Introduction to Methods of Applied Mathematics

Prof. Vivek Aggarwal & Prof. Mani Mehra
Department of Mathematics
Indian Institute of Technology - Delhi/DTU

Lecture – 12
Adjoint Linear Differential Operator (Contd…)

Welcome back viewers. So, today, we are going to go further and we are going to try to find the adjoint and about the Green function of the adjoint operator. So, this is lecture number 12.

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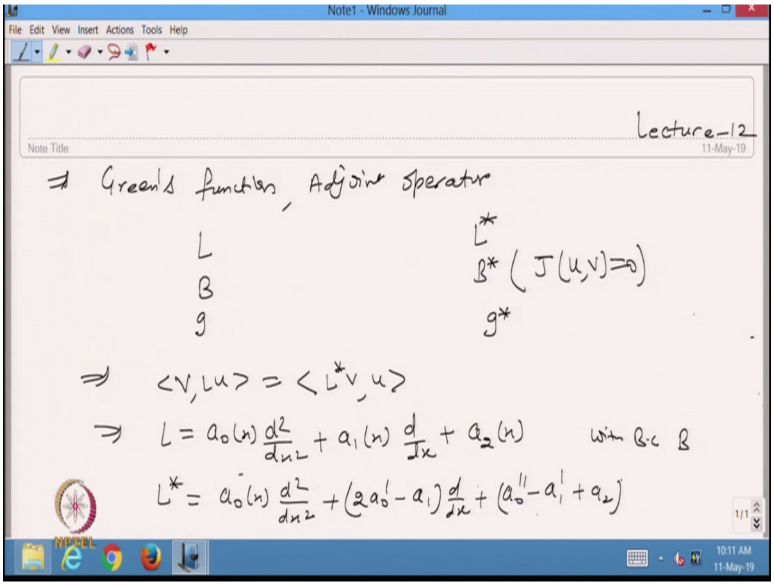

So, today, we will talk about that. In the previous class, we have studied the Green functions and the adjoint operator. So, we know that, suppose I have a linear differential operator L and then I know that I can write the corresponding adjoint operator with this B as the boundary conditions and B star is the corresponding boundary condition such that my bilinear concomitant is equal to 0.

Then, I know that I can find out the Green functions, that Green function I represent by g, and then I can find out the Green function for the corresponding adjoint operator. Then, what I want do, I want to find out the relation between g and g star, what is the relation between the Green function of the linear operator and, g star, the green function of the corresponding adjoint operator.

I also know that V of L u, the inner product, this is equal to L star v u. So, this is the way we have defined the inner product and from there we are able to find what is the L star, and I also know from here that if L = a 0 d square by dx square + a 1 d by dx + a 2. So, this is my linear differential operator with some boundary condition that we represent by B.

Then, we are able to find my corresponding L star, the adjoint operator. So, that adjoint operator we can define as a 0 x d square over dx square +, then we have defined two times a 0 dash – a 1, that is a function of x, d by dx + a 0 double dash – a 1 dash + a 2. So, that is the corresponding adjoint operator. So, from here we can find out the corresponding adjoint operator.

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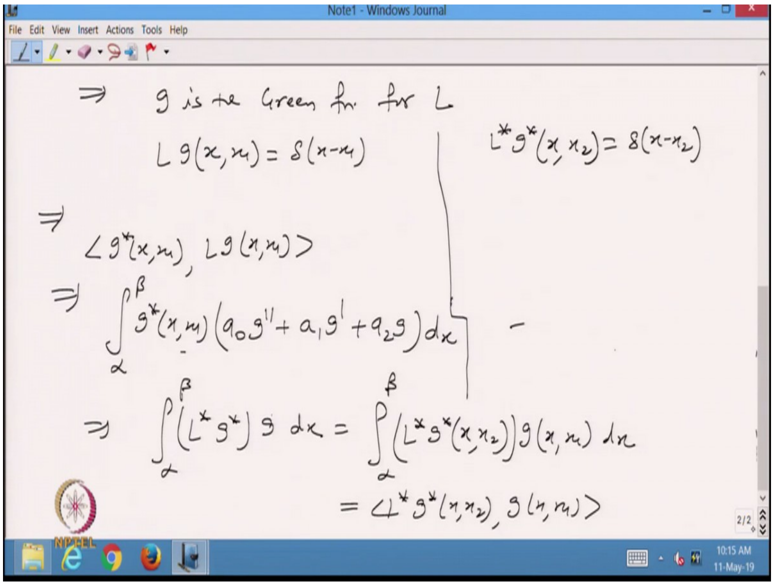

So, g is the, we are taking that as Green function for the operator L, then I know that L of g, so I call it x x 1, so that is equal to the Dirac delta x – x 1. Then, I define that L star of g star, and that I represent by x x 2. So, this is x x 2, so that is a Dirac delta defined at x 2. Now, from here, I will just try to find out what will happen to g star to L and then g x x 1 inner product.

