Introduction to Methods of Applied Mathematics
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Lecture 15
Laplace Transformation (Contd…)

So, welcome viewers. Back to this course. So, in the last class we have started with the Laplace transformation and then we have discussed the existence for the Laplace transformation. So, this was the slides, which was the last slides in the previous lecture. So, in that the M was skipped. So, this M also there.

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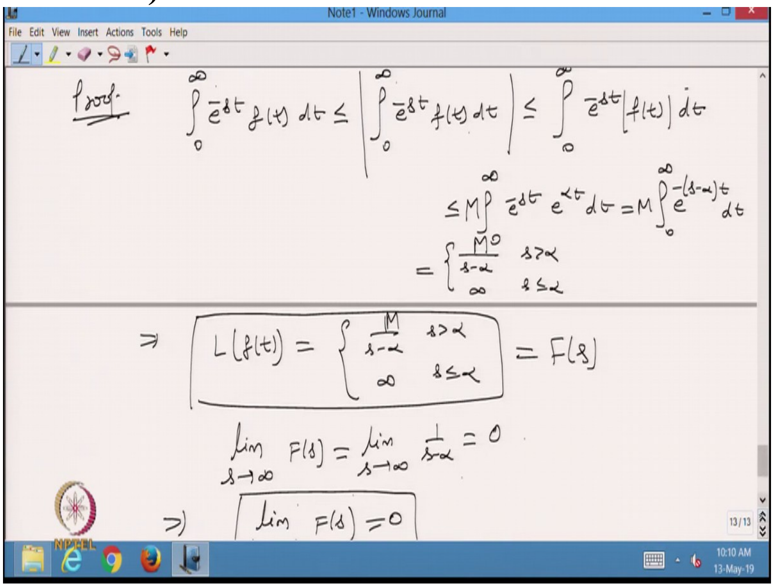

So, you just correct that this M is there and then other things are same. So, now, we are going to start with the next lecture and that is the lecture number 15.

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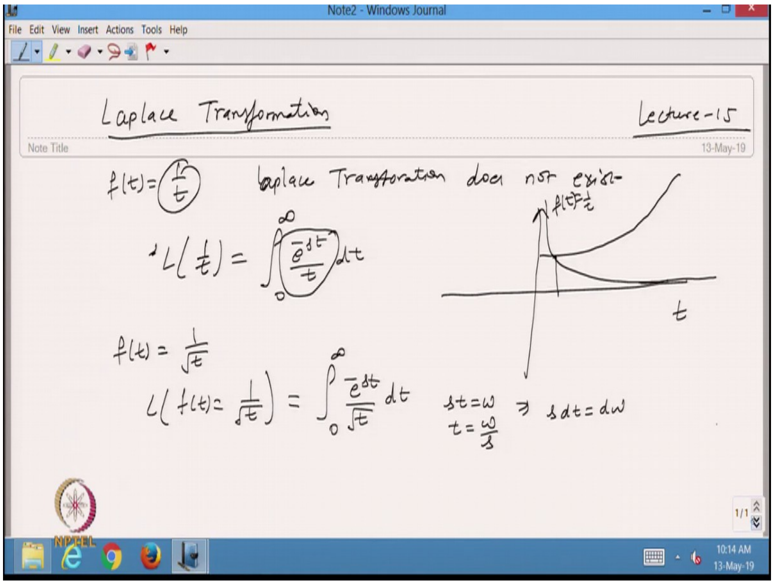

So, in this case also we will discuss the various properties of the Laplace transformation. So, in the last class also we have discussed that in my function f t is 1 by t. So, in that case, we have found that the Laplace transformation does not exist. So, this one we have discuss in the last lecture. Now, what about so, if you see this one, I can say that from the existence theorem, my function 1 by t is like this one, of this type, never touching this t axis and this my f t = 1 by t and if I take the any exponential function for any value alpha that is always of this type.

So, if you see that, at this point after this value this function is satisfying that this my function f t = 1 by t is of exponential order, but as we are going close to 0, it is unbounded function and in that case we say that this function f t is not of exponential order and then in that case and also in the Laplace transformation we are dealing with 0 to infinity. So, this function is unbounded as we go close to the 0.

So, for the integration actually we have to see that if I take the Laplace of 1 by t, then we have to see that from 0 to infinity, e raise to power -st over t dt. So, how this behaves when we are going close to 0. So, that is the main thing. So, in this case we say that the Laplace transformation does not exist. But if I take the another function of the similar way of 1 over under root t. So let us see that this function is also this type, if I plot this function, only thing is that this function is falling much slowly as compared to the function 1 by t.

So we want to see that how it behaves when we multiply by e raise to power -st, near 0 because the functions is the problem and t tends to 0. So let us take the Laplace transformation of this function f t over 1 by under root t. So let us take the Laplace transformation this one, so this is from 0 to infinity. e raise to power -st over under root t dt. Now, I will take the variable change of variables. So let us assume that my st is some w, I take. So from here I can say from here that the t is w by s. So from here and also my sdt = dw.

