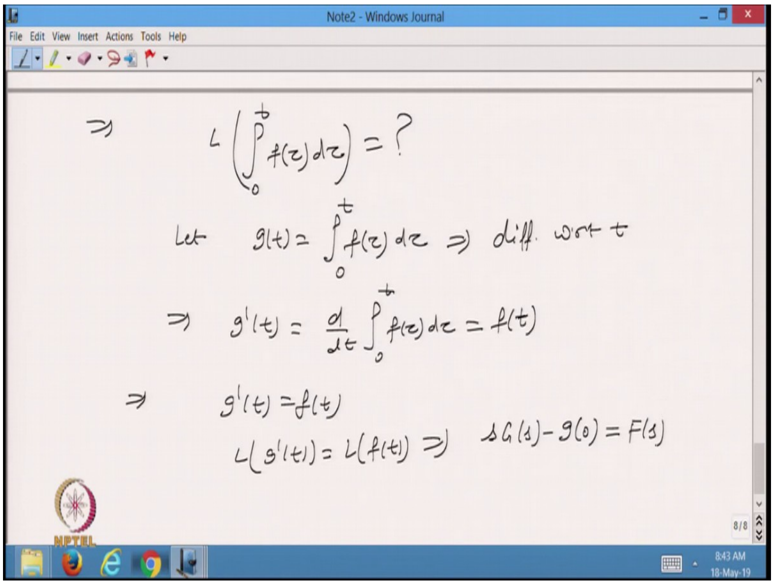

So, suppose I have the integration of f t like I have my function defined as this one 0 to t, I have f tau d tau. So, I here I am have the integration of the function f from 0 to t and I want to take the Laplace transformation on this function. So, what will happen if I want to take the Laplace transformation or this one? So, from here the proof I can say that that what will be this one. So, let it is not to prove it is just that I want to find out that what will be Laplace transformation of this one.

So, let I choose that my g t with the function from 0 to t f of tau d tau, so I choose this one. Now from here I take the derivative with respect to t, both sides. So take the differentiation, differentiate with respect to t, both side. So from here I will get it will be g't on the left side and the right side it will be d by dt 0 to t f tau d tau. Now in this case, what is happening? I am here I am doing the integration of the function from 0 to t and then I am differentiating this one with respect to t.

So by the first fundamental theorem of calculus, from here, I will get f t. So this is my f t. So from here, I can say that my g't is equal to f t . Now I know that the Laplace transformation. So, Laplace transformation of g't is equal to the Laplace transformation of f t. So, from here Laplace transformation g't is what? It was a Laplace s G of s – g of 0 is equal to the Laplace transformation of f t. This is my, I can write as f s.

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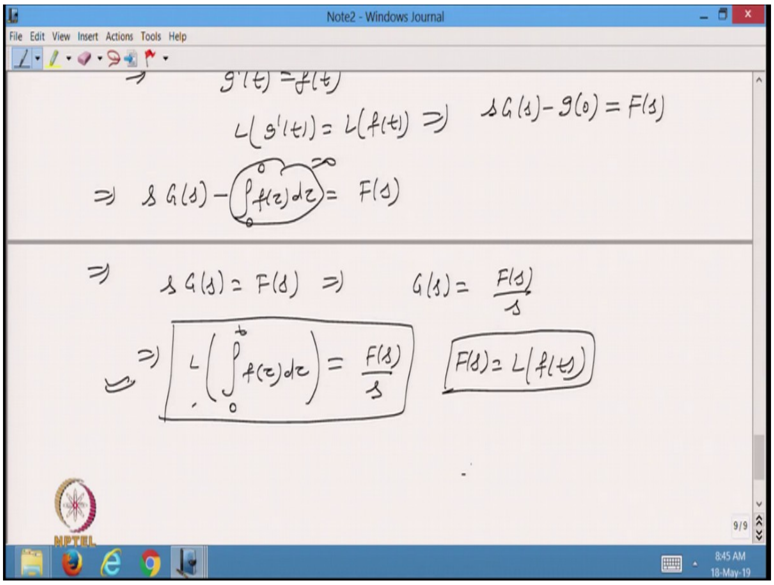