And I know that this will be again equal to, taking the integration over the domain, so the domain we are taking in this case, the boundary domain we have taken, and that is between alpha and beta. So, this is from alpha and beta, g star, and then my differential operator and that is a 0 g double dash + a 1 g dash + a 2 g, and then the integration with respect to x.

And now, I can find out the solution for this one and then we found that this will ultimately lead to alpha beta and then it will go to L star g star into g dx. So, this is a function of x. Actually this can be written as alpha beta L star g star, that is x x 2, and this is g x, x 1 dx. So, this we can write as equal to L star g star and g x x 1 inner product.

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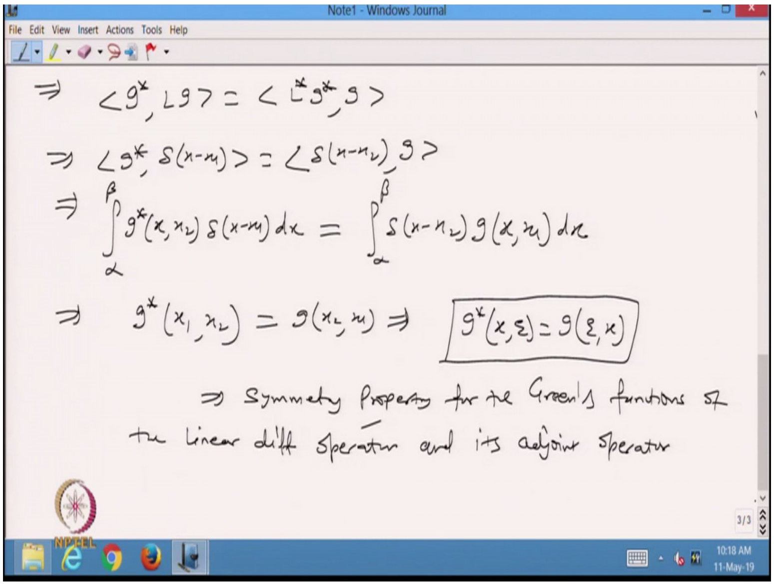

So, from here I can say that g star L g inner product is equal to L star g star g, this one. Now I know what is g star L g. So, L g is a Dirac delta function x – x 1 and this one is L star g star, so this I know is a Dirac delta function into g. So, this is the inner product. Now if you see this one, now from here, this one I can write as the integration. So, if I write the integration, this will be from alpha to beta g star, g star is x x 2 Dirac delta x – x 1 dx.

So, this is my left hand side, and right hand side it will be alpha beta Dirac delta x – x 2 and then g x x 1 dx. Now, by the property of the Dirac delta function, this integration on the left hand side will become g star x 1 x 2, and on the right hand side, this will become g x 2 x 1. So, this one we can write as g star, if I write it so it is x and xi, that is equal to g of xi x. So, now, this is the relation between the Green function of the linear differential operator and the Green function of its adjoint.

You can see that if we are able to find the Green function for the linear operator, then I just interchange the variable x and xi and that should be the green function of the corresponding adjoint operator. So, this is called the symmetry property for the Green's functions of the linear differential operator and its adjoint operator. So, this one we can write down. So, let us verify how we are able to find the Green function for the given operator and its adjoint operator.

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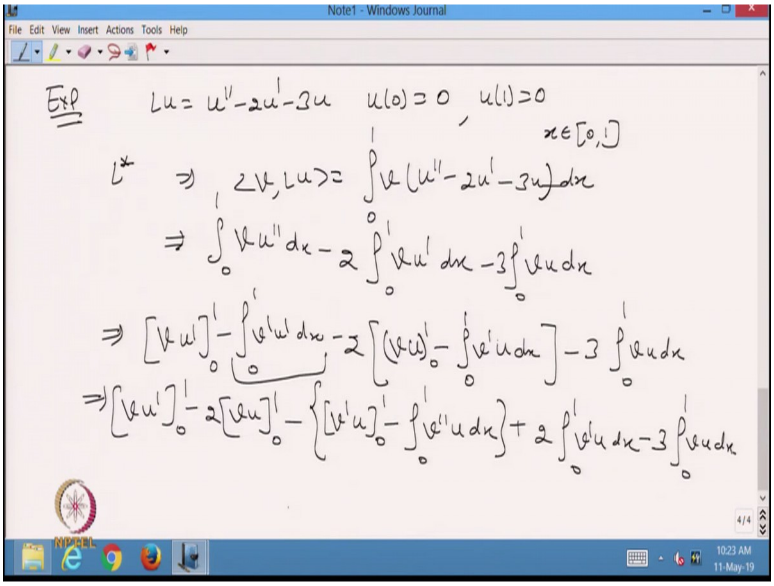