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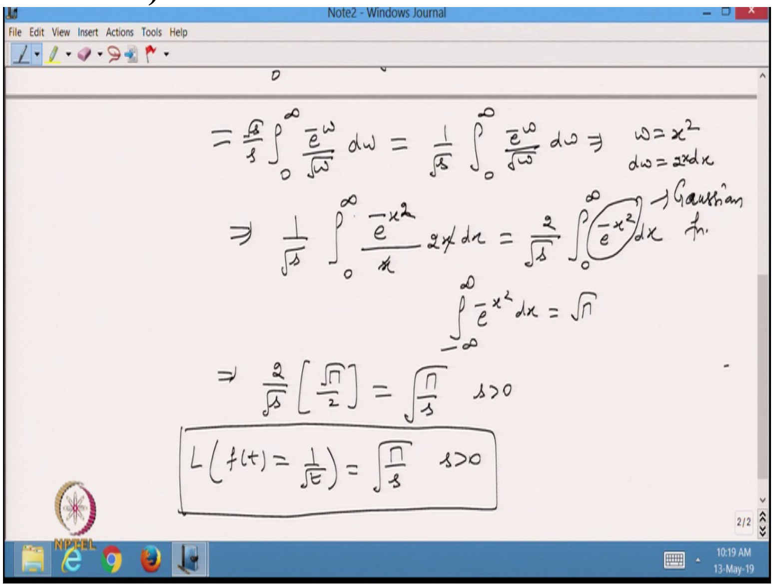

So from here, I can define my function equal to and when t = 0, w is also 0. When t = infinity, the w is also infinity. So from here, I get my integration from 0 to infinity, e raise to power -w. So, this is a w I have taken. dt is dw over s and under root t. So under root t from here my under root t from here will be so under toot t will be w by s. So from here and this one so I can write from here. So t will be this one. So from here this can become 0 to infinity.

So under root s and because I am taking the integration with respect to w. So s I can take common. So it will be under root s over s and inside it is -w by under root w dw. So, this becomes a 1 over under root s 0 to infinity, e raise to power -w over under root w dw. And this one again I can write in this form as taking the change of variable w = I can take as x square and from here I can say that dw will be 2dx. Right.

And then from here, the integration is same when w is 0 x is 0 and w is infinity x is infinity, from here I can write as 1 over under root s and the integration from 0 to infinity e raised to power - x square and under root is x and from here I dw is 2 x dx. So, this will cancel out and from here I can write that this is equal to under root s from 0 to infinity e raised to power - x square and dx and you know that what is this? So, it is a Gaussian function.

And I know that from the integration that if I take the integration from - infinity to infinity e raised to power - x square dx then it = under root 5. So, from here I can say that and this function is always positive function and this is a symmetric also. So, from here I can say that this will be = the integration of this one will be = from because this is given from - infinity to infinity, but here in this integration is from 0 to infinity. So, from here I can say that this is = under root pi by 2.

So, these 2 will cancel out and from here I can write that this is = this one for S > 0. So, from here, I can say that the Laplace transformation of the function f t =1 by under root t is = pi over s when s is positive. So, in this case the Laplace transformation I just for 1 by under root t because I told you that it depends on that how the function is behaving, when we multiply e raised power -st near the point = 0. So, from here we are able to see that the Laplace transformation of the 1 by under root 2 is under root pi by under root s when s is positive.

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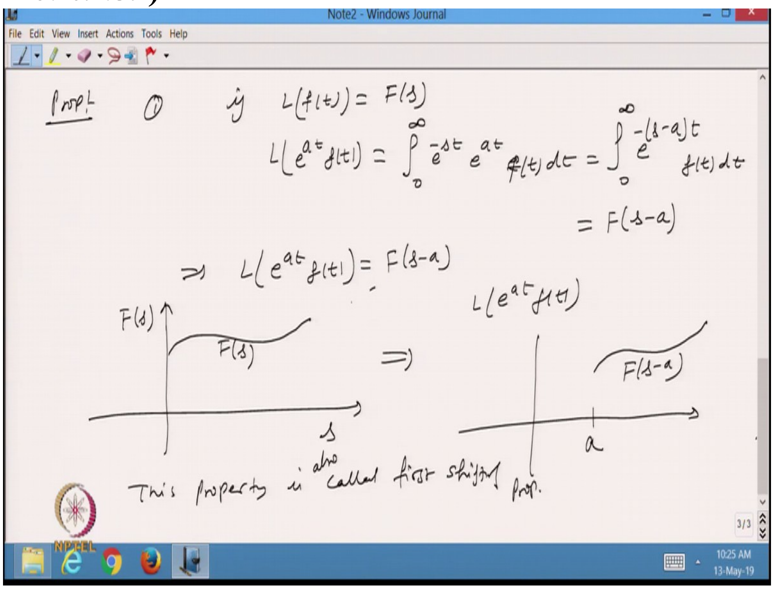

So now, we will define some another properties of the Laplace transformation. And that another properties a Laplace transformation is the first one is that if my function f t is there and I already know the Laplace transformation of that function, FS is given. So, what will happen sometimes that I have multiply by this function f t by some exponenation, at f t so what will happen? And then we from here, just I want to find the integration is e raised to power –st e at this is f t dt.

And which can further we return as e – s-at f t and from here I can say that this is again the Laplace transformation of f t only thing that we have changed my s to s - a. So, from here I can say that this is the Laplace transformation with the change in scale of this. So, here we have written that f is s-a. So, we are changing the scale of the function s by a. So, from here I can say that the Laplace transformation of e raised to the power of at f t it will be f s-a. So I am changing the scale.

So, like, So, like I can define now, that if I take the function f t and my this is my s and this is my F s . So, for the Laplace transformation of the function f t my transformation suppose was this one for s > 0, then the Laplace transformation of e at f t will be again the same function will be there, only thing is that it will move by (a), so, it will be this value. So, this is my F s-a and this is my f s. Other things will be same it just shifting the scale on the right hand side.

So, this is sometime we also call that this property at the first shifting theorem. So, this property is also called first shifting property or the theorem. So, in this case they no need to find the Laplace transmission again and again I will use this property.

