Introduction to Methods of Applied Mathematics
Prof. Vivek Aggarwal & Prof. Mani Mehra
Department of Mathematics
Indian Institute of Technology – Delhi/DTU

Lecture – 13
Sturm-Liouville Problems

So, welcome back to the course. So, in today's lecture we will go further and we will try to find out how the self-adjoint operators are used to find out the Green functions, and then we will go for how the Sturm-Liouville problems can be solved, and then their properties.

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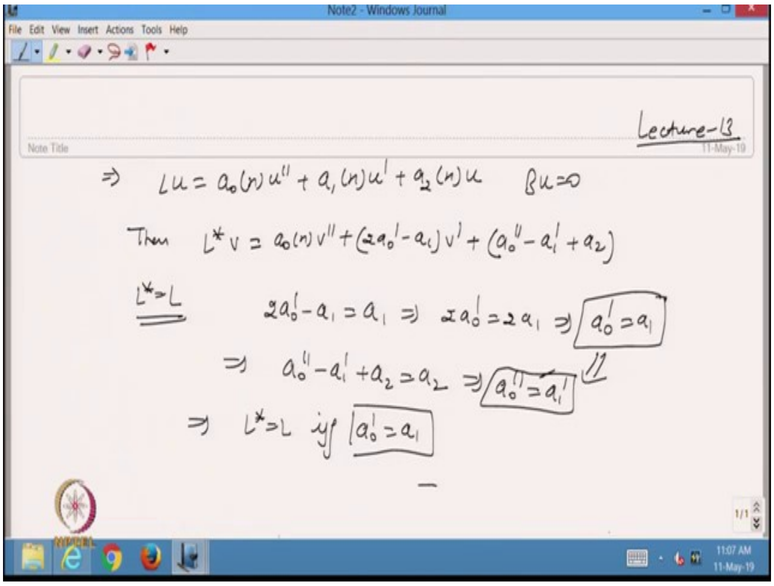

So, this is lecture 13. So, in the previous classes we have discussed that if my operator Lu= a0(x), so this is a0 x a 1 x u dash + a 2 x u, so this is my linear differential operator with some boundary condition B u = 0, then we know that it is adjoint operator. The L star v can be written as a 0 x v double dash + 2 times a 0 dash – a 1 v dash + a 0 double dash – a 1 dash + a 2.

So, that is my corresponding adjoint operator. Now, what about if I want to find out that L star = L. So, this is the definition of the self-adjoint operator. So, from here I can say that this is possible when my 2 a 0 dash – a 1 should be = a 1. So, from here, I will get 2 times a 0 dash should be = 2 times a 1. From here I can say that my a 0 dash should be = a 1.

So, in that case, from this also I can find the a dash – a 1 dash + a 2 should be = a 2, and from here I can say that my a 0 double dash would be = a 1 dash. This is again taking the derivative of this a 0 dash. So, from here I can say that the L star = L if and only if my a 0 dash = a 1. So, if this is true, then I can say that my corresponding differential operator is a self-adjoint operator.

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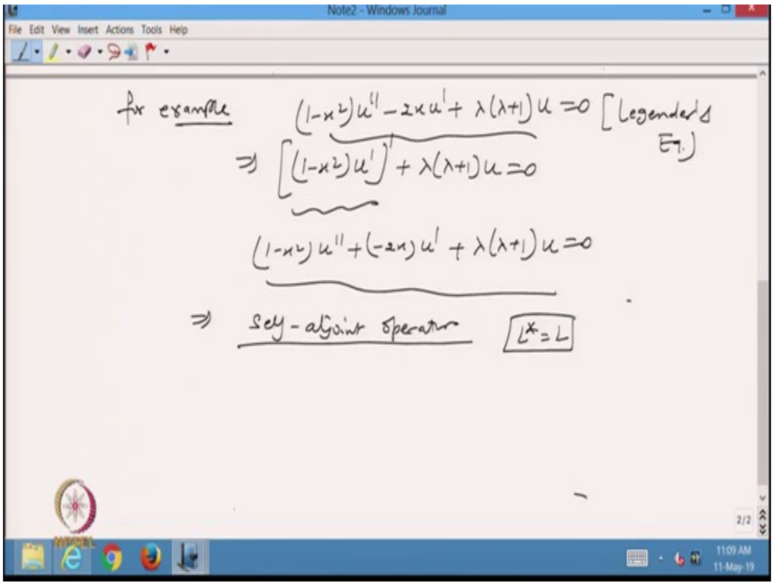

For example, I take the differential equation 1 – x square u double dash – 2 x u dash + some lambda lambda + 1 u = 0. So, this is the Legendre’s equation we know. So, in this case, this is my a 0 x, this is my a 1, so if u see that the a 1 is equivalent to the derivative of this 1. So, this equation I can write as 1 – x square u dash, the whole dash + lambda lambda + 1 u = 0.

Now, from here, if I take the derivative of this, then I will apply the product rule. So, it will be 1 – x square, then the derivative of u dash. So, this will be u double dash + the derivative 1 – x square. So, this will be – 2 x and then u dash and then the remaining part lambda lambda + 1 u = 0.

So, from here, we can write down that this is same as this equation. So, in this case, I can say that if I find out the corresponding adjoint operator, then we will find out that, so, this is we can say the self-adjoint operator. So, in this case, we can check that if I find out the L star, then it will be same as the L.