So, let us do one problem, then it will be more clear. So, let us take one example. So, let us solve the problem L u = u double dash – 2 u dash – 3 u u at 0 is 0 and u at 1 is 0. So, this is the linear second order differential equation, boundary value problem we are taking. So, this is my x belongs to 0 1. Now I want to define what is my L star.

So, if I want to find the L star for this one, then I know from here I can write down my v L u the inner product. This one can be written as 0 to 1 and then v is a function. So, we know that v is a continuous function which has the same characteristics as this u has. So, v is the function and then we are taking the inner product.

So, it will be u dash – 2 u dash – 3 u and then taking the integration with respect to x. So, from here, I write this integration 0 to 1. So, it is v u double dash dx – two times 0 to 1 v u dash dx – 0 to 1 three times v u dx. So, now, I will try to solve this integration with by parts. So, from here if I apply this one, then this will become v u dash, taking the integration from 0 to 1, – 0 to 1 v dash u dash dx – two times, so again I will solve this one with the by parts.

Then it will be again v, integration of u dash that is u, 0 to 1 –, then it will be v dash u dx. And the last is, this is same, v u dx. So, this can be written again as this one. So, again, I will solve this integration. So, this one is same as v u dash 0 and 1, and this one I can take here – two times v u 0 to 1 –, now I want to integrate this one. So, this will be v dash u 0 to 1 – v double dash u dx, so this will be there, – 2 times we have taken, so this one is there, and then it will become + two times 0 to 1 v dash u dx – three times 0 to 1 v u dx.

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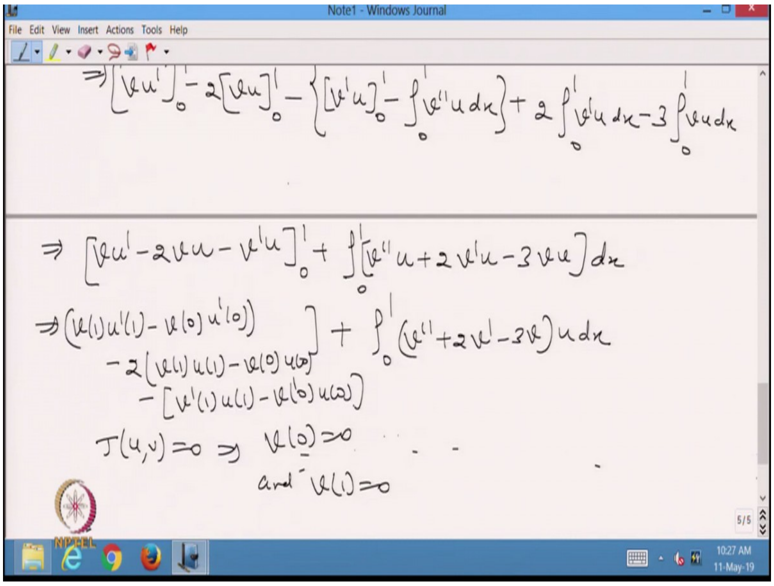

So, from here, we can write further, and then I can take this limits together. So, this will be v u dash – two times v u – v dash u 0 to 1. So, this one I have taken, +, I can write it as 0 to 1 v dash u from here + two times v dash u – 3 v u dx. So, this is the corresponding integration we are getting. Now, from here, I know that u 0 is 0 and u 1 is 0. So, if I put the limit here and here, this will be 0 except only in this case.

So, if I put this one, it will be v 1 u dash 1 – v 0 u dash 0, so I am left with only this one, +, and this one I can write as 0 to 1 v double dash + 2 v dash – 2 v u dx. So, this is my bilinear concomitant. So, this I can write as g, this one. Now, to define my adjoint operator. So, from here, just to make this bilinear form is equal to 0. From here, I will define that if I am able to take my v 0 as 0 and v 1 as 0, so in this case, just to make it a little bit more complicated, I can take this as u dash 1 = 0.

