Introduction to Methods of Applied Mathematics
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Lecture – 14
Laplace Transformation

Welcome viewers, welcome back to this course. So, today we are going to start with a new topic because in the previous classes we have all developed the technique to solve the differential equations. Basically, till now, we have solved the second order linear differential equation, homogeneous and non-homogeneous with initial condition or the boundary condition. So, today we are going to start with a new topic and that topic we also use to solve the differential equation. So, today we are going to start with lecture 14.

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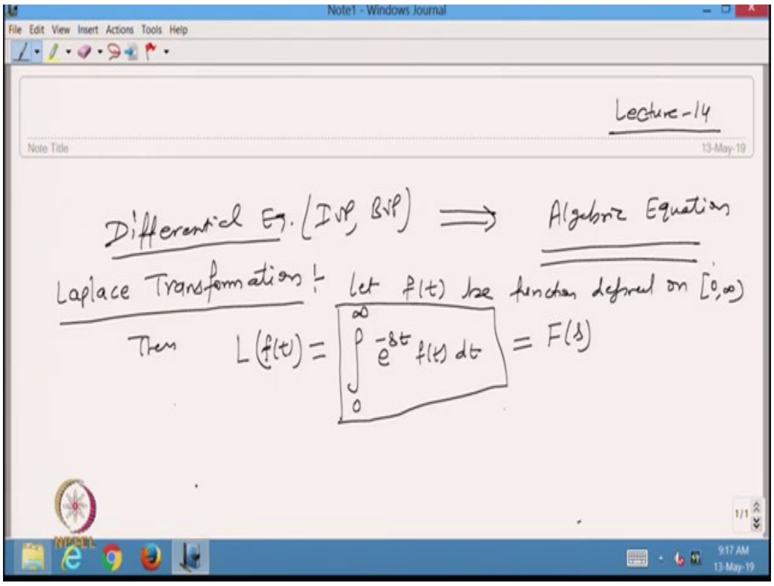

So now, as I told you, we know that I have a differential equation, it may be initial value problem or the boundary value problem, and on the other side, I have an algebraic equation, this is the algebraic equation and I ask that which one is easier to solve. If you have a differential equation on the one side on the one hand and algebraic equation on the other hand and you want to solve, which one is easier to solve? So, obviously, the answer will be that algebraic equation will be easier to solve.

So, today in the topic, we are going to transfer this differential equation, generally the initial value problem, into the algebraic equation, and then we will solve that one. I have a differential equation and I am converting this differential equation into the algebraic equation using some transformation. So, obviously, that transformation should contain the sign of integral because the anti differentiation is integration.

From here, I will start with a new topic and that is the Laplace transformation. So, for the Laplace transformation, I define that let I take a function f t., so now we are defining the function in terms of t, that let f t be a function well defined on the interval 0 to infinity, then I take the Laplace L of f t by taking the integration from 0 to infinity. Here, I am taking a new variable e raised to power – s t and then multiply by f t d t.

So, this is the integration we are going to take. Now, if this integration finite, the integration this one is finite value, then we say that this integration exists, and in that sense, I will say that the function f t has the Laplace transformation. And if you see from here that we are doing the integration with respect to t and then putting the limit.

So, after doing all this calculation you will see that I will get the function on the right hand side in the form of s. So, I call this one as a function of s, and then I am taking the Laplace of small f t, then we generally represent this one by S. So, F is a function of s and from here I can say that this f is a Laplace transformation of this f t.

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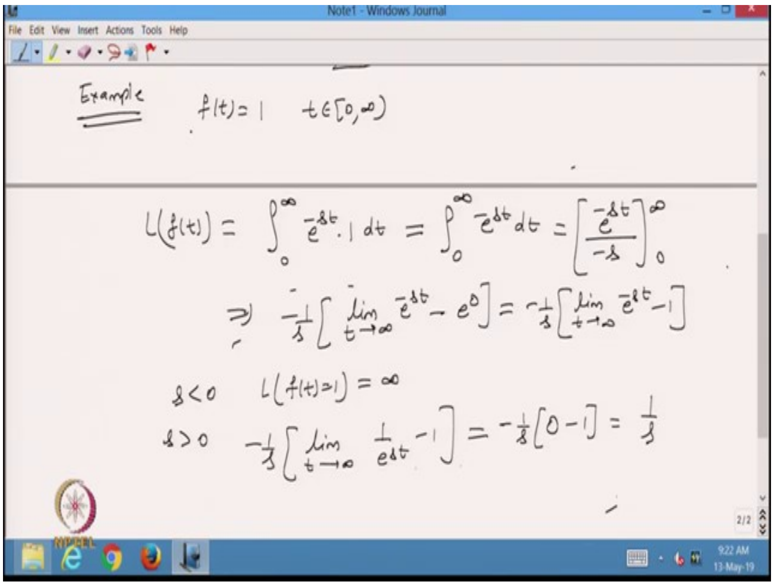

So, let us take few examples to understand more how this integration happens. So, suppose I start with a very simple example. So, take some examples. Let us take a function, my f t = some constant function 1, this is the simplest function I can define. So, f t = 1 and t belongs to 0 to infinity. Now, I want to take the Laplace of this one. So, the Laplace of my f t in this case will be integration from 0 to infinity e – s t into 1 d t.

So, from here I can solve this integration and the integration of exponential will be e – s t divided by – s and now putting the limit from 0 to infinity. So, this one I can define as, so 1 over – s I can take common, then I put the limit t tends to infinity e raised to power – s t –, and putting t = 0 so it will be easier.

So, from here, I can define that this is – 1 over s and now this limit I want to find out. So, in this case if you see exponential is a – SD. So, T is my always positive value starting from 0 to infinity. So, in this case if you see exponential e is – s t. So, t is always positive value, starting from 0 till infinity. So, in this case it depends on what is my s.

So, if you see that if I take, so this one I will just define, now limit t tends to infinity e – s t, it will be – 1. So, now I take for s, if I take s less than 0, so in that case I am choosing my s to be negative. So, if s is negative, then you can see from here, if s is negative, then this will be a positive value. And in that case, if I put the limit t tends to infinity, then in that case I will say that my Laplace of f t = 1 will be infinity or even I am taking s = 0 also. So, this is the case.