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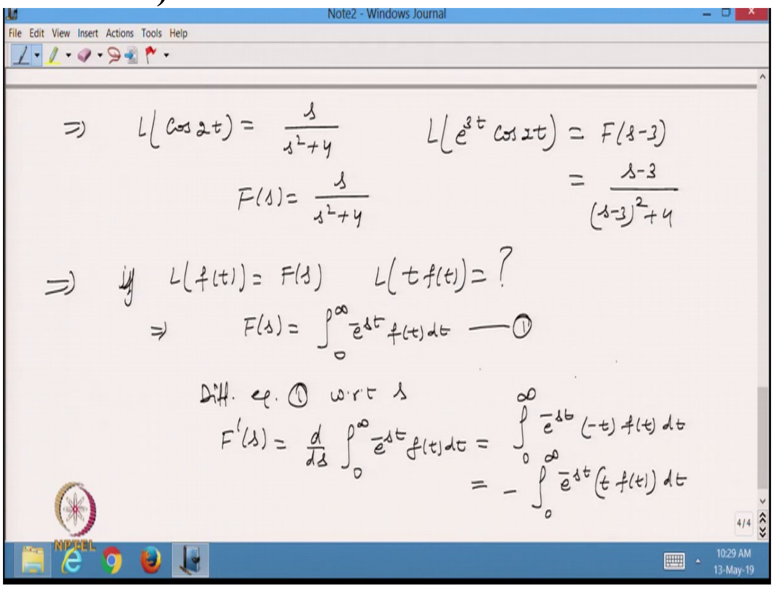

For example, I have the Laplace transformation of cos 2t, that is s over s square +4. Now, I wanted to find the Laplace transformation of e 3t into cos 2t. So, now they no need to define the Laplace transformation and again and doing all this calculation, I will use the first shifting theorem. So, from here I know that my f s is s over and now, I know that the Laplace transformation for this function will be f s-3. Because this is a a = 3 here and from here I can write that this is = s-3 over s-3 whole square +4.

So, whenever we are having this type of functions and this s - something is there then we should be understood that we are using somewhere we are using the first shifting theorem. Now I define another property is that what will happen? If so, I have if my function L of f t is f s, then as I told you that this Laplace transformation will be used when solving the differential equation, so, what will happen when I have the Laplace transformation multiply by some t f t So, this one I want to find out what it is.

So, this is a given to me, but what are this one. F t is the solution of the some differential equation. So, I know that sometime the differential equation the coefficients are also function of t and may be suppose, I have a differential equation in which some t into f t coming up. So, what will be the Laplace transformation of t f t So, this one I want just want to define now I know that my f s is 0 to infinity e raised to power –st f t dt. Ok.

Now what I do is that I so just call it equation number 1. So, differentiate equation 1 with respect to s on the left side I will get f' s and on the right hand side I will use the Leibnitz rule. So, I just want to take d over ds of this f t dt. So, from here I can define taking this into differentiation inside. So it is a function of s and t and this is a function of t only so I am taking the derivative with respect to s. So, this will be taking the derivative with respect to s it will be e raised to power –st into –t f t dt. And which can further written as - t can take – sign I just take here. So, this will be 0 to infinity e raised to power –st t f t dt.

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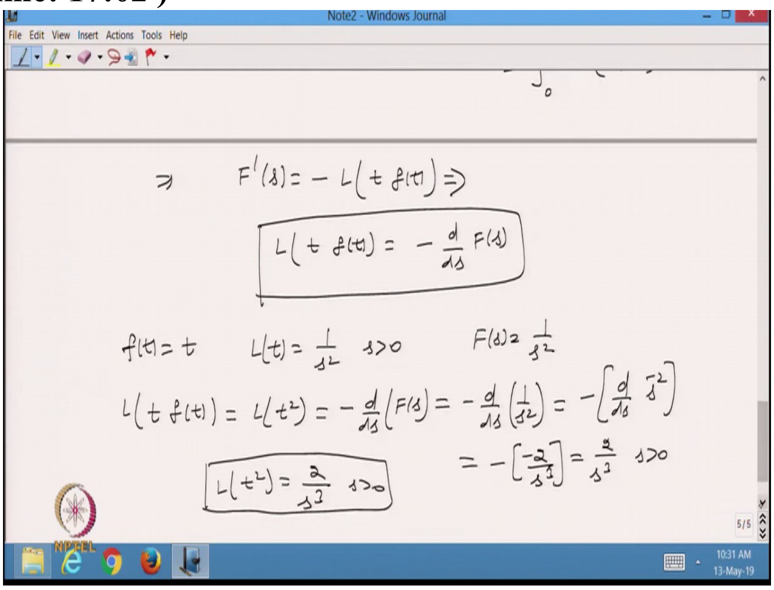

And if you see carefully, then from here I can say that f' s is equal to - the Laplace transformation of t f t I can say from here that the Laplace transformation of t f t will be - of the by ds F s. So, this is just now we are able to define this one. So, we can use this property. So, let I take my function f t is equal to suppose I just want to verify this one. So, let us take my f t = t and I know that the Laplace transformation of t is 1 by s square. This is already we know.

What will happen if I just want to verify this one and Laplace of t into f t. This one I have to do. So from here, I just want to find this one. So it will be t square and I will use this property so it should be - d over ds of F s and F s I know is 1 over s square. So I am taking - d by ds of 1 over s square. So this will be what? So I am taking d by ds of s - 2. So this will be - into - 2 over s cube. I am taking the derivative.

So from here I can to find it will be 2 over s cube when s is positive and that is all right, we already found that the Laplace of t square will be 2 over s cube. So, this is verified that whatever just now we have done it is true. So, we can use this property to define the various Laplace transformation of the various function just multiplying by the some t. Now, the next things we are going to discuss is, because so let us do 1 example, that how we are able to use this Laplace transformation to solve the differential equation.

Because, our main purpose is to solve the differential equation using the help of Laplace transformation.