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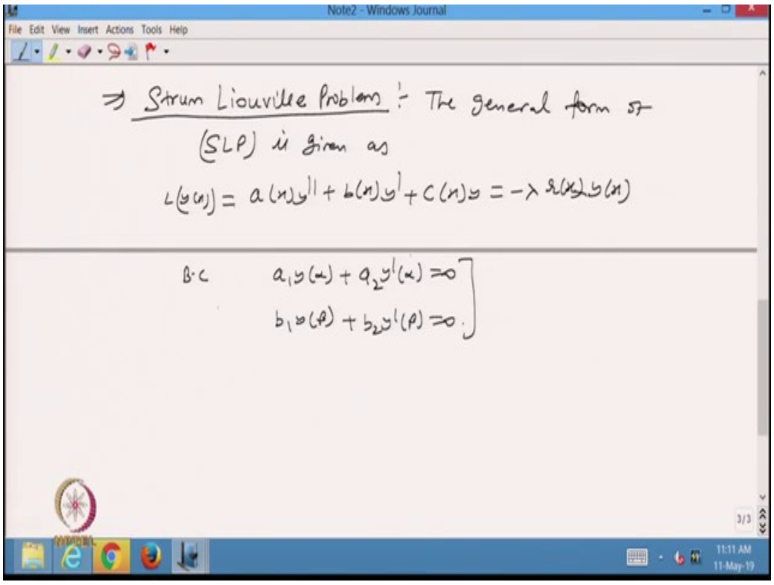

Now, we will try to go further and then we apply another equation that is called the Sturm-Liouville problem. So, the Sturm-Liouville problem is that, the general form of Sturm-Liouville, so short form this is SLP is given as L of y x, I will write like this one, so this will be a x y double dash + b x y dash + c x y = – lambda r x y x. So, with the corresponding boundary condition, so boundary condition is a 1 y at alpha + a 2 y dash alpha = 0 and b 1 y at beta + b 2 y dash at beta = 0. So, this is the mixed boundary conditions for alpha and beta.

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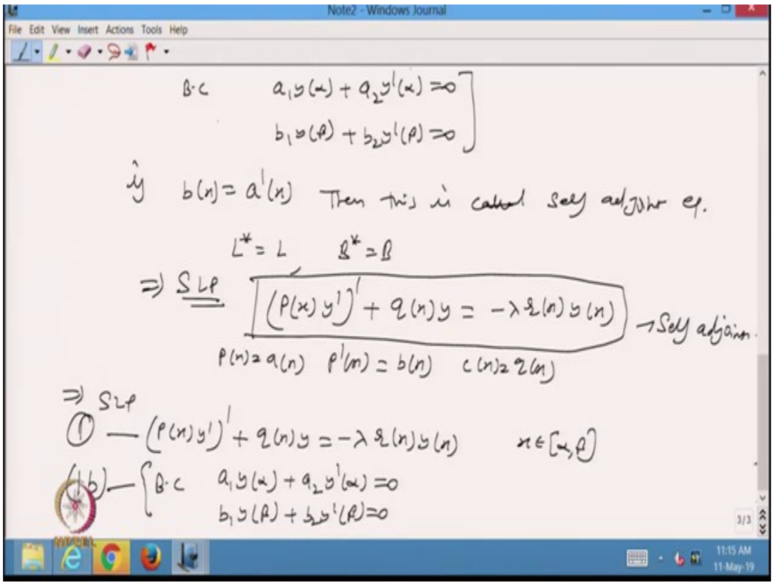

So, this is my corresponding boundary condition. Now, if in this case my b x that is a coefficient of y dash = a dash x, then, this is called self-adjoint equation, means, in that case my L star will be = L and my boundary condition B star will be = B. Then, I will say that that is is self-adjoint equation or the system.

So, in this case, the given Sturm-Liouville problem will reduce to, so I will call it now some p x y dash dash + q x y = – lambda r x y x, okay? So, in that case, I will just replace this one. So, you can say that now my p x is a x, p dash x is b x, and then my c x I will just replace by the q x. So, now my equation will be this one and you can see from here that this is self-adjoint. So, this is in the form of self-adjoint.

So, from here, now my SLP p x y dash dash + q x y = – lambda r x y x with boundary condition, so I am taking the boundary condition that y alpha a 1 + a 1 y dash alpha = 0. So, homogeneous boundary condition I am taking, then the b 1 y at beta + b 2 y dash at beta = 0. So, you can say in this case that x belongs to alpha beta. So, this is the system of equation. I can just write that this is my equation number 1 and this is a corresponding boundary condition, I can call it 1B.

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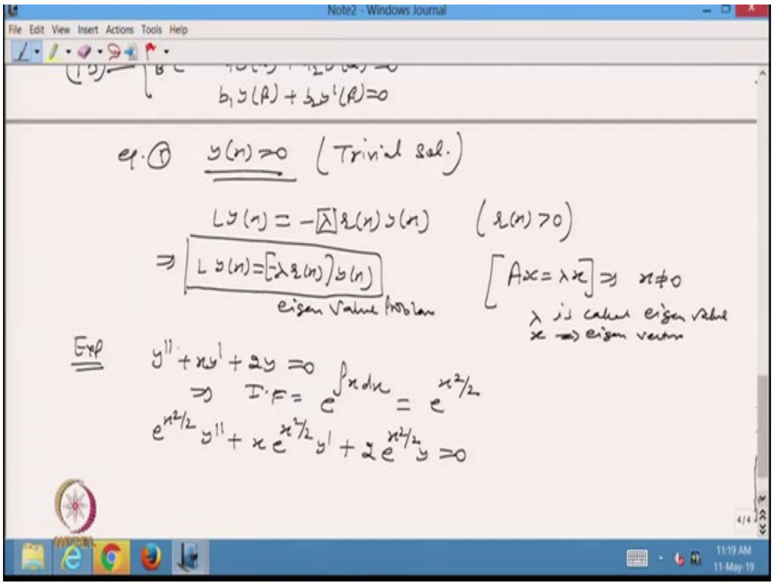

Now, I want to find out this equation I want to solve this equation. So, from here if you see this, the equation number 1, y x = 0 is always a solution, because y is here. If I put equal to 0, this will be 0. Derivative will be 0 and right hand side is also 0, then y x = 0 is always a solution of this equation, the Sturm-Liouville problem 1. So, this solution is called the trivial solution.