Now, what about if I take s positive. So, if I take s positive, for the limit if s is positive, then this one I can write as – 1 over a limit t tends to infinity. I can put this one as 1 over e s t – 1, and then this is positive value, the s is positive and t is positive and this is going to be infinity, and from here if you see, then this will be – 1 over s and this value will be 0 – 1. So, from here this value will be 1 over s.

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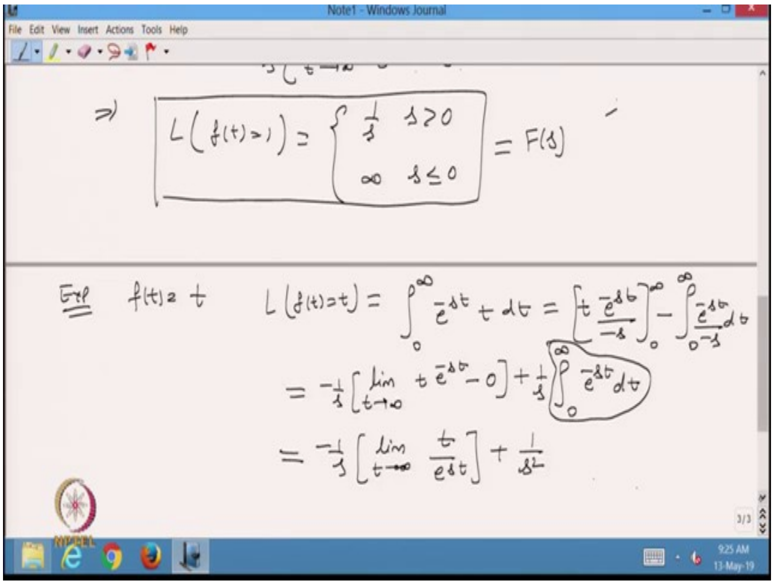

So, from here I can write that the Laplace of f t = 1. This will be 1 over s if s is greater than 0 and it is infinity when s is less than or equal to 0. So, that is the Laplace transformation for function f t = 1. So, the same example I can take. So, let us take another example. I take another example. Now I choose f t is equal to some polynomial t.

So, what will happen in this case? I want to define the Laplace of f t = t. So, this one I can write as F s. So, it is my 0 to infinity e raised to power – s t into t d t. So, here I have to take the by parts rule to find out the integration because in the Laplace transformation always you will get the function of this type and then you have to apply the integration by parts.

So, in this case, what I will do, I will take t as the first function integration of, and then putting the limit – 0 to infinity. And here I will take the derivative t that is 1, so it will be e raised to power – s t d t by – s, so integration of the second function. Now, from here I will get – 1 by s I will take common, and here I will put the limit t tends to infinity t e raised to power – s t – and I am putting 0, so it will be 0, and this – s I can take outside, so this will be + 1 by s 0 to infinity e raised to power – st d t.

And you know that this is the Laplace transformation of the function 1. So, this one we have just done, finished this one. So, this is the Laplace transformation of this one and that value is 1 by s. So, from here, I just define 1 by s and then solving this limit. So, I will take the same, two cases.

So, let us take the case when s is positive, s is greater than 0. So, in that case, this will be limit t tends to infinity, t by e raised to power s t +, so here 1 by s and this will become 1 by s, so I can take it as s square, okay, because I am choosing that s is positive, s is greater than 0. So, in that case, what will happen?

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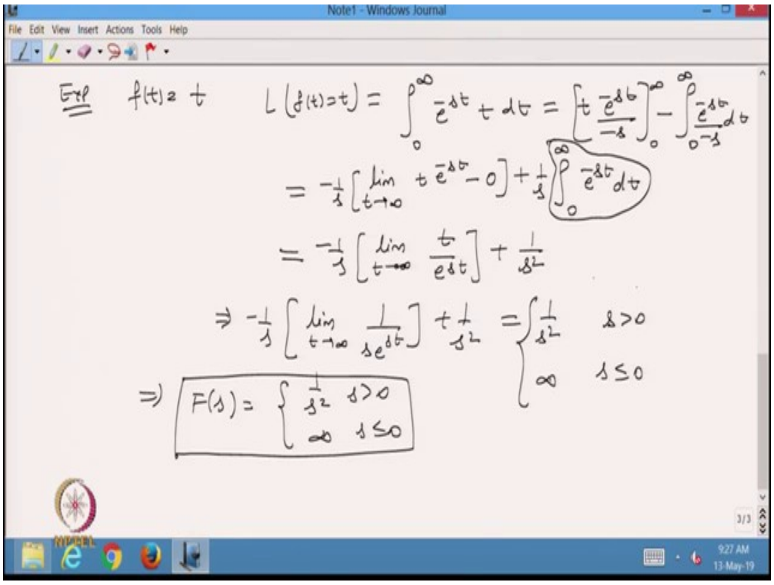

I will get infinity by infinity form and then by the L'Hospital rule I will get the limit t tends to infinity 1 over s e s t, the derivative, + 1 by s square, and then I am putting the limit t tends to infinity becomes infinity and then 1 by infinity will be 0. So, from here I will get my s square when s is positive. Similarly, if I take s negative, then its value will be infinity. So, this will be equal to 0.

So, from here, I can say that my F s will be of this type, infinity when s is less than 0. So, this is the Laplace transformation of the function f t. And if I take the Laplace transformation of t square and doing the same thing again what we will get, I am getting, from the by parts integration, from this place I will get t into e raised to the power – s t and that is the Laplace of the t just we have done. So, the same way we can do.