So from here I can find out. So what about my g 0? So, s of g s – g 0 , what is my g?, g this one. So, if I put instead of t, I put 0. So, you will say that it is integration from 0 to 0 and that value will be always 0. So this is equal to 0. So from here I will get as g s is equal to f s. Okay. And so this one, so from here my g of is f s divided by s. So from here I can say and what is my g, g s? So, it is a Laplace transformation of g. Here my g is, integration, so from 0 to t f of tau d tau, so I know already know the function f t.

So, I know the Laplace of f t that is f s. So, Laplace of integration of that function is always equal to the f of s by s. Where the f s is the Laplace transformation of f t. So, this function the Laplace transformation of this function already known to me, if I already know this one by this formula, I can find the Laplace transformation of the integration of that. So, this is also one of the important properties that we use some time in solving the differential equation.

Now, in the last class, we also discuss about the unit step function and then we discussed that the Laplace transformation of the unit step function, so in this I will go a little bit further. And you know, already know that we in the previous classes, I have discussed about the dirac delta function. So, dirac delta function we already know that this is equal to that is equal to 0 when x is not equal to a and infinity when x is equal to a.

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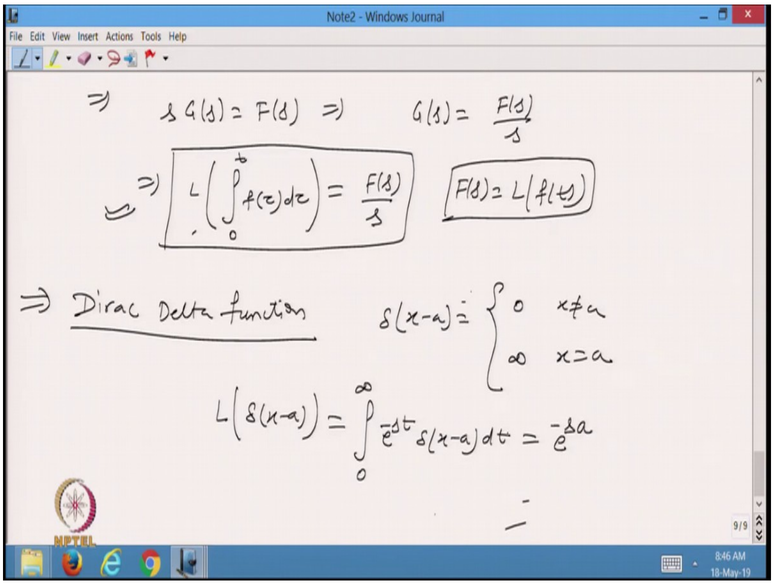

So, that we already know. Now, what I do, I want to find out the Laplace transformation of the dirac delta. So, what about the Laplace transformation, so, Laplace transformation I will apply the Laplace transformation for this function, e raised to power – st dt. And if you remember that, whenever I have some function f t multiply by the dirac delta function and doing the integration then we already know then this is always equal to the function the value of the function at the point a.