So, we always go to finding out the non-trivial solution of this equation. Further, the Sturm-Liouville problem, I can write as this one. This equation I can write as now L, then my y x = – lambda r x y x. So, the Sturm-Liouville problem can be written like this one. And now I am just assuming that the function r x is always positive.

So, from here, if you see, then I can write that, if you just remember the matrices, then we know that if I right matrix A x, A is a square matrix of n cross n and then if I write A x = some lambda x where this x is the vector, then, from here I know that if my x is not equal to 0, then this lambda is called eigen value and then x is called corresponding eigen vector. So, that we know now.

Now, this linear operator is also analogous to the matrix in the matrix theory. So, from here I can say that for this value of lambda I can write this equation as L of y x = – lambda times r x y x. So, this equation, if you see it can be written as the eigenvalue problem depending on this r x. Now, the question is where is this r x coming from? So, let us take one example. Suppose I have a differential equation y double dash + x y dash + 2 y equals to 0.

So, this type of equation suppose I have, then from here I just, because you know that this is not a self-adjoint type because the coefficient y dash is 1 and the coefficient of y dash is x. So, from here what I do is I can take the integrating factor as e x dx where x is the coefficient of y dash. So, from here I can write down e x square by 2, and then I multiply this equation with that e raised to power x square – by 2. So, it will become e x square by 2 y double dash + x times e x square by 2 y dash + 2 e x square by 2 y = 0.

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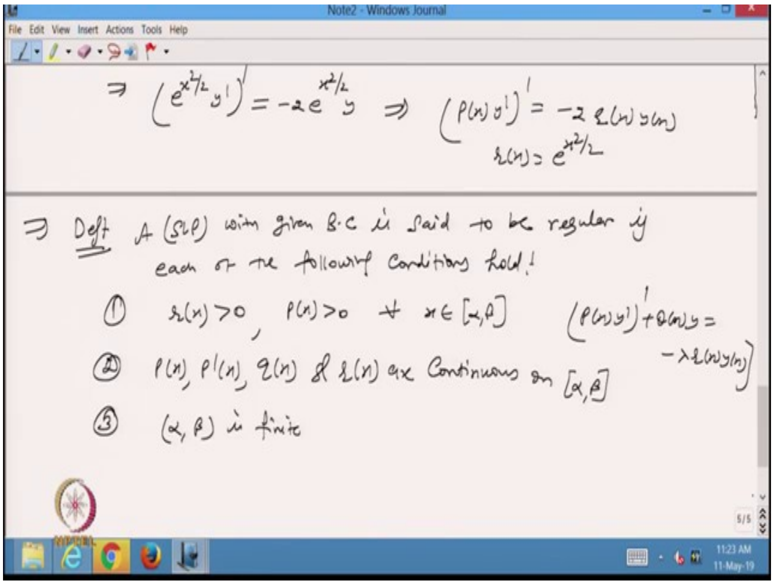

So, from here I can write this equation as e x square by 2 y dash dash = – 2 e x square by 2 y. So, from here I can write like this. Now, if you take the derivative of this one, it will be the first function derivative of second, y double dash +, then taking the derivative of this one, so e x square by 2 itself and then taking the derivative of x square by 2, so that will be x y dash and on the right hand side this one.

So, if you see this one, so this equation becomes now of the type p x y dash = – 2 r x y x. So, in this case my r x is exponential function. And you know that the exponential functions are always positive. So, this r x is coming from there. Whenever the given equation is able to convert to the self-adjoint form, then we are able to find the value of r x. So, there is a one observation.

So, just write down, definition, a Sturm-Liouville problem, SLP with boundary is said to be regular if each of the following conditions hold. So, what are the conditions? The first condition is that my function r x should be positive, in the previous case it was the exponential function, and if I solve this equation, so my p x, this one, for all x belongs to alpha beta.

So, p x is the coefficient, if you see this form, so it is a coefficient of p x y dash dash + q x y = – lambda r x y x. So, this is the form we are taking. The second one is that p x, p dash x, q x and r x are continuous on the given interval alpha beta. So, these are the continuous functions. And the third one is that the alpha beta if you take the interval is finite.

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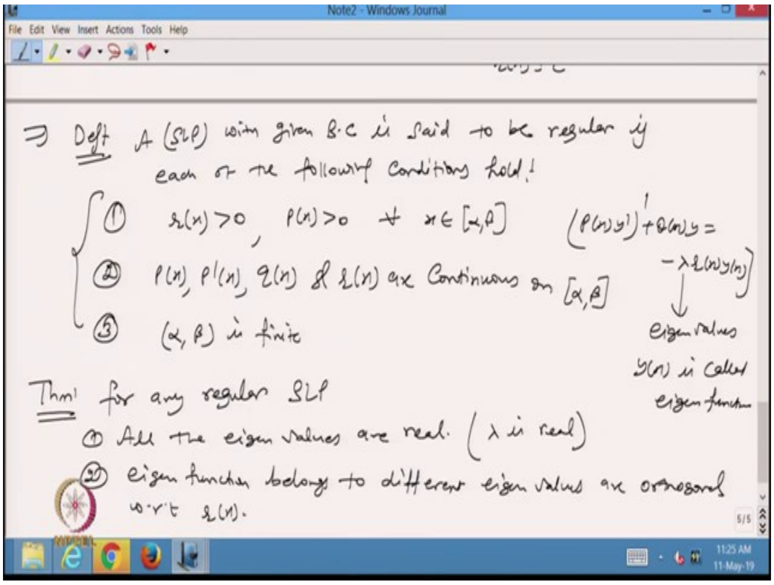

So, if you are able to satisfy all these three conditions, then my corresponding Sturm-Liouville problem is called the regular problem. So, there is one theorem we want to define.

For any regular Sturm-Liouville problem, first thing we want to define, because I told you that this is a similar type of the eigen value function, so this is called the eigen value problem.

So, all the eigen values are real. It means from here I can say that lambda is real for all. Second one is that eigen functions, because in this case if you see, then this lambda is called eigen values and then solution we are getting corresponding to this lambda, so this v x is called eigen function.