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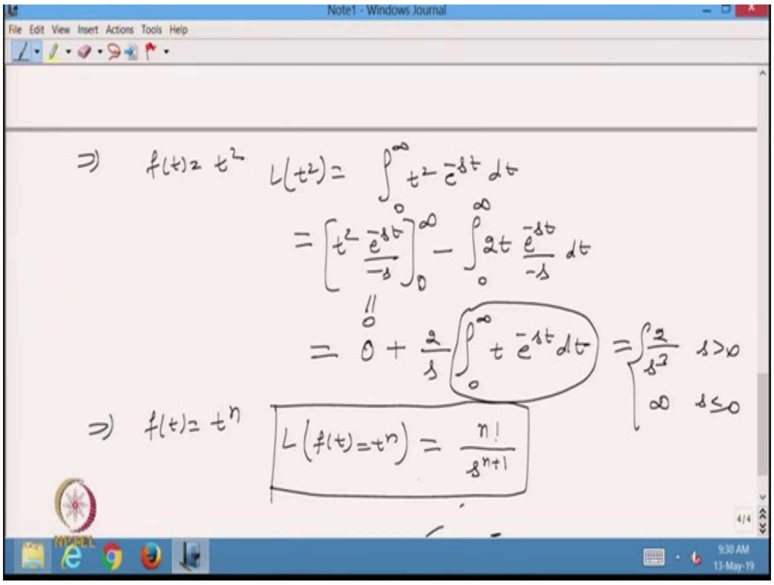

Now, suppose, I take the Laplace of f t = t square. So, the same thing will work here. So, now my Laplace of t square will be 0 to infinity t square e raised to power – s t d t, just now we do that. So, it is a t square e raised to power – s t by – s, 0 to infinity. So, this will be becoming 0, –, then I will take the integration and it will be 2 t, and then e raised to power – s t over – s d t.

So, this part will be 0 again by putting the limit for s positive, and in that case I can say that this is going to be 0 and this will be, so we are getting here 0 this value, and – and – + sign will be here and 2 will be taken common. So, it will be 2 by s and then inside I will get infinity t e raised to power – s t d t. This is the Laplace of the function t that we have just found as 1 over s square.

So, from here I will say that the value of this will be s cube when s is positive and infinite when s is negative. So, from here, I can define my Laplace for the general function. So, now, if I have f t = t raised to power n, then I can define from here that the Laplace of f t = t raised to power n. So, here, I am getting 2.

If I take the t cube, so I will get taking the derivative, so that will be 3 t square and then this value. So, again the same thing, so 3 will come. So, here it is 2 by s cube. There it will come 3 into 2 to 6. So, from here I can say that it will be s and + 1, because when I am taking t square the s cube is coming. So, n + 1, t square 2. So, from here I can say that n factorial.

So, 2 is 2 factorial and then it will be multiplied by 3 so it will be 6, so 3 factorial and so on. So, from here I can define my Laplace for any polynomial of degree n and its Laplace transformation will be n factorial by s, n + 1. So, this is the simplest function we have taken, the Laplace transformation, then I can define the Laplace transformation for any other function.

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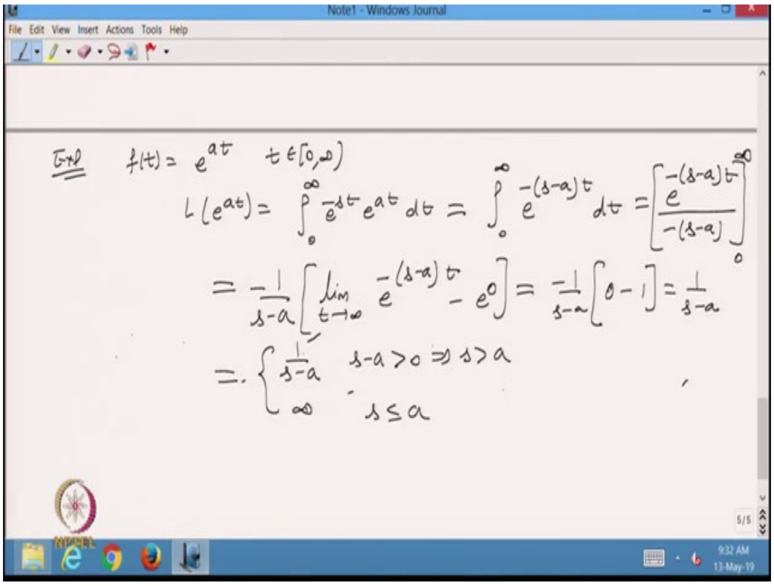

So, let us take another example, and I take the function f t is equal to some exponential function, e raised to power a t. So, this function we are defining and this function is well defined for t belongs to 0 to infinity. And then in that case, if I want to define the Laplace of e a t, so it will be 0 to infinity e – s t, e a t, d t where a is a real number.

So, from here, I can define from this, e raised to power – s – a t d t and this will be – s – a t over – s – a and then putting the limit 0 to infinity. So, from here I can say that, I just take the common s – a and then putting the limit. So, it is the limit t tends to infinity e raised to power – s – a t and then putting the value 0 here.

So, – it will be e 0. Now, the same case will happen, so now I will choose that my s – a is positive value. So, in that case, I can say that this will be 1 over s – a, and if I am choosing my s – a is a positive value, then this will become 0. So, this will be 0 – 1, and from here I can write it as s – a.

So, from here, if I say that this is equal to 1 over s – a when s – a is positive, or I can say that s is greater than a and this will be infinity when s is less than or equal to a. So, from here, I am able to find the Laplace transformation of any exponential function. So, we are able to find the Laplace transformation of any polynomial or the exponential.

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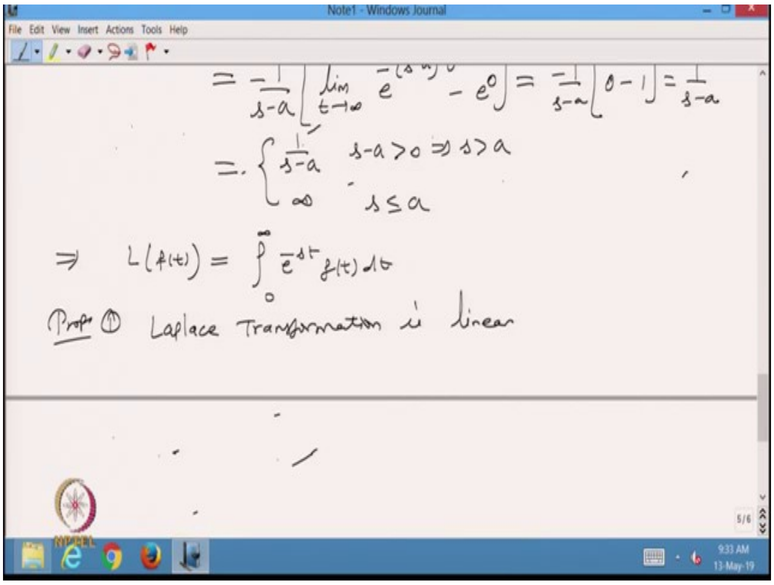

Now, I want to define one property of the Laplace transformation. So, if you see that the Laplace transformation of a function f t where f t is a well defined function in the interval from 0 to infinity, so this is 0 to infinity e raised to power – s t f t d t. So, from here I give property 1 of the Laplace transformation that is Laplace transformation is linear. So, what is the meaning of that?