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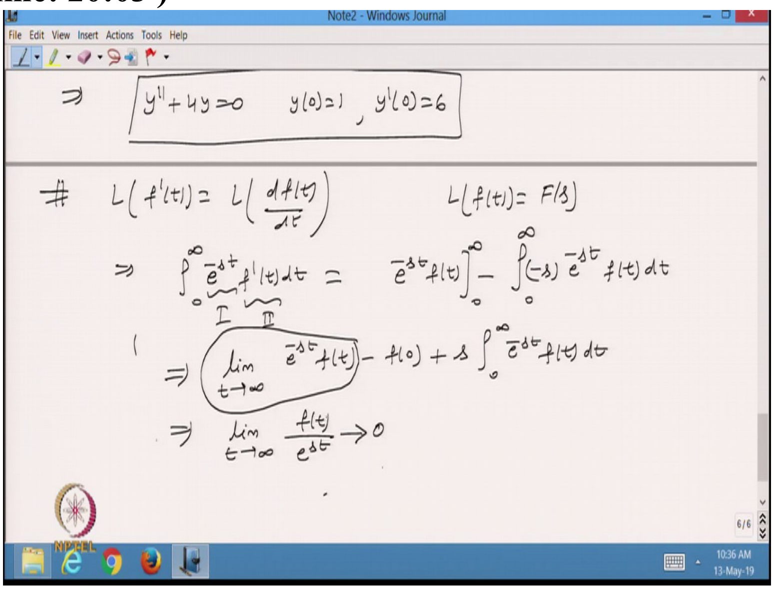

So I just take a very simple example. Just take y'' + 4 y = 0 y 0 = 1 and y' 0 = 6 . So this is a very simple linear second order initial value problem. And this equation I want to solve. So in this case, first thing is that I have to do take the Laplace transformation of the derivative. So before solving this differential equation, we should be able to find that what is the Laplace of the derivative with the function?

So let us define this one before solving this problem. What is the Laplace transformation of the function f' t means I want to find the Laplace transformation of this one, where the Laplace transformation of f t is known to me and this is equal to F s so that is given to me. okay. So from here I will put the definition, I will go by definition. So taking the integration from 0 to infinity and then e raised to power -st f ' t dt.

Now, this is again we have to use the by parts, but only thing is that the derivative is there. So, I have to use this as the second function now, and this as the first function. So, from here I will get e raised to power -st and integration of the second function. So, that is f t 0 to infinity - 0 to infinity. So, taking the derivative of e raised to power –st. So, it will be -s e raised to power -st and then integration f t dt. Now, from here I will get my function.

So, this is the integration we have to solve. So, I have to define what is the limit t tends to infinity e raised to power –st f t and then putting 0 – f 0. Because this is 0 value be 1 + and from here I can take my s outside. So, this will be s and then 0 to infinity e raised to power –st f t dt. Right. So, this is the integration we are able to solve. Now, I know that my function f t is of exponential order. So, in this case and I am also taking that S > 0. So, from here, now, I want to solve this factor that what will happen to this one.

So, in this case, I now I have the limit t tends to infinity f t e st. Now, my function f t and e st is there. So, if t tends to infinity, I just want to check that what is the value of this function and my function f t if we go by the existence dt this function is of exponential order. So, suppose this is of exponential order. So, it is alpha t and e st. So, from here, if you see that st tends to infinity, this will goes to 0, because this e st because e alpha t I am taking and e st is I can say that it is taking advantage over the function f t .

So, when the t tends to infinity will go because this function going faster than f t . So, when the t tends to infinity this value is going very small and tending towards 0.

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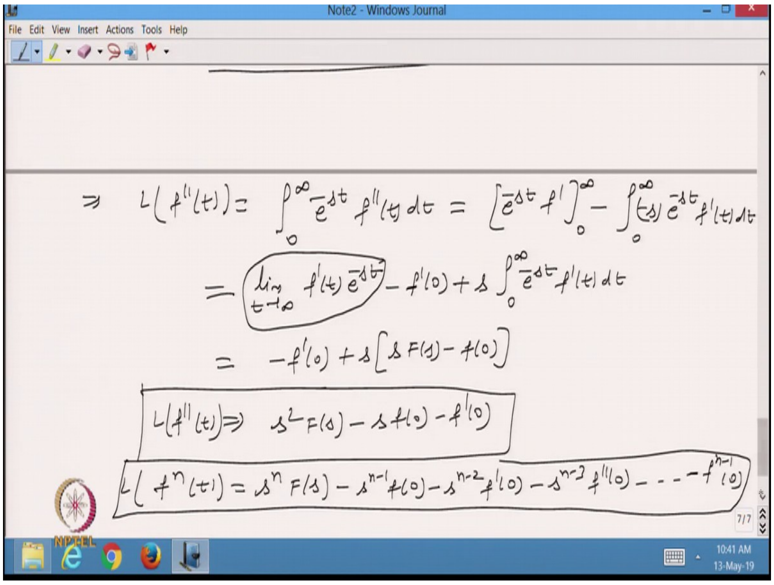

So, this factor will be 0. So, from here I can say that my s 0 to infinity e –st f t dt –f 0 . So, from here I can write that the Laplace transformation of f ‘ t will be s f s . So this is f s – f 0 . So, we are able to define the Laplace transformation of the function f ' t . Similarly, I want to take the Laplace transformation of L of f'' t . So, let us see what will happen in this case. So, this function is again I am going by the definition.

So, e raised to power -st f'' t dt. So, this will be there. And then again applying this formula, so e - st the first function and then the integration of the second function - 0 to infinity - s e raised to power -st f ' t dt. So this factor is again, I am putting this value, so limit t tends to infinity f ' t e raised to power - st. You can see from here that I am taking the Laplace transformation of f ' t or f'' t . So, we are assuming that f ' t is also exponential order and f'' t also is an exponential order or may be piecewise continuous function.