So, the eigen function belongs to different eigen values are orthogonal with respect to the function r x, so they are orthogonal basically, under the inner product. So, I am now defining the inner product. So, inner product, suppose I have two functions, y n x and y m x, so these are the two functions I am taking or you can just take simple any function.

So, I have two functions f x and g x. So, I am taking the inner product. So, I am defining the inner product in this case taking the integration from alpha to beta r x, the function we are taking, this is called the weight function here, and then f x g x dx. So, this is the inner product I am defining.

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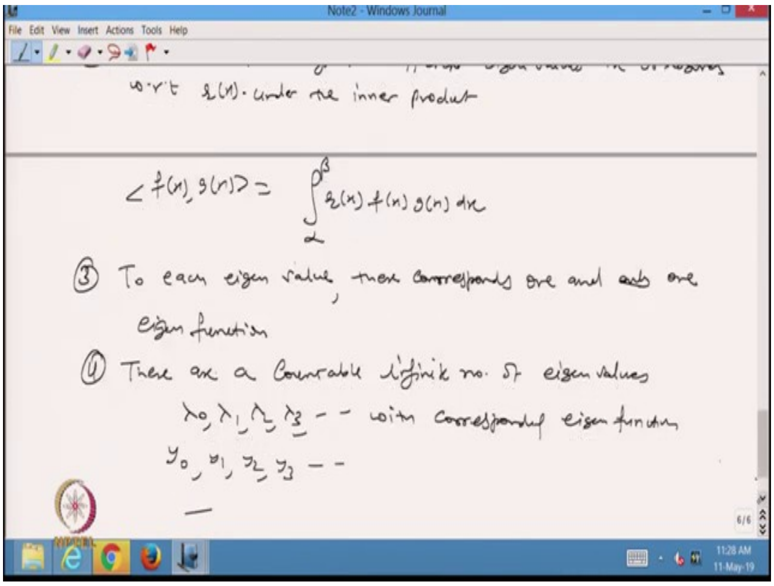

So, next third one is to each eigen value there corresponds one and only one eigen function. So, this is the third property. And the fourth property is that there are countable infinite number of eigen values, that is we call it lambda 0, lambda 1, lambda 2, lambda 3, with corresponding eigen functions that we call it y o, y 1, y 2, y 3, and so on, and also because just now we are telling that all the eigen values will be real, so then we can define the ordering and then we can put that one that.

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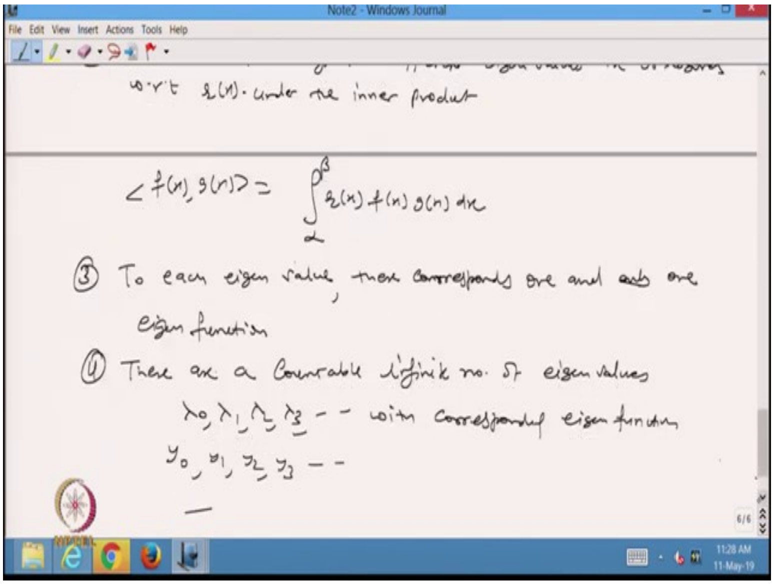

So, we can put the ordering of the eigen values and then we can define the ordering that lambda 0 is less than lambda 1 less than lambda 2 less than lambda 3 and so on. So, we can put them in the ordering. And the fifth one is that let I take any function f x, any continuously differentiable function on the interval alpha beta.

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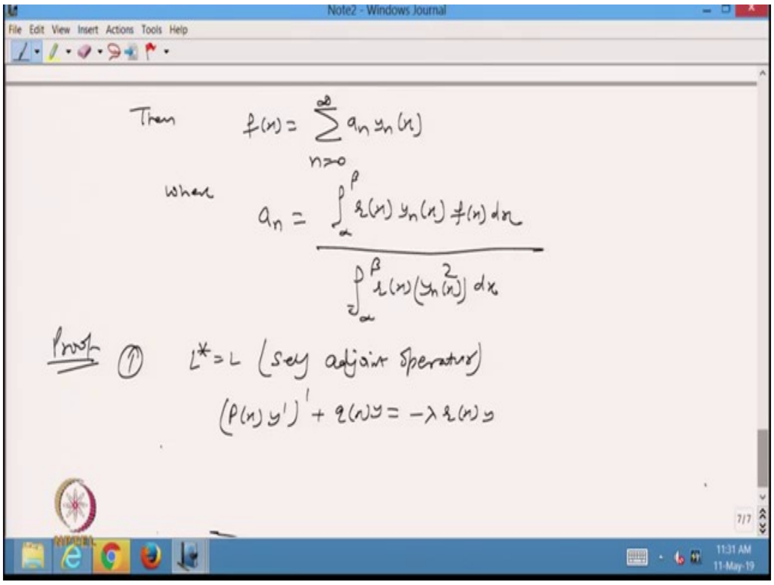

Then this function f x can be written as a summation of a series with coefficient a n and then y n x where y n x are the eigen functions corresponding to the eigen values where the coefficient a n can be written as, so I am taking integration r x y n x f x dx divided by alpha beta r x y n x square dx. So, the coefficient I can find out like this one and then its value will be a n.