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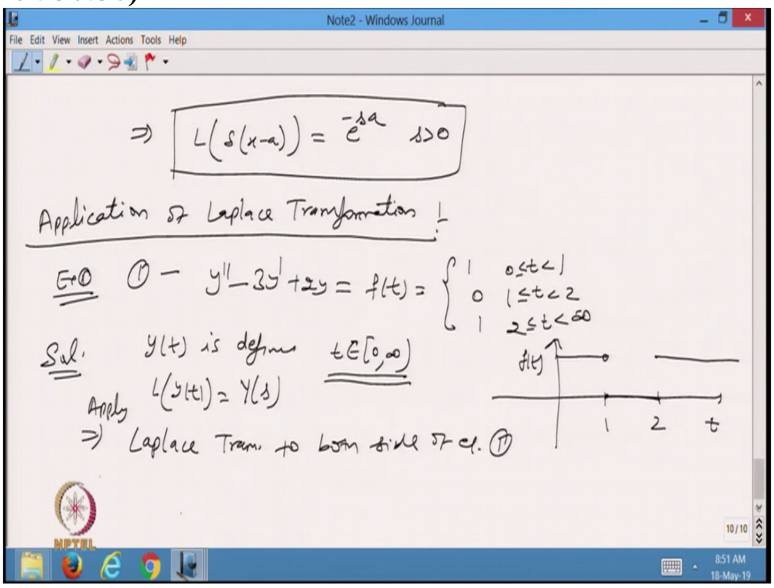

So, from here I can say that the dirac delta, the Laplace transformation of the dirac delta is always equal to e raised to power -sa where s is > 0. So, this is the Laplace transformation of the dirac delta function. So now after find out the Laplace transformation of the dirac delta, so let us do the some application of the Laplace transformation. So, let us take 1 first 1 the example I just take 1 I have a differential equation y'' -3 y' + 2 y = f t.

And f t is given to me as a function 1 when t is, its value is 0, when t is and its value is 1 and t is = 2 infinity. So this function is given to me in this form. So if you want to find out the solution of this second order linear differential equation, you can apply the method but if you see on the right hand side, the function f t is not a continuous or differential function as we were doing for in the previous classes, but this is a piecewise function and if you see this function this function if I plot so it is t and this is my f t. So, from 0 to 1.1 is this 2 this 1.

So, from 0 to 1 its value is 1. Because it is open on that 1 and then from 1 to 2 is value is 0 and from 2 to infinity its value likes this. So in this case my function f t is given to me in this form. So, it is a piecewise continuous form. So, I in this case, if I want to solve this differential equation using the previous methods, then we do not know that how to apply the how to find out that because I in this case, I am able to find the solution for the homogeneous part that I can put = 0 and then I will solve this one, but what about I want to find a particular solution.

So, in the particular solution, you know that we have applied the operator theory to find out the particular solution or the variation of parameter. So, in that case, we do not know what will happen if I have the function f t on the right hand side in the form of a piecewise continuous form. So, in that case, this method is these differential equations seems to be very difficult, if you want to apply the previous method, but if I have the knowledge of the Laplace transformation, then you will see that it is very easy to find the solution for this type of differential equation.

So, what I do is that and now in this case you know that my function y t is well defined, I assume that and between the T belongs to 0 to infinity. So, that should be there that my t should be the function y t should be well defined and of the exponential order for all t from 0 to infinity, because this is their only then I can apply the laplace transformation and I also consider that the laplace transformation of the function y t is y s . So, this one I already assume. So, from here what I will do that this equation number 1.

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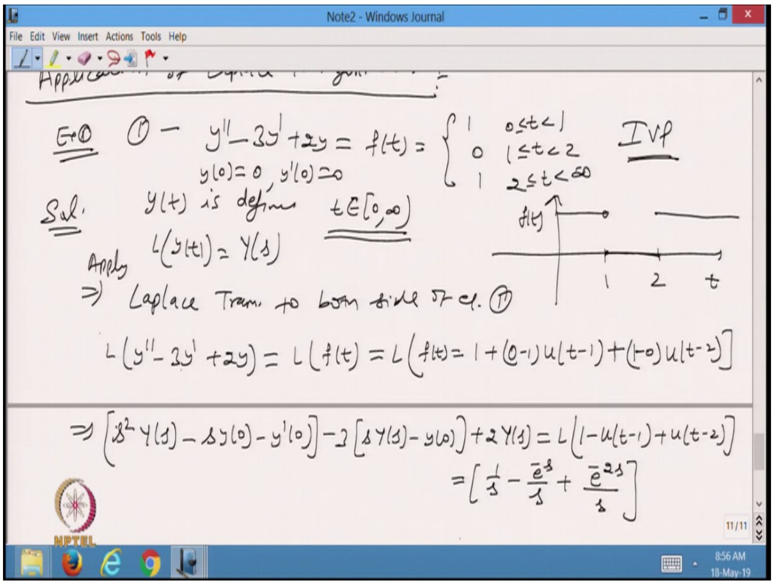