So, that we should be able to take the Laplace transformation - putting the value 0 here. So it will be f ' 0 and then + s and then I am taking the Laplace transformation e raised to power – st f ' t dt. So, this factor will again go to 0, because our exponentially e raised to power -st is going much faster than the function f ' t , so this will be 0. So from here it will be - f '0 this function + s and this is already we know that this is equal to s.

So, this = s F s - f 0 just now we calculated this value and from here I can write that this is equal to s square f of s - s f 0 - f '0. So, this is my Laplace of f second derivative. So, whenever we are taking the Laplace transformation of any nth degree nth derivative of the function, so, in the process we always using the Laplace transformation of the n – 1th derivative like this one we are using here and this is being multiplied by S. So, from here I can define the Laplace transformation of any nth derivative.

So, from this one if I take the Laplace transformation of any nth derivative, that then from here I can write like this one. So, it is second derivative multiplied by s cube. So, it will be s raised to power n and then function F s - which should be s n - 1 f 0 then next should be s n - 2 f '0 next should be s n - 3 f " 0 and so on. So, in the last I should get the value f n - 1 at 0. So, this is a Laplace transformation of any nth derivative of the function f t .

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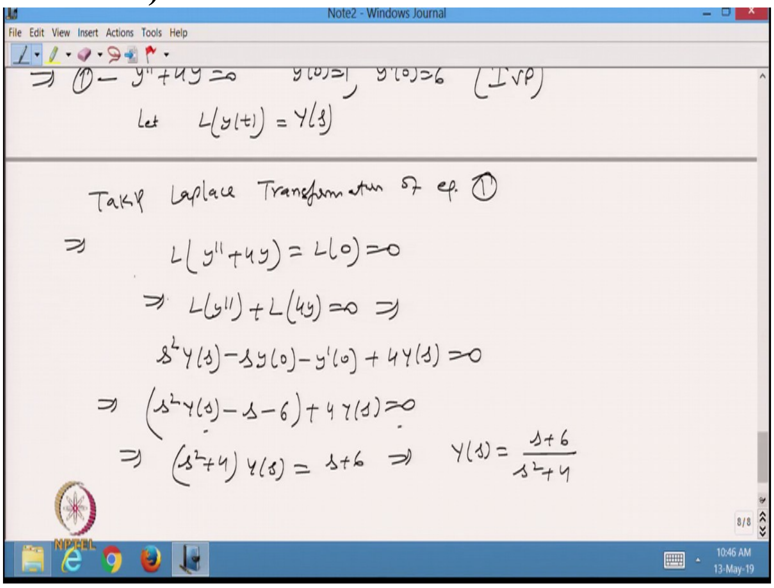

So once I am able to find the Laplace transformation of the function. So, now we are ready to solve the differential equation. So, just now we have defined the differential equation y" + 4y = 0 y 0 = 1 and y '0 = 6. So, this is my second order initial value problem. okay. We know that we this equation we can solve very easily as we can define the characteristics equation and then we can find the solution and based on this initial condition, I can find the value of the coefficient and then we get the unique solution.

So, the question comes that why we are applying the Laplace transformation. So, this equation we are just solving to show that the the application of the Laplace transformation, but later on we will be able to find the solution of the those equation which we were unable to solve using the previous method. So, but if you will see that using the Laplace transformation we are able to solve quite difficult problems, which was not in fact unable to solve using the previous methods.

So, to solve this one, so, I call it equation number 1. So, let the Laplace of function y t , so, it is a function of y t here = y s . So, this is I am considering. Now, taking Laplace transformation of equation 1. So, we are taking the Laplace transformation of the equation number 1. So, from here I can say that Laplace of y '' + 4y should be equal to Laplace of 0 on the right hand side and Laplace of 0 is 0 that we already know.

So, and also know that the Laplace transformation is a linear transformation. So, this can be written as Laplace of y '' + Laplace of 4y = 0. So, from here I can define the Laplace of second derivative. So, should be just now we have found the Laplace transformation of the second derivative of the function. So, from here I can define this that should be = s square y s - sy of 0 - y ' of 0 just now we have found this one and it should be equal to I can write this one as + 4 times y of s = 0.

Now, the initial condition is 0 at 1 and y'0 is 6. So, that will be used here. Sometimes we are unable we have the problem which where the initial conditions are not given, then we have to assume this values to solve this equation. So, from here I can write this one.

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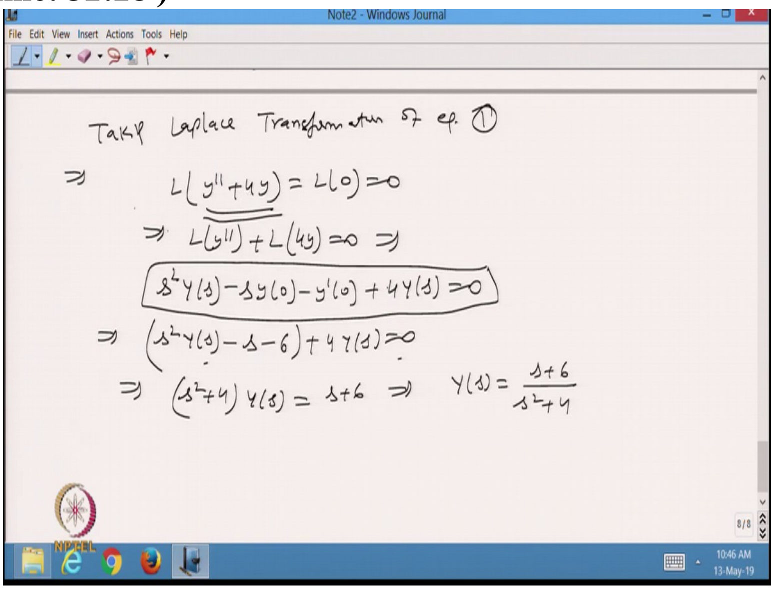

So this become s square y s - I can write this as s - and this is 6 + 4y s = 0. So, using this one I take just collect the terms corresponding to y s , so, this will be s square + 4 y s and this one I can take on the right hand side so it will be s + 6. So, from here I can define my y s = s + 6 s square + 4. Now, we know that this is the Laplace transformation. Now, I want to find the function because the solution of this differential equation is small y t , but we got the solution.