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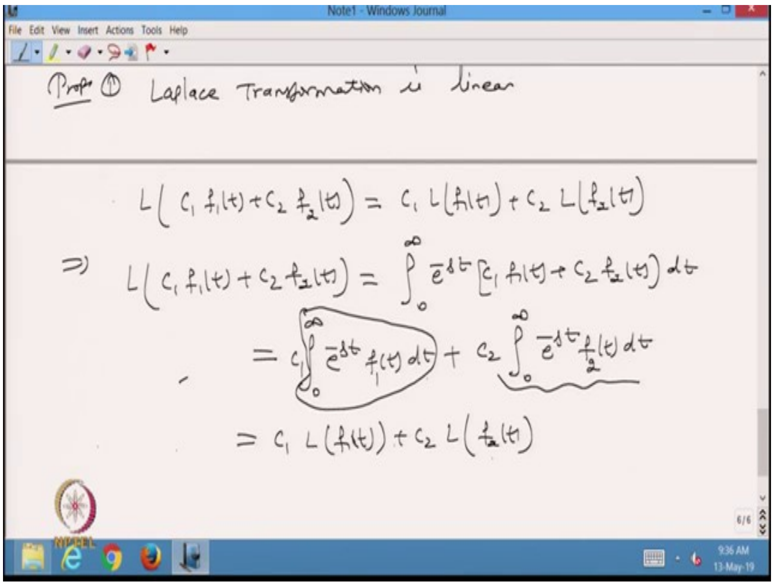

The meaning of that is that a transformation is said to be linear if I want to take the Laplace of C 1 f 1 t + C 2 f 2 t. So, this is a linear combination of the two functions. So, if this comes equal to C 1 and the Laplace transformation f t + C 2 the Laplace transformation of f 2 t, then we say that the Laplace transformation is a linear function.

And now, you know that in the Laplace transformation we are dealing with the integration and the integration itself is a linear transformation. So, from here, if I just want to find out what is the Laplace transformation of this linear combination of two functions where C 1 and C 2 are the constants. So, this will be becoming 0 to infinity and this is e raised to power – s t. So, C 1 f 1 t, C 2 f 2 t d t.

So, from here I can write that, just separating the things because this integration I just separate this one as C 1 I can take outside and then I can apply e raised to power – f t d t + C 2 0 to infinity. So, this is my f 1 and this will f 2 t d t, and then from here you can see that this is the Laplace transformation of f 1 t and this is the Laplace transformation of f 2 t.

And then I can write this one as C 1 the Laplace transformation of f 1 t + C 2 the Laplace transformation of f 2 t, and then from here I can say that Laplace transformation is a linear transformation. Now, one more thing I want to just define. So, we have taken the Laplace transformation of the function and the exponential. So, now I want to take the Laplace transformation of any trigonometric function.

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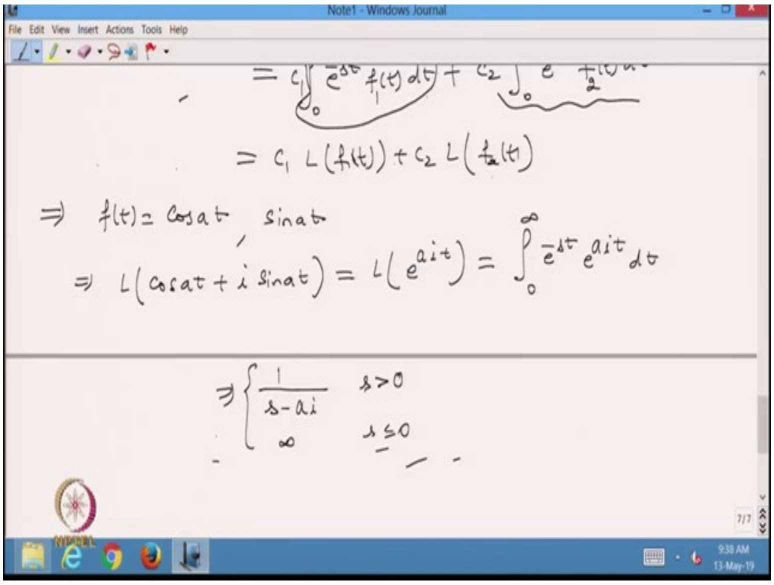

So, I just want to define what is the Laplace of a function f t = cos a t or what is the Laplace transformation of sin a t. So, this one I want to find. So, now, this one we can define like this one. So, I just want to find the Laplace of cos a t + iota, iota is the complex number, sin a t. So, this one I want to find. So, just know I told that the Laplace transformation is a linear transformation.

From here I can define that cos a t + i sin a t, I know that it can be written as e a i t, this is the exponential function, and this one I can just write 0 to infinity e raised to power – s t, e a i t d t. And I know that the Laplace transformation of an exponential function of an exponential function is equal to, so this I already know, that this is equal to 1 over s – a i when s is greater than a i, so it is 0 and this is infinity when s is less than or equal to 0, because a i is just the imaginary number.