And if I take u dash 1 = 0, and u at 1 is not defined, this one, so I will just rewrite this part. So from here, I will write v 1 u dash 1 – v 0 u dash 0, so this is the first part, – two times v 1 u 1 – v 0 u 0 – v dash 1 u 1 – v dash 0 u 0. So, this I will be getting from here. So, this is my bilinear form.

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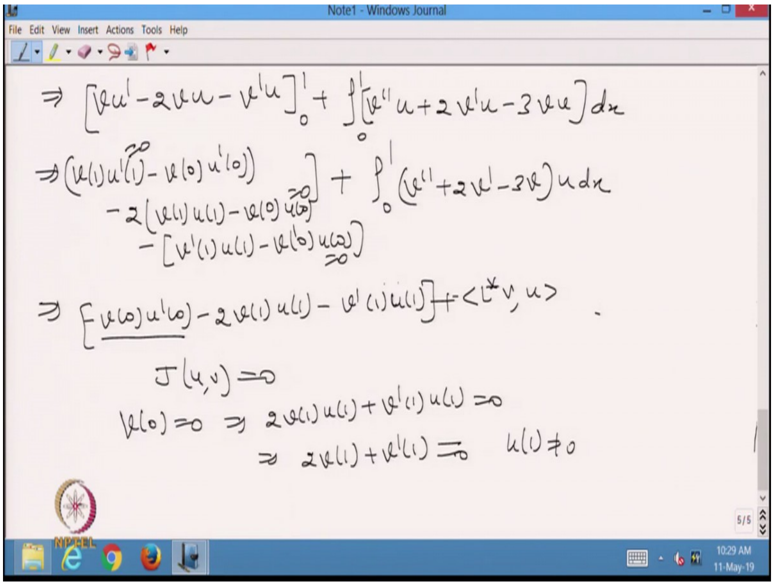

So, to make this one equal to 0, now just try to do this one. Now, this part is given to me that this is 0. U dash 0 I do not know. My u 0 is 0. So, this is 0, this is 0. So, from here, if I put these values, so u dash 1 is also 0. I am left with only this part. So, I will get from here that – v 0 u dash 0 – two times v 1 u 1 and – v dash 1 u 1, so this is my factor, +, and this one I can write as L star v u, that is the inner product. So, this is my bilinear form. So, bilinear form u v.

So, just to make it is equal to 0, then what I choose is that v 0 I choose as 0. So, if I choose v 0 = 0. From here, this will be 0. And then, from here I will choose that two times v 1 u 1 + v dash 1 u 1 = 0; u 1 is taking common, so from here I will get 2 v 1 + v dash 1 = 0. We are assuming that the u 1 is not equal to 0 because we have given that u dash 1 = 0. So, from here we will get this value.

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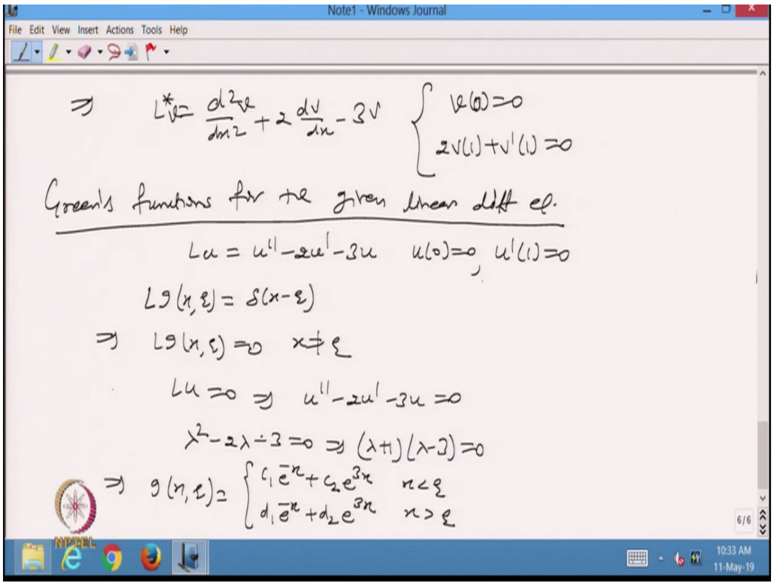

So, from here, I will get my corresponding adjoint operator. So, the adjoint operator for this one is, in this case, my L star will be v double dash + 2 v dash – 3 v. So, + 2 v dash – 3 v. So, that is my corresponding adjoint operator. So, from here, my B star, the boundary condition is that the v 0 is 0 and 2 v 1 + v dash 1, that is equal to 0. So, this is my corresponding boundary conditions. Now we want to find out what will be the Green's functions.