Now, if you from here will take the Laplace transformation this one way differential equation, and once I take the Laplace transformation of the differential equation, I come up with this algebraic equation. So, that is the power of Laplace transformation that changing this differential equation into the algebraic equation. So, here there is no derivative involved, only the algebraic value is there. So, I get this algebraic equation, and from here I am able to find my solution, but that solution in the form of y s .

Now, I want to find the function y t such that this is the Laplace transformation of that function. So, I can say from here that now I want to find the value of y t . This is the which one is the Laplace inverse transformation of the function y s .

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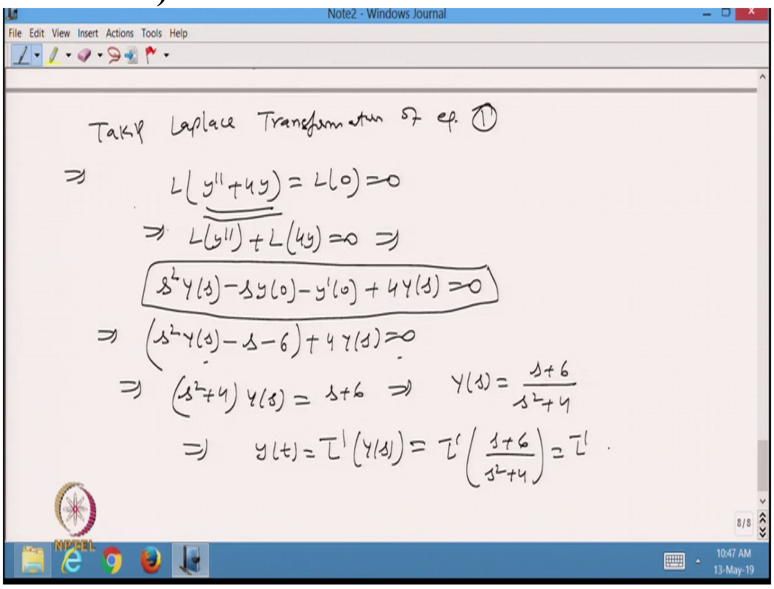

So here there is no specific formula for the inverse. But just now we can say that we are finding the inverse of the function such that the Laplace of that function y t is y s . So from here I can say the. So I am defining L inverse s + 6 s square + 4. So from here I can write and now L inverse if I know that the L is the linear property satisfying the linear property. So similarly, L inverse also will satisfy the linear property.

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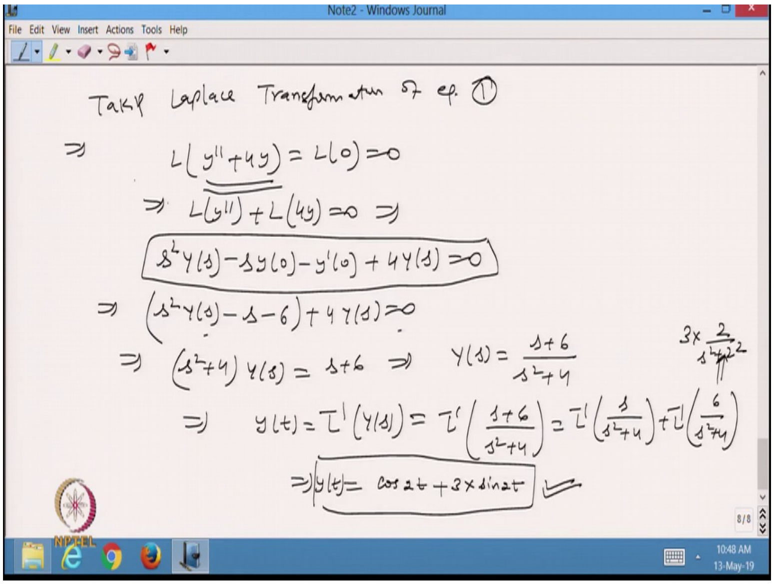

So from here I can say I can separate this one as s square + 4 + L inverse of 6 over s square + 4. And from our, the value of the functions we can say that it seems that this = So, from here I can write my y t is in. So, this is cos 2 t if I take the Laplace transformation of cos 2 t, this is the value and this +, so, it coming 2 square here and 2 should be there. So, far from here I can write that this as 3 multiply by So, this one is 3 into 2 divided by s square 2 square.

So, from here I can say this = sin 2 t. So, from here my function is this is the solution of the initial problem, y t = cos 2t + 3 sin 2t and that is the unique solution we were looking for the initial value problem. So, in this case, we are able to solve this differential equation. But this differential equation was very quite easy to solve, even we can apply my previous method and we should be able to find the solution within the same time as we are able to find using that Laplace transformation.

But we should be able to solve quite difficult problems using the Laplace transformation. So, before that, I will take some advantage of the previously defined some functions.