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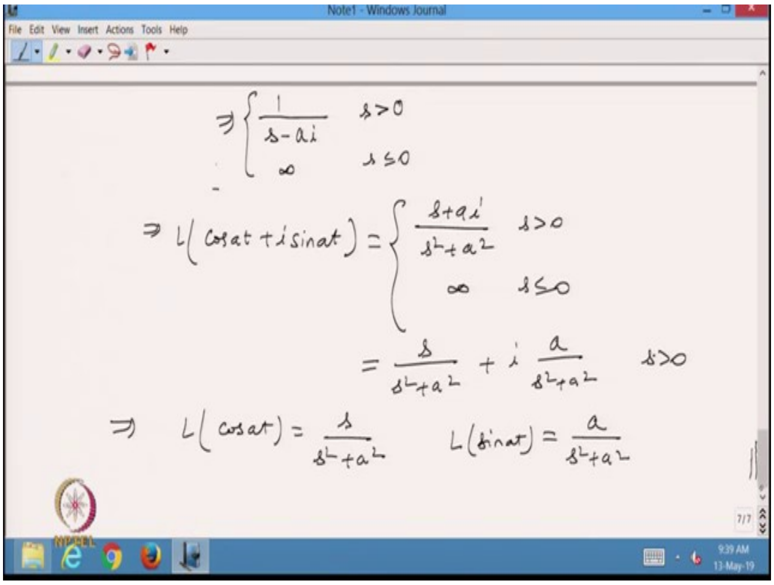

So, from here I can write down the Laplace of cos a t + i sin a t is equal to, so this one I will separate this complex number into the real part and the imaginary part. So, I just multiply by s + a i and divide by s + a i. So, from here I can write that s + a i and dividing by s + a i. So, that will give me s square + a square when s is positive.

So, from here, I can write this function in the simplest form and I can separate this one as a real part, so real part is s over s square + a square. And then I can write iota a over s square + a square when s is positive. Now, comparing the real part and the imaginary part, so from here I can say that the Laplace of cos a t is, taking the real part equal, so this will be s over s square + a square and the Laplace transformation of sin a t will be a over s square + a square. So, this one we are able to find.

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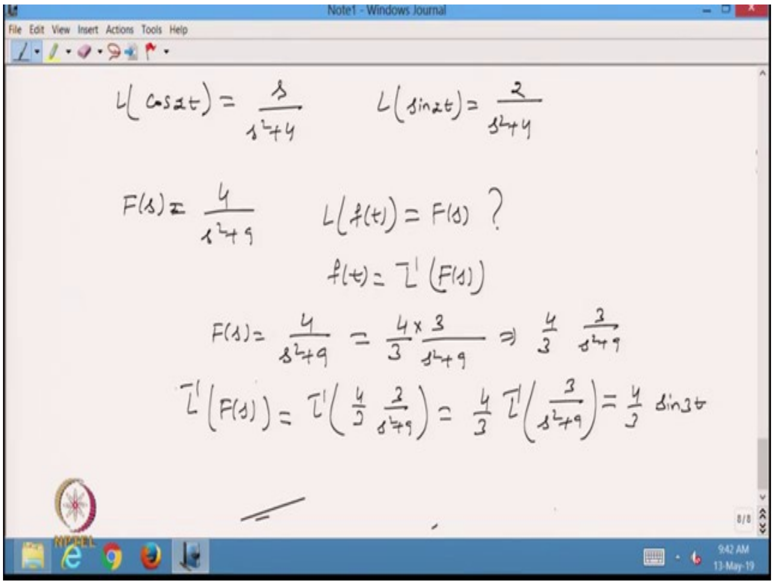

So, if somebody ask that what is the Laplace transformation of cos 2 t, so this will be s over s square + 4. What is the Laplace transformation of sin 2 t, so this will be 2 over s square + 4. Now, if somebody ask that, my F s is 4 by s square + 9, then which function f t we can find such that the Laplace of that function is equal to this function. So, basically, I want to find what is this function f t such that this is equal to my f s.

So, in this case, we can just say from here that I want to define the function f t such that the Laplace inverse of F s is this one. So, from here, I can say that I can define a Laplace inverse such that if I take the inverse of F s then I should get the function f t. So, from here, we define another way of finding the inverse and that we represent by F L inverse. So, from here, now my F s is 4 by s square + 9 and I know that if I have my Laplace of sin 3 t, then it should be 3 by s square + 9.

So, from here, I can say that 4 cross 3 divided by 3, and then I will write s square + 9. So, from here I can write that this is equal to 4 by 3, 3 by s square + 9, and I want to take the inverse of F s. So, this will be L inverse 4 by 3, 9, and this one I can take common. So, I want L inverse 3 over s square + 9. So, if I define this one, so this will be equal to 4 by 3 and then this is Laplace of sin 3 t. So, from here, I get the solution.

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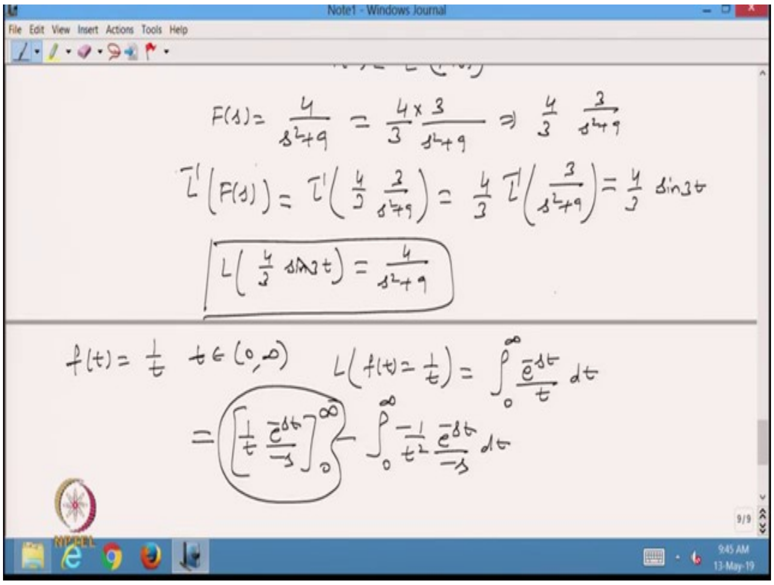

And that solution gives me, so from here I can write that the Laplace transformation of 4 by 3 sin 3 t = 4 by s square + 9. So, now we are able to define the Laplace transformation of the trigonometric function also. So, let us define another function, because till now it seems that we are able to take the Laplace transformation of any function, it may be a polynomial, it may be a trigonometric function, it may be an exponential function.

So, let us take the Laplace transformation of the function. So let us take another function f t, it is 1 by t. So, let us take this function 1 by t. You know that my t belongs to 0 to infinity. So, at t = 0, this function's value is infinite, so this function is undefined at the value t = 0. But we are dealing with the integration, so let us take that and try to find the Laplace of f t = 1 by t.