So, these are the 5 properties. So, just now I will try to prove few properties, so I can say that the proof. So, property 1 is that, now I know that my L star = L, this is a self-adjoint operator because it is regular, and if there are regularities then self-adjoint operator, and then I can say that in this case my p x y dash dash + q x y = – lambda r x y.

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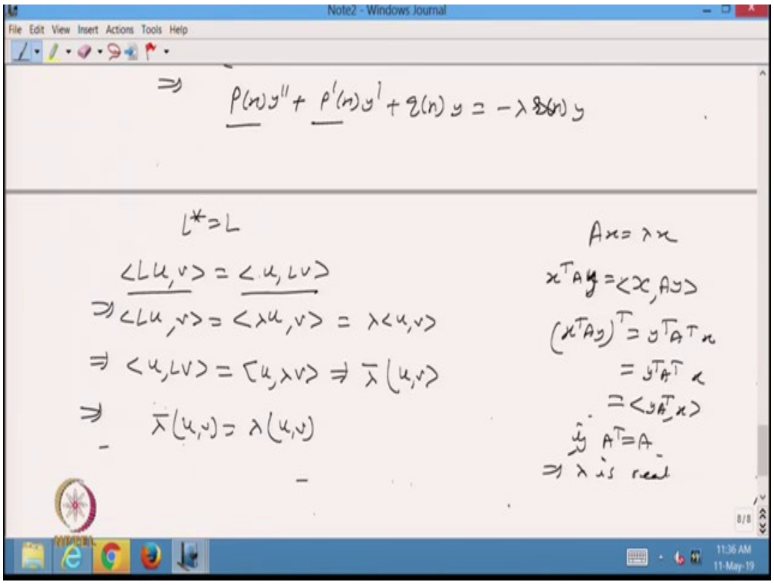

So, from here if I try to take the derivative, so again this will be = p x y double dash p dash x y dash + q x y = – lambda r x y. So, this one I can define, and from here I can say that this and this, the derivative of y dash is a derivative of the coefficient of y double dash. So, from here this is the self-adjoint operator.

Now, if you will remember that the same thing is going to happen when we take the analogous form in the matrix theory. So, in the matrix theory we know that my A x = lambda x, then I just define the inner product in that form. So, in that case I know that the x transpose A y = x A y inner product. So, this inner product we have defined.

Now, this becomes = y A transpose x, this one. So, from here I can write this as, so if I define this one, then x transpose A y, taking the transpose, so this can be written as y transpose A transpose and x. So, from here I can define my y transpose A transpose and x. So, this one I can write as y A transpose and x. So, this one I can write.

And from here, if I know that if A transpose = A, then I can write that the corresponding eigen value lambda is real. The same thing we are going to do here. So, I know that my L star = L. So, I am now defining my L u v. This is = u l v by the definition the self-adjoint operator. Now if you see L u v, so this one is a type of eigen values problem, so I can define from Lu, I can define the lambda of u v.

And this can be written as lambda I can take common and then it is u v. Now, let us see what will happen if I define u and L v. So, in this case I can write this as u and L v is lambda v because this is given to me, so I will take the conjugate of lambda and then u v. So, from here, I can say that my L dash u v = lambda u v. So, this is a lambda bar. So, lambda bar is the conjugate of lambda because we do not know still what is the lambda, lambda is real or complex. So, I am applying the properties of the inner product.

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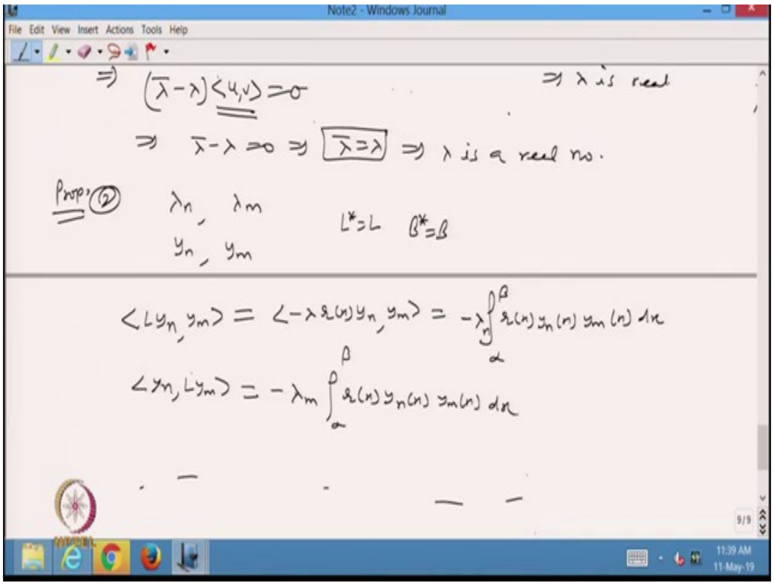

So, from here, I can take lambda bar – lambda and then the inner product u v = 0. Now I know that the inner product of u v is not equal to 0. So, from here I can say that the lambda bar – lambda = 0. So, from here I can say that the lambda bar = lambda. So, that shows that lambda is a real number and lambda bar is the corresponding eigen value. So, from here, I am able to show that this is real.

Now the second property, 2, I just want to define the proof of that one. Now let I take two eigen values, so lambda n and lambda m and then the corresponding eigen function. So, eigen function I am taking y n and y m. Now, I want to find what is L y n and y m. So, this one I want to find, because if you see that L star = L and the corresponding B star is also B, so this is self-adjoint system. So, in this case y n is also solution and y m is also solution of the equation.

So, I can define the L y n y m. So, this one is in inner product I want to define. So, this is – lambda r x y n y m. This one is the inner product I am taking, from alpha to beta, so – lambda r x y n x y m x dx, okay? So, this one I am just defining in this case. Now, what about y n L y m. So, again, the same thing will happen. So, this will be equal to, this I just call it n, it will be lambda y m alpha to beta r x dx. So, from here, and this is self-adjoint so both are same.