I take the Laplace transformation, apply Laplace transformation to both sides of the equation 1. So, in that case I will take the Laplace transformation, so Laplace transformation of y'' - 3 y' + 2 y equal to Laplace transformation of f t. Because the Laplace transformation if you see, whenever I take the Laplace transformation with a derivative, then the value of the function and its derivative at some point should be known to us, then I take the initial problem. So, for me I also define that y at 0 is 0 and y' 0 is also 0.

So, this is my initial value problem. And I am taking the initial value as y 0 and y' 0 0. Now I apply the Laplace transformation. So, because I know that Laplace transformation is a linear ah transformation. So, from here I can apply now, my f t is there. So, first of all I will apply try to make the write the function f t in the form of a single function with the help of a unit step function. So, in this case what I do my f t I can write in this form as 1 + then 0 - 1, this - this and defining the unit step function at the point which is common to both + this value - this value.

So, 1 - 0 and then unit step function ah u at 2. So from the left hand side, I will get s square y s - s y 0 – y' 0. So, this is my Laplace transformation of y'' -3 and then the Laplace transformation y' will be s y s – y 0. Okay, + 2 times the Laplace transformation y that is already known to me that is y s. And on the right hand side, I want to take the Laplace transformation of the function. So this function can be written as 1 – u t -1 + u t -2. So, this one this Laplace transformation I want to define.

And just to verify that whether we have written this function correctly or not, you can put the you can check the value of the function for different value of t like I have a value whenever t belongs to 0 to 1, in that case this value with 0 this will be 0 will get only 1 when T lies between 1 to 2. So, this value would be 1, so, in that case this 1 -1, so, it will be 0 and this will be 0 and the value will be 0. When t is lie between 2 to infinity in that case, this will be 1 this will be 1, so, the value will be 1. It is okay.

Then from here the Laplace transformation of 1 is 1 over s - and I wanted to find the Laplace transformation of unit step function at t equal to 1. So, this is a unit step function. So, I know that the Laplace transformation is e raised to power –s over s in this case. And in this case, my Laplace transformation will be e raised to power -2s over s. So, this is the Laplace transformation we are defining here. And you can also see that just now we have calculated the Laplace transformation of dirac delta function.

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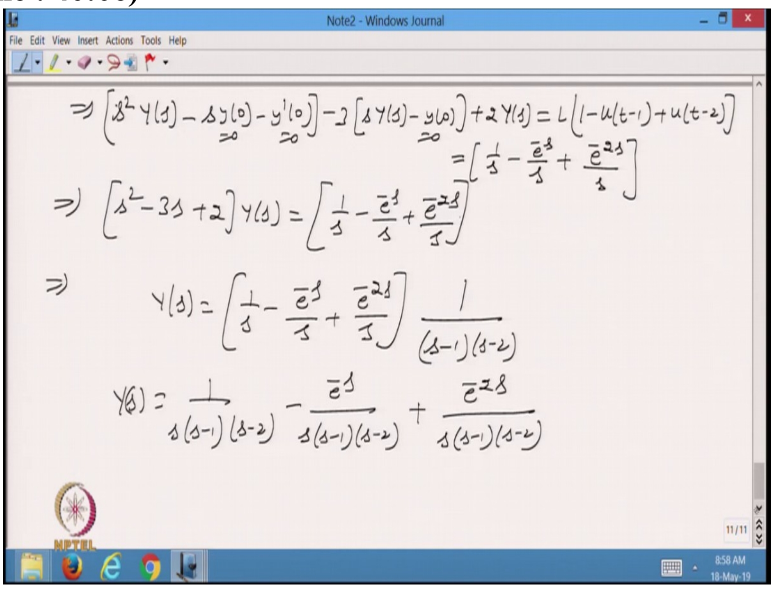