So, if you take the Laplace transformation of this one I can define this integration from 0 to infinity, t d t, and in this case, you will see from here that this function if I want to take the integration and taking this 1 by t as my first function and e raised to power – s t as the second function that we are doing, then if I take the derivative, so let us do this one, I go by the same way. So, let us take 1 by t as the first function and e raised to power – s t over – s as the second function 0 to infinity.

So, what is the differentiation of 1 by t, so it will be – 1 by t square, and then it will be – s t over – s d t. So, this will keep increasing like this one. Next will be – 1, 2 over t cube and so on, and here also I am dividing by t. So, if I take that as we are taking s is positive, so if I just take the integration of this function, for infinity it is okay but when I take t = 0 then it becomes infinity.

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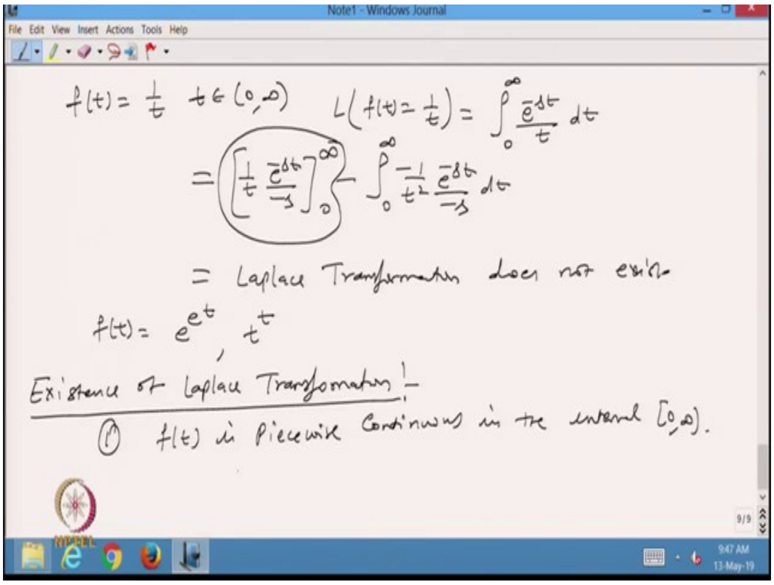

So, in that case, I can say that for this type of function this Laplace transformation does not exist or I can take another function f t = e raised to power e t or I just take another function e raised to power t. So, like this function, if I just take the Laplace transformation, you can say that the Laplace transformation does not exist. So, for this one we just want to find out which functions we are able to take the Laplace transformation and then we get into the existence of the Laplace transformation.

So, before going to the existence of the Laplace transformation we want to define two terms. The first terms we want to define, and that we already know in fact, is that what do you mean by a function f t is piecewise continuous in the interval 0 to infinity. So, what do you mean by the piecewise continuous in the interval 0 to infinity? It means that my function, so let us define one function, so this function means that this function is continuous but in the pieces in the whole interval from 0 to infinity.

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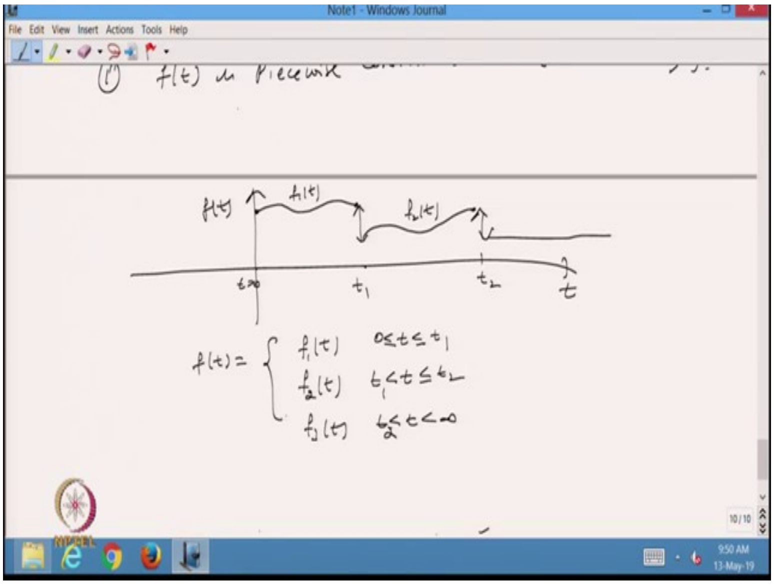

Like suppose I have some function, this is my t x and this is my f t. So, I define a function, so from 0 its value is this one, this value here. So, this is t 1 I say, and after that the value of the function is this one and then this I call it as t 2, and after that the value of the function is like this one for all value of t.

So, in this case, I know that my value of the function at t = 0 is this value, because we are discussing here that my function is a continuous function, but in the pieces. So, if the function is continuous it means its value is well defined at any value of the T. So, at t = 0, so this is my t = 0, the value the function is this one.

So, let at t = 1 the value of the function is this one, and starting from this, then at t = 2 the value of the function is, let us take this value as a function of the value, and then this one. So, from here, I can say that my function f t is this function, so I call it as f 1 t when t is defining 0 to t 1. So, at t 1 also I am taking this value. So, let us take equal to sign here.

Then, I will take my function f 2 t. So, f 2 t is, so t is greater than t 1 and less than or equal to t 2 because t 2 value is this one, and then I have taken f 3 t. So, t is greater than t 2 up to infinity. So, in this case, my function f t is in the pieces but this is continuous function and in fact, if you see from here, it has discontinuity at the point this one which we generally take as a jump discontinuity.

So, we can say that my function f t in this case is a piece wise function with finite number of jump discontinuity. Now, you know that in the Laplace transformation we are dealing with the integration and for the integration even the function has a jump discontinuity, we are able to take the integration.