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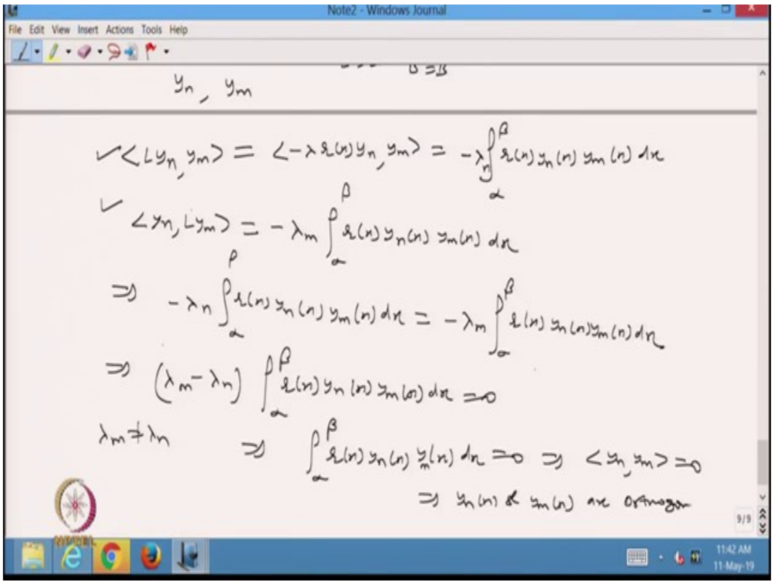

So, from here I can write down. So, this and this both are same because y n and y m both are the solutions of the L and L star and they both same, so L star n = L. So, these are the solutions. So, from here I can write down, so this becomes – lambda n r x y n x y m x dx = – lambda m r x y n x y m x dx.

So, basically this is a inner product we have defined. If you see that this is the inner product of y n m. So, from here I can write down, just taking on the left hand side. So, it can be written as alpha beta r x y n x y m x dx. This is we are defining, and that is equal to 0. Now, I am taking lambda m 0 = lambda n.

So, from here, because these are distinct, so, from here I can define alpha beta r x y n x y m x dx. So, this is equal to 0. So, from here I can say that the inner product of y n y m = 0. So, from here I can say that y n x and y m x are orthogonal because the same concept comes for the self-adjoint matrices.

In the self-adjoint matrices we know that, self-adjoint means symmetric matrices we know that the eigen values are real and in that case we find out the eigen vectors, so the eigen vectors are orthogonal to each other. So, same things are going on in the case eigen values and eigen functions.

So, then, the third property is that for the one eigen value there cannot be two eigen functions and that is obvious that we have one eigen value and there is a corresponding two eigen functions, then the linear combination is also the solution of the given equation satisfying the boundary condition and that is not possible because of the uniqueness. So, that is not possible that the one eigen value can have the two distinct eigen functions.

So, now, taking the next one is the third and fourth. So, fourth also, in this case in the property fourth also we can have an infinite countable number of eigen values and then the corresponding eigen function and then we have just defined that the eigen values a real. So, if they are real we can put it in the order and then we can have the sequence of the ordering that we have defined, that the lambda 0 is less than lambda 1 and less than lambda 2 like that one. Okay? So, this one we can do.

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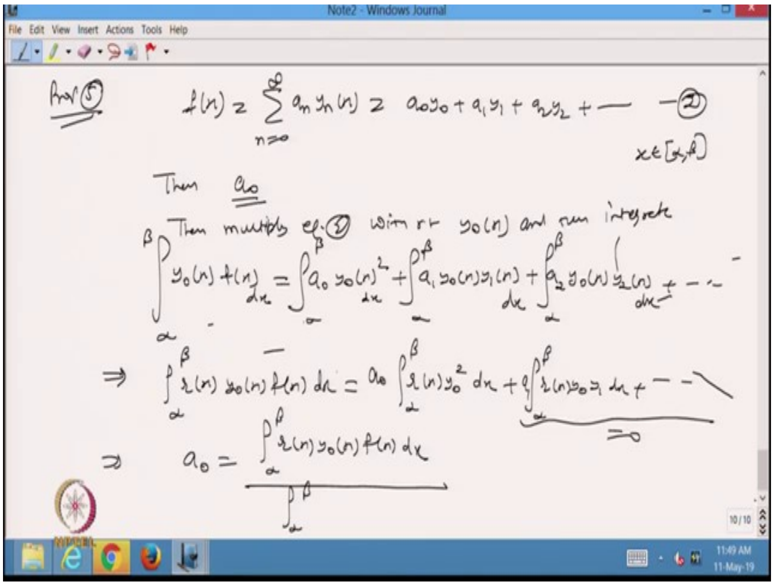

And then property number 5. So, this is another I just want to discuss here. So, in that case, we have a function f x, any function, just I am taking a real function f x which is continued and differentiable and I am putting this function is equal to summation a n and this is y n x and starting from 0 to infinity, like a 0 y 0 + a 1 y 1 + a 2 y 2 like this one. And what is this y 0, y 1, y 2, these are the eigen functions.

So, in that case, I can say that I am able to write a function f x in the series solution or in the summation of y n x, and also we know that this y 0, y 1, y 2, all are orthogonal functions. So from here, I am able to write like this one and I am considering here that this series is convergent, in fact it is pointwise convergent and it converged to the function f x in the given interval that x belongs to alpha beta. So, this I am considering.

So, I want to find the value of a n, like a 0. So, suppose I want to find the value of a 0. So, I will find out the value of a 0. What I will do is that I will multiply the equation. I just give it the name 2. So, I multiply this equation by a 0. So, what I will get, a 0, so this is a constant I want to find, f x = a 0 square y 0 + a 0. I just want to find the value of a 0. Then, multiplying this equation 2 with respect to the function y 0 x, and then integrate over the interval alpha beta.