So, that was e raised to power - as, but here we are dividing by s. So, that was a Laplace transforms the unit step function. So, from here I just find out a solution this value is 0, this is 0, this is 0 that we have chosen to make it simpler. So, from here I will get my i t will become s square -3 s + 2 y s. This is 0, so we will get this value and on the right hand side we get this one. So from here, my y s becomes so this value I can take on the right hand side. So this will be 1 over s – e raised to power -s over s + over 1 over.

So this one I can make the factor. So, this is coming s s square. So, this will be s -1, s - 2 because if you see from here it will be s square -3s + 2. So this is my y s. Now I want to solve this one from here I will get that if I multiply this factor, so my y s I can choose as here 1 over s s -1 s -2 - + e -2s s s-1 s-2 . So, this is the factor we are taking here. Now, I wanted to find the Laplace transformation of this factor.

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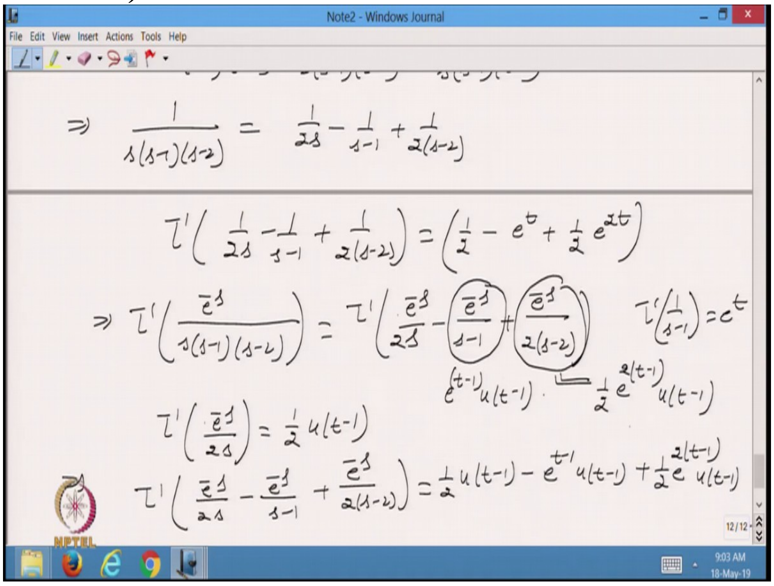

So from here, I will use the partial fraction. So, this factor as s s-1 s-2. It can be written as using the partial fraction. So I can write this as 1 over 2s - 1 over s-1 + one over 2 s-2. So, using the partial fraction I can write this fractional part in this form. Because whenever we are solving this type of differential equation, generally we have to apply the partial fraction to make this because only then I am able to take the Laplace transformation inverse Laplace transformation. So, from here, I am able to find this value.

Now, suppose I want to find the Laplace inverse of this one. This value and I know that the Laplace inverse is also linear this form. So, it will be 1 over 2 – e raised to power t + 1 by 2 e raised power 2t. So, this is my inverse Laplace of this function. So, from here I can define the Laplace inverse of e - s divided by this factor again s s-1 s-2 and this one I can take similar way. So, it will be again l inverse of e raise to power –s over s 2s, - e raise to power -s s-1 + e raise to power –s 2 s-2.

And You from here you can see that I have a function of s and that is being multiplied e raise to power –s. So, if you remember, then this form leads to the use of somewhere we are using the unit step function. So, from here if I want to take the Laplace transformation inverse of the Laplace transformation, so, the Laplace inverse of e raise to power –s over to 2s. So, this if you take this one it will be half u t-1. Laplace of this one it will be e t-1 u t-1.