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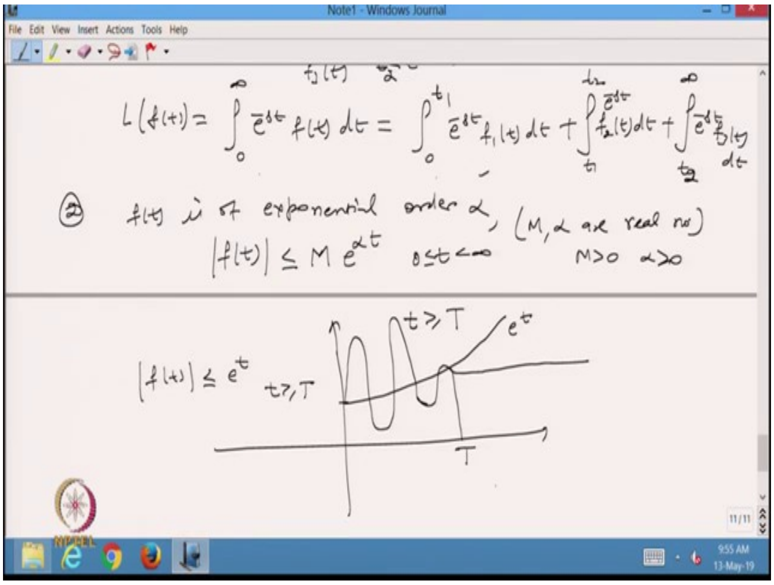

So, suppose I want to take a Laplace transformation of this function f t, so it will be 0 to infinity e raised to power – s t f t d t. So, because of the properties of the integration, I can extend, split this integration from 0 to t 1. So, this will be e raised to power – s t f 1 t d t + t 1 to t 2, it will f 2 t d t + t 2 to infinity, it will be f 3 t d t.

So, in fact, I am able to define the Laplace transformation for any piecewise continuous function which has the jump discontinuity that is finite in numbers. So, this is the one of the properties that is there. And, the second one I will define, the term I will define and that is called the function f t is of exponential order and that exponential order alpha. So, what is the meaning of this?

So, if I am able to write my function f t, so this is my function f t, suppose the value of this is always less than some constant M e raised to power alpha t for all t infinity, or sometimes we are unable to take this, so my alpha you can say that M and alpha are real numbers and positive.

So, in this case, I can say that this function has a bound, that bound is by M e raised to power alpha t and t is we are defining this one. So, sometimes we are also able to write this for t greater than or equal to T because it may happen like, for example, so let my f t is there, now exponential, so let us take alpha = 1, so e raised to power t. So, at t = 0 its value should be 1.

So, suppose this in my exponential function e raised to power t and I have my f t of this type, so add this value, its value is like this one, then this one, then this one, then after some time its value is always less than e t. So, in that case, this is the point. So, I just derive this function again. So, my function is like this one, and then after this value, so from here I just choose the value of this one and this one I call it T.

So, from here you can see that my function is always below the function e raised to the power t. So, from here for this function I can say that my function f t is less than or equal to, M I just take 1 and alpha I will take 1, so it is always less than t for t is greater than or equal to T. So, sometimes we are able to find this T also. So, what is the meaning of this?

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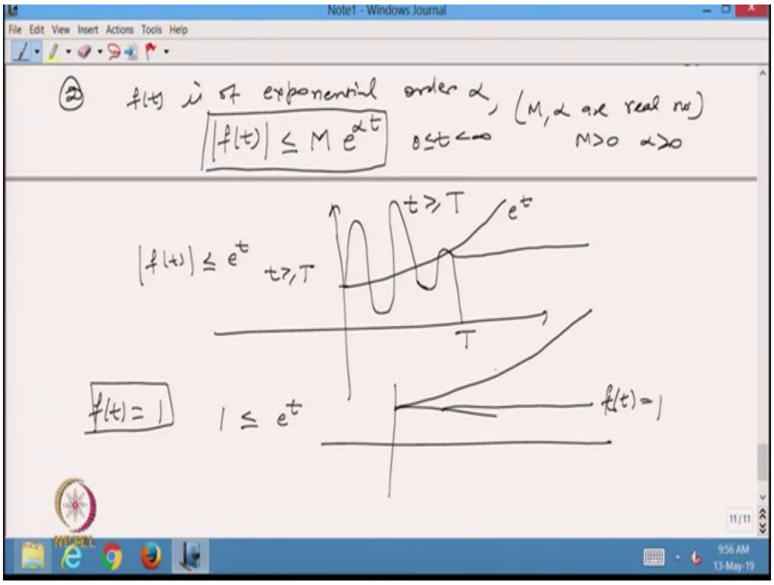

It means that whatever the function I have and if I take the positive value, the modulus value of that function, so that is always less than or equal to some number M e raised to power alpha t, means e raised to power alpha t is an exponential function, so my function should lie below this exponential function.

Suppose I take my f t = 1, so in this case I can say my 1 is always less than, just I take 1, M is 1 and alpha is 1. So, I can take e t. And I also know that my function is 1 here, this is my f t and the exponential function is like this one. So, for all values of t my function f t is always below the exponential function.

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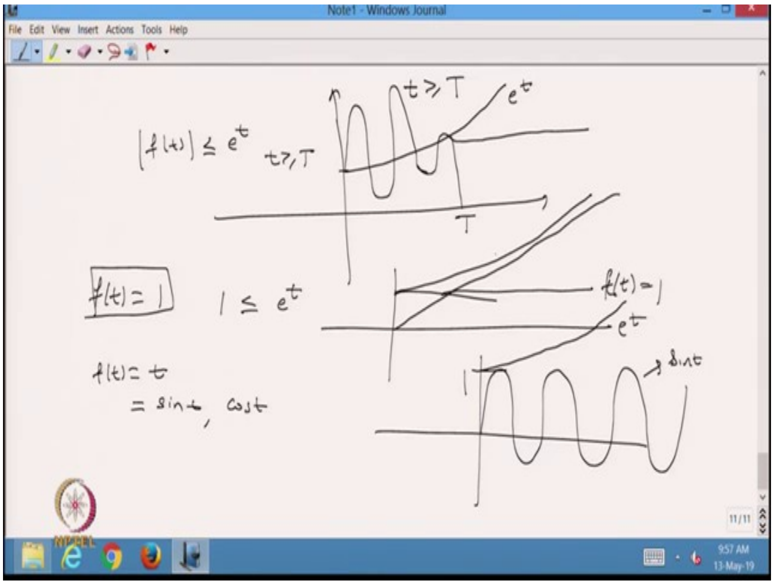

So, this function f t = 1 is of exponential order 1 or I can take my f t = t. So, in that case, I have the function that is this one and this function is always below the exponential e t or I can choose my function sin t cos t and its value is always 1, maximum value is 1, so it is always less than this one.