So, what I am doing now, I am just taking y 0 x f x it will become a 0 y 0 x square + a 1 y 0 x y 1 x + a 2 y 0 x y 2 x and so on like this one and then I am taking the integration with respect to x from alpha to beta. So, again alpha to beta dx, alpha to beta dx, alpha to beta dx and so on. So, I am taking the integration now. Just now we have found that this function y 0 x and y 1 x they are orthogonal to each other with respect to the weight function r x.

So, now, I can multiply again with the r x here. So, I can define this one as, again multiplying each with the r x. So, I can write from here alpha x and then r x y 0 x f x dx. Because sometimes the r x can be 1 also, so in that case the calculation of this integral becomes very easy. But what I am doing, I am multiplying like the way I multiply the y 0 x, I am multiplying by r x into y 0 x.

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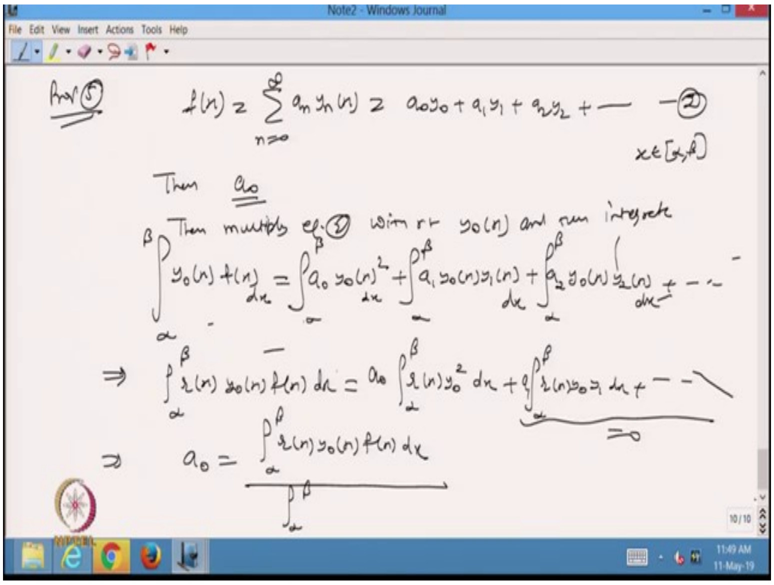

So, then it becomes, so this is a 0 alpha to beta r x y 0 square dx + alpha to beta, a 1 I am just taking outside, it will be r x y 0 y 1 dx and so on. Now this part all will be 0 because they are orthogonal. So, from here I can find out that my a 0 will be alpha to beta r x y 0 x f x dx divided by alpha to beta r x y 0 square dx. In this way I am able to find my value of a 0.

So, this is possible only when we have the property that all the eigen functions corresponding to the different eigen values they are orthogonal to each other. So, from here, if I want to find any general value of a n, then what I will do, I will again apply the same thing. I will pre-multiply equation number 2 by r x into y n x. Suppose I want to find the value of a n, so I will multiply it by r x into y n x and then doing the integration with respect to x between the interval alpha to beta and from there I am able to find the value of a n.

So, a n will be in this case it becomes alpha to beta r x y n x f x dx because I have multiplied by r x into y n x, divided by alpha to beta r x y n square dx. So then, in that case, my value of the a n will be there, and then I am able to find all the coefficients, and this integral we can define very easily. From there I can find the value of a n.

And then we are able to write the function in the terms of the series and that series is made up of the functions y n that is the eigen functions corresponding to the eigen values. If you just remember this one, this concept looks similar as the Fourier series. So, that way we will do the Fourier series in the future, but this is just another series in which we are able to write in function in the terms of a eigen function corresponding to the eigen values.

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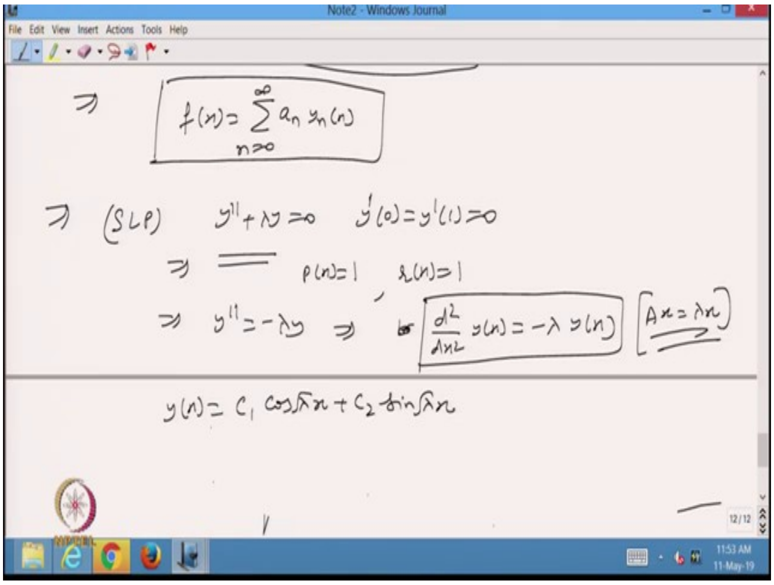

So, in this case, by this way, we are able to find all the values of f n and then I am able to write the function f x as summation and from 0 to infinity a n y n x. So, this is the concept we are doing here. Then, I will take a quick example that how we can solve a Sturm-Liouville problem.

So, let us take example, Sturm-Liouville problem like y double dash lambda y = 0 and then I take the boundary condition y dash 0 y dash 1 = 0. So, you can say that this is a self-adjoint form and in this case my operator is just d square over dx square. So, from here I can find out the solution.

Now, from here if you see my p x will be 1 and my r x in this case will be 1. So, this equation I can write in this form y double dash = – lambda y. And if you see from here I can write this as d square by dx square y x = – lambda y x. So, if you just write this one, I can write this A x = some lambda x, this form, and this we know that it is an eigen value problem.