Because, the Laplace 1 over s-1 if I want to the inverse e power t, but I have factor e raised to power -s. So, it is a unit step function and we know that this is becomes f t -1. Then Laplace inverse of this function it will be 1 by 2 e raise to power 2 t-1 u t-1. So this will be there. So, from here, if I find out the Laplace inverse, if I just write here directly then Laplace inverse of the whole factor -2 will be in this case it will be = 1 by 2 u t-1 - e raise to power t-1 into u t-1 + half e 2 t-1 u t-1 . So, this will be the, the inverse of the Laplace transformation for this factor.

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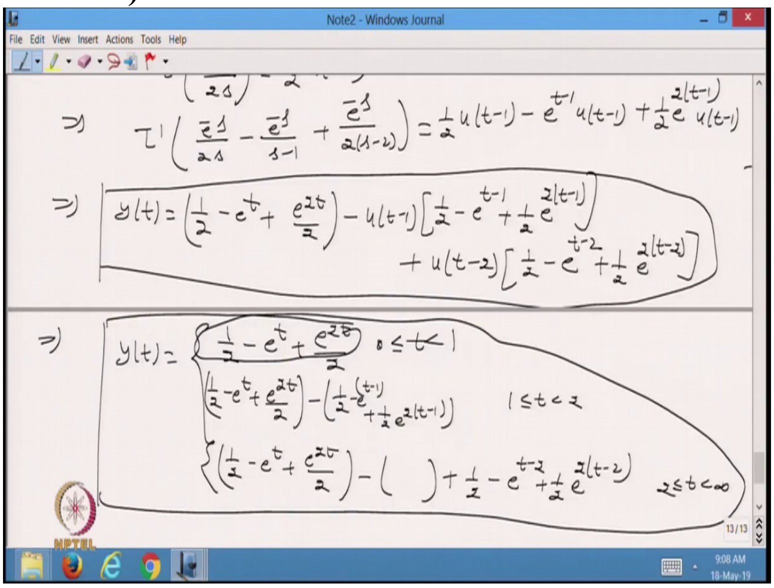

So, from here and the same way I can go for the another one only thing difference is that if you see it clearly, then here only we are multiple by e raised to power -2s. It means the unit step function has the value at the value at so, a is equal to 2 in that case. So, from here now, y s is equal to the factor we have taken. Now, I want to find the f t. So, f t in that case will be the solution in that case will be y t will be.

So, I can write my factor now, half – e t + e 2t by 2 and then the inverse - so I can take u t-1 common here from this one, so to be half – e raise to power t-1 + half e raise to power 2 t-1 last factor. So, that would be unit step function define at 2. So, this will be half - e t -2 + half e 2 t-2. So, from here this one. So, that is my ultimate solution of the given differential equation, when the right hand side function was the piecewise function. So, from here I can say that my solution y t will be, I can write this function in the piecewise form.

So, this will be equal to half – e t + e 2t by 2 when t is 1 because this factor will be in that case. So, now then when I whenever I have t. So, in that case it will be the same factor half -, so, in that case this value will be 1 and then I will get this value. t-1 + half e 2 t-1 . So, this is a factor. When t is 2 and then whole value will be equal to always. So, this will be half – e t - this factor + +, so I am taking now when the t is > 2, so t is > 2 its value will be 1 and then it will be equal to half – e raise to power t-2 + half e raise to power -2. So, this is equal the t.

So, from here you can see that this function, if you see that this functions is almost same here, in this case, only we are transferring this to by 1 on the right hand side, and then again transferring this one to the right hand side by the one. Otherwise, we have the same function, but it is just doing the transformation by 1 and again 1 and we get this solution. So, that is the solution of the corresponding differential equation we are trying to solve.

So, today we are try to find out the solution of a differential equation where the right hand function was the piecewise function. So, in the next class will go further and try to find out the application of the Laplace transformation for other problems. So, thanks for watching. Thank you.