So, from here, I can say that my exponential function is like this one and the maximum value is 1 and my exponential function is going from here. So, this is my e raised to power t and this is my function sin t. Similarly, I can take cos t. So, this function is always less than this one. So, I can say that my function sin t is of exponential order 1.

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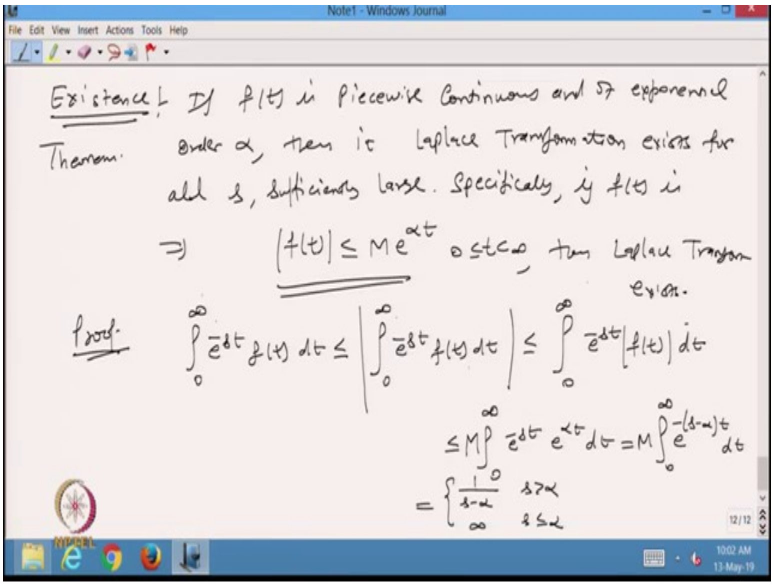

So, now I will define that word existence. So, in this case, I will say that if my function f t is piece wise continuous and of exponential order alpha, then its Laplace transformation exists for all s, sufficiently large as we have seen that in the Laplace transformation we have taken the different different value of s. When s was positive the Laplace transformation was valid.

When s was less than or equal to 0 it was infinity. So, that is the value of s we are talking about. And specifically if f t is, so from here I can say that if f t is piecewise continuous and exponential order, then f t will be M e raised to power alpha t for t, then Laplace transformation exists. So, if my function f t is piecewise continuous and of the exponential order alpha, then its Laplace transformation exists for all s sufficiently large.

So, this one. So, I just want to take the proof that function or that existence theorem, you can just say that this is existence theorem. So, for the proof, now if you see that here exponential order function means that the modulus of the function should be bounded by this factor M e raised to power alpha t. So, now, I will take the help of, because here we are dealing with the Laplace transformation and that is the integration, and integration you can say that this is the extension of the summation.

And as we were doing in the case of series solution and we know that the absolute convergence of the series is equal to the convergence of the series. So, from here, I know one thing is that my integration from 0 to infinity e raised to power – s t f t d t, this is always less than or equal to f t d t modulus value because if my exponential function is always positive and if my function f t is also positive, so in that case, I will get this equal sign.

If it is a negative then it is less than. And from here, I know that this can be written as 0 to infinity, it is always positive. So, I can write from here. I am just taking the modulus value inside the integral and this value I can write down as, so this is less than this one and this is given to me that this is equal to 0 to infinity e raised to power – s t and this value is given to me that function is exponential order.

So, my modulus is value of f t is less than or equal to M e raised to power alpha t. So, I can take M outside here, and this will be e alpha t d t. So, from here, I will get the value M e raised to power e – s – alpha t d t. So, from here, I know that this is equal to 1 over s – alpha when s is greater than alpha and infinite when s is less than or equal to alpha.

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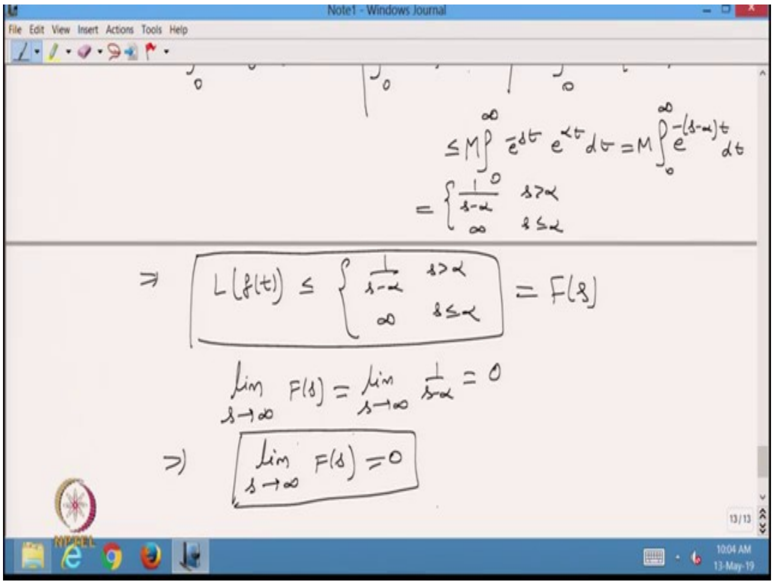

So, from here, I can say that the Laplace transformation of the function f t which is either a piecewise continuous function or the exponential order, exists and this is equal to 1 over s – alpha when s is greater than alpha and infinite when s is less than or equal to alpha. So, that is the existence of the Laplace transformation of the function f t.

And from this one I can say that, one thing also I can observe from here, that if I choose, so this is equal to F s and s is greater than alpha, alpha is any positive number. So, in that case my s is always greater than the positive number. So, what will happen if I take limit s tends to infinity F s in this case, because this we have done for any general function.

So, what will happen if we take the limit s tends to infinity F s. So, from here you can see that alpha is some real number, so from here, I just take limit s tends to infinity, it means we are dealing with this value of the function, so this is s – alpha and from here I can say this one. So, from here, I can write down another property of the Laplace transformation is that if I take s tends to infinity limit putting on the F s, that value is always equal to g.

So, this is another observation we should keep in mind about the Laplace transformation. So, in this class, we have started with the Laplace transformation and then we have discussed the existence of the Laplace transformation. In the next class we are going further from this one. So, thanks for watching.