So, this becomes the eigen value problem. So, from here I can find the solution and if you solve this one, I can find the solution of this equation. This is a simple linear equation. So, from here I can find out the solution. So, this is C 1 cos under root lambda x + C 2 sin under root lambda x. So, this one I can find out.

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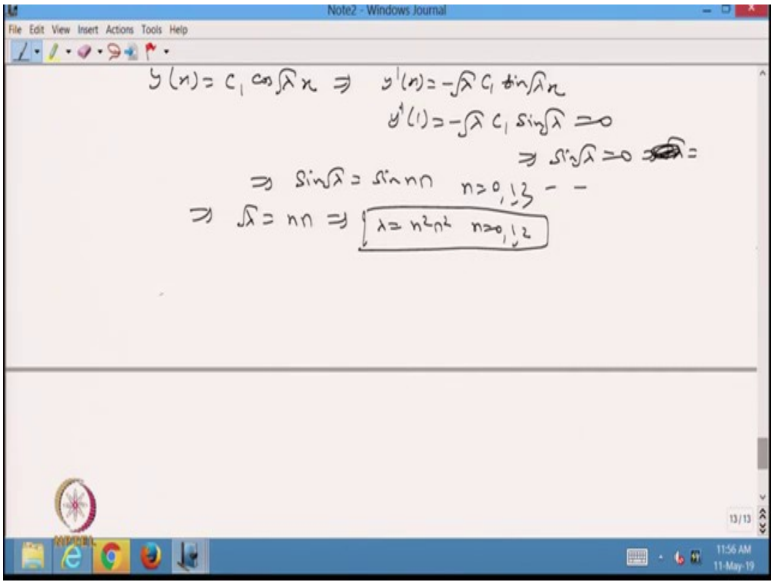

And then I put the boundary condition. Then my y dash x in this case will be – under root lambda C 1 sin under root lambda x + under root lambda C 2 cos under root lambda x and then I can put the boundary condition. So, from here I can put the boundary condition is y dash at 0.

So, this will be – under root lambda C 1 and this is sin 0 + under root lambda C 2 cost 0 and that value = 0. So, this part is always 0. So, from here I can say that, because we are looking for the non-trivial solution, y x is always a trial solution for this equation. So, we are looking for non-trivial solution and my lambda is not equal to 0.

So, from here I am taking that my C 2 is 0. So, if my C 2 is 0 I can find the value of C1. So, my y x is now C 1 cos under root x. Then from another boundary condition that is at 1, so from here my y dash x is – under root lambda C 1 and then I will put my boundary condition y dash at 1. So, this will be under root C 1 sin under root and this is given to me equal to 0.

Now, I cannot have my C 1 as 0 because if I take the C 1 0, then it will be just the trivial solution. So, from here lambda is also not 0, from here I can say that my under root lambda = 0. So, from here I can say that my under root lambda can be written as, because we know this is 0, then from here I can write that my sin under lambda = sin n pi and n is just value 0, 1, 2, and so on.

So, from here I can write that my under root lambda = n pi. That gives me that the lambda is n square pi square and n is 0,1, 2. So, this is my eigen values, all eigen values we are able to find. So, this is infinite number of eigen values we are able to find. And then with the help of the eigen values I can find my Eigen function.

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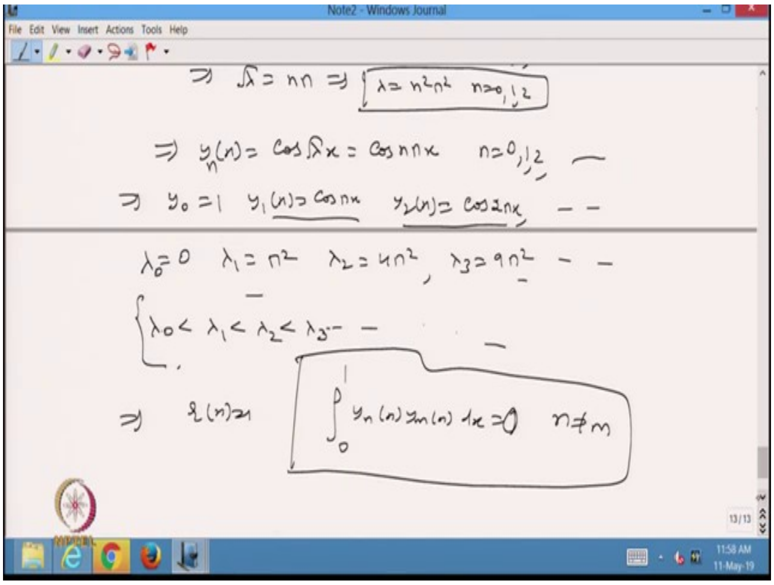

So, in that case my solution y x will be, because C 2 was 0, so from here my solution is just cos under root lambda x or I can call it cos n pi x and n is 0, 1, 2, 3. So, this is the value function of n. So, I can call it as y n x. So, from here I can say that my y 0 is just 1, my y 1 x will be cos pi x, my y 2 x will be cos 2 pi x, and so on.

So, in this case my lambda will be, so lambda 0 will be corresponding 0, should be 0, my lambda 1 will be pi square, my lambda 2 will be 4 pi square, my lambda 3 will be 9 pi square and so on. So, from here I can say that my lambda 0 is less than lambda 1 and so on and this is the corresponding eigen function. So, this is the property third we have defined, so this is true for this case.

And then we can also show that this function, the eigen functions corresponding to the different different eigen values are orthogonal. So, in this case, my r x is 1. So, I can check that from 0 to 1 if I take the integration, and then I take any y n x and any y m x dx, that value will be always 0 when n is not equal to m. So, this is my orthogonal properties and that we can verify.

So, this is the example of the Sturm-Liouville problem and we are able to verify that in the Sturm-Liouville problem our operator is a self-adjoint operator, it is a regular form, then the eigen values will be real and for the distinct eigen value the corresponding eigen function will be orthogonal and this will also form a sequence of the corresponding eigen values. So, thanks very much for this lecture. Thank you.