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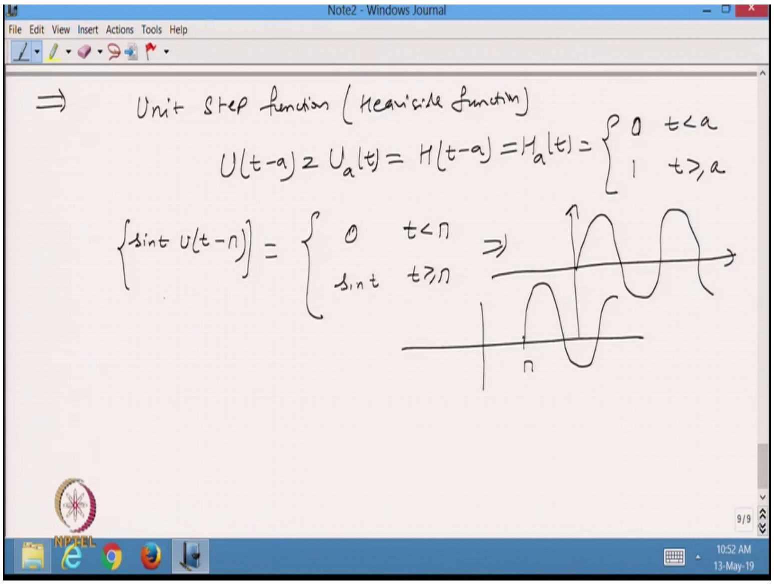

So, if you remember in the previous class, we have defined the unit step function or because whenever we in the previous class is when we developing the theory of green functions, so, at the time we have discussed the unit step function or the heavy side function. So, I know that unit step function u t-a sometime you write as u a t sometime we write as heavy side a t - a or H a t. So, this function is 1 its value is 1, when t is so it is value is 0 when t < a and it is value is 1 when t is > = a.

So, this is a step function we know already know with the step width is one. So, this is we already know. Now, I will take the use of this one. So, suppose I take define a function sin t into u t-pi suppose I define this function. So, what is this function? So, if you see this function will be when t is < pi, its value should be 0 and when t is > = pi, it should be sin t, because this is a function, u t – n So, its value is 1.

So, this one. So, in this case, I can see that, if I have the function sin t and suppose I want to transfer this function on the right hand side, then I will multiply this one with a step function u t-pi and my function become this one. So, this function you can say that, earlier my function was sint like this one and I have transfer this function on the right hand side and now, it is my suppose I take pi, so the function is moved from here to this side. So, in this function as we moved on the right hand side. So, what is the use of this function the unit step function. So, let us take the advantage of what is the advantage of unit step function.

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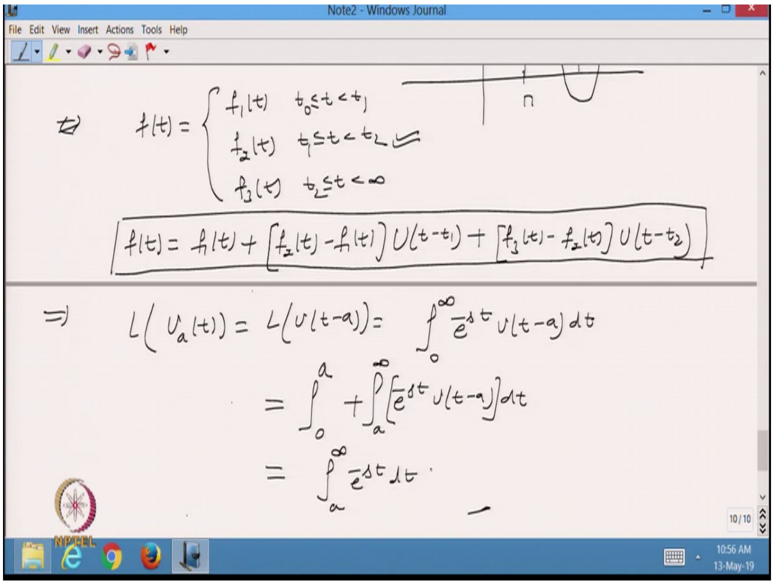

Suppose I have a function my f t and that function is given to me in the terms of piecewise function. So, suppose this function is given to me like this one it is f 1 t t is t naught t 1. So, this is a given to me now then I have my function value the function f 2 t when t > = t1 and < t 2. And suppose this is equal to f 3 t when t > = t 2 and < infinity. I take just 3 functions for this one. Now, what I want to do I want to write this piecewise function in terms of a single function, because it may have a n number of piecewise function.

So, and then suppose I want to define the Laplace transformation for this one. So, I first of all I will what I will do, I will I want to write this function as a single function. So, what I will do in this case, I can write this function as f t is equal to f 1 t I will I like this one, then second one I will do f 2 t – f 1 t. So, I take the difference of this one multiply by so, here the function f 1 f 2 separated to add the point t 1. So, take the unit step function at t1.

So, I multiply this one + then I will go further and I will take f 3 t – f 2 t and then taking a unit step function t – t 2. So now let us see that whether we have done the things the right way or not. So what I will do, let us take my t when t lying between t 0 to t 1. So, when t lying on to t 1, what are the value of this unit step function, because t < t 1 its value would be 0. So, in that case, I will get only value function that f t is equal to f 1 t what will happen if I my t is lying here.

So, if my t is lying here, then I will this value will be t is lying > t 1 and < t 2. But t is greater than t 1, it means its value will be 1 and this value will be 0. So, f 1 t and f 1 t will cancel out in that case and I will get f 2 t that is the value of the function. Now, what are we going to get t > t 2. So, this is also 1 This is also 1. So, this will cancel out with this one this is cancelled by this one and I will get f 3 t. So, if I take the value of t defined in to the different intervals of interval, then I will get this function.

So, this is the advantage of the unit step function that the function which were given to us in the form of piecewise continuous form. We are able to write that function as a single function with the help of unit step function. Ok. The next thing is that because now what I will do I want to take the Laplace transformation this function. So, I should be able to find the Laplace transformation of the unit step function.

So, let us define the Laplace transformation of the unit step function. So, u a t u t-a. So, this is again we go by the definition e raised to power - st u t-a dt. Now, this u t-a is a unit step function. So I know that I can split this integration into 2 parts e – st u t-a dt. So like this one, so in this case when t is between 0 and a, its value will be 0 so this interval, this integral will reduce to this form e raised to power –st dt. Because when t was > a it is value will be 1. So I can I like this one.

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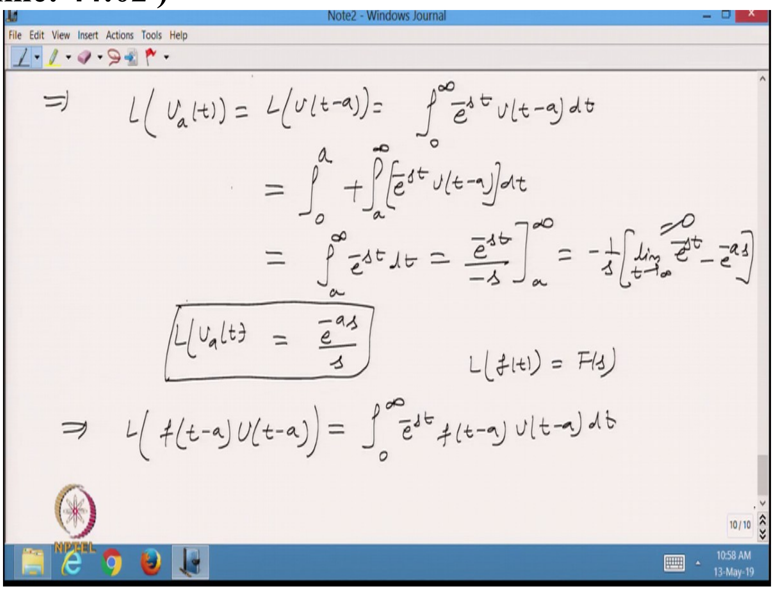

So from here, so if I do the integration, it will be e raised to power - st over -s a to infinity. So it is - 1 over s. And the limit t tends to infinity e raised to power - st – e raised to power - as. So, this will get, now this is again this is equal to 0. So, from here I can write that this is equal to e raised to power –as over s. So, that is the Laplace transformation of u a t. So, we are able to find the Laplace transformation of the unit step function.

So, let us see what will happen if I want to the Laplace transformation of if any function t – a. So, that function is also as being scaled as t - a. So, that function is multiplied by unit step function t – a. So, what I now I have suppose I have a Laplace transformation or the function f t. So, this is given to me and I want to define the Laplace transformation of f t - a u t-a. So, this is again I will go by definition.

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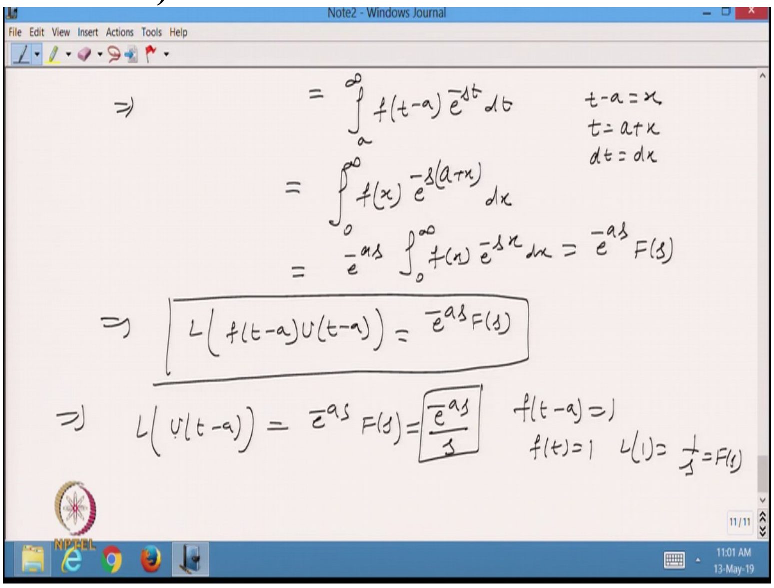

So, it will be f -a u t-a dt and then again splitting the integration and from here I can write it that this will become from a to infinity f t-a e raised to power –st dt. Now, I will change the variable. Let put t – a = x. So t will be a + x. And the integration in this case, when t is a x will be 0, when t is infinity x will be infinity. So, from here this integral change to 0 to infinity and f t - s f x - s and t will be a + x dx. Ok.

Because from here you can see that my dt will be the dx. So, from here we are able to define this one, then I can take this e raised to power -s common outside because it is a integrations respect to s. So, I can take this and this one. So, it is my f x e raised to power -s x dx and this is the Laplace transformation function of x. So, from here I can write that this is equal to e – as f s. So, from here, I can say that the Laplace transmission of the function t - a multiplied by t - a = F s.

It means whenever I getting the function in the term of s and that function is multiply by some exponential form, e raised to power -as. It means if I need to find the inverse that involves will contain the unit step function okay. So, from here, I just take one example, that in the previous case, I just define what is the Laplace transformation of u t-a. It means, in this case, I am assuming that my f t – a is 1.

It means I am assuming that my f t Laplace transformation of 1 and Laplace transformation of 1 I know that it is equal to s. So, that is my F s. So, if I take the Laplace transformation of this one, it should be used for - as F s and F s is 1 -. So, it should be equal to e raised to power –as F s. And this is what we have just defined, it means, that in this case I was taking the Laplace transformation are the function 1 multiply by the unit step function.

So, in this today class, we were able to go further we have solved 1 differential equation using the Laplace transformation and then we have defined the unit step function that how if function which is given to us in the form of piecewise form, we are able to write that function in the form of a one function using the unit step function. And then we have defined the Laplace transformation of a function multiply by the unit step function.

So, in the next class will go further and solve some harder problem using that Laplace transformation. So, thanks very much for you watching this lecture. Thank you.