So, let us start finding the Green's function for the given differential equation. So, we have started with this equation, that is my L u u double dash – 2 u dash – 3 u and with the condition that u 0 is 0 and u dash 1 is 0. So, from here, if I want to find the Green functions, if you see this one. So, we know that L of g x – xi. So, this I can say from here that L of g x xi is equal 0 when x is not equal to xi.

So, from here, I can find my relation, that g x xi I can find for corresponding homogenous part. Now, I know that my L u = 0, it gives me u double dash – 2 u dash – 3 u = 0. So, from here corresponding characteristic equation will become lambda square – 2 lambda – 3 = 0. So from here, the value the lambda comes – 1. So, if I want to factorize this one, this will give me lambda + 1 lambda – 3 = 0.

So, from here, this will be lambda square – 3 lambda + lambda – 2 lambda and – 3. So, from here, I can find out my Green functions. So, Green function in this case will be, I can find out my Green function x xi. So, this can be written as c 1 e – x + c 2 e 3 x when x is less than xi and d 1 e – x + d 2 e 3 x when x is greater than xi. So, this is the Green functions corresponding to the given equation.

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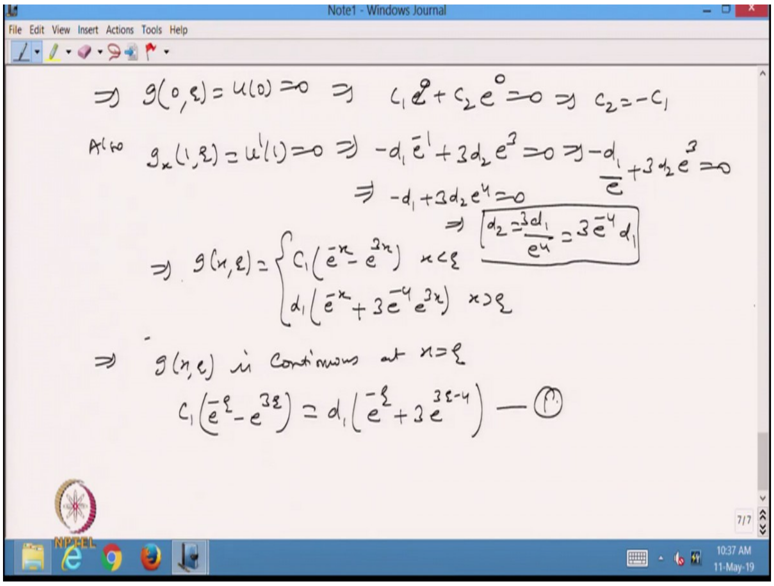

And, from here, I want to find the value of c 1 c 2 and d 1 d 2. I know that my g of 0 xi, that is given to me, so this is given to me, u 0 = 0. So, from here, I can find my c 1 e 0 + c 2 e 0 = 0. So, from here, I can find out my c 2 is – c 1. Also, g of x at 1 xi, so this one will be equal to the given condition.

So, from here, I can find out that – d 1 e – 1, taking the derivative of this, + three times d 2 e 3 = 0. So, from here I can write down that – d 1 over e + 3 d 2 e 3 = 0. So, that gives me my – d 1 + 3 d 2 e raised to power 4 = 0. So, from here, I can find that my d 2 will become, so I can take the d 1 here, three times by e raised to power 4.

So, this will be three times e raised to the power – 4 d 1. So, that will be my value of d 2. So, after inserting this value in the given equation, so from here I can write down that my Green function will become, so c 2 is this one, so from here I can write down my c 1 will be c 1 e – x – e 3 x when x is less than xi and this one I can write as d 1 e – x and d 2 is this one, so + three times e raised to power – 4 into e 3 x when x is greater than xi.

So, this is the Green functions we are able to find. Now, I have two constants to find out, c1 and c2. So this one we can find out with the help of the properties of the Green functions. First thing is that the Green function is continuous at x = xi. So, from here, I can write down that the c 1 e – xi – e 3 xi = d 1 e – xi + 3 e e xi – 4. So, this one we can find out. So, this I can take as equation number 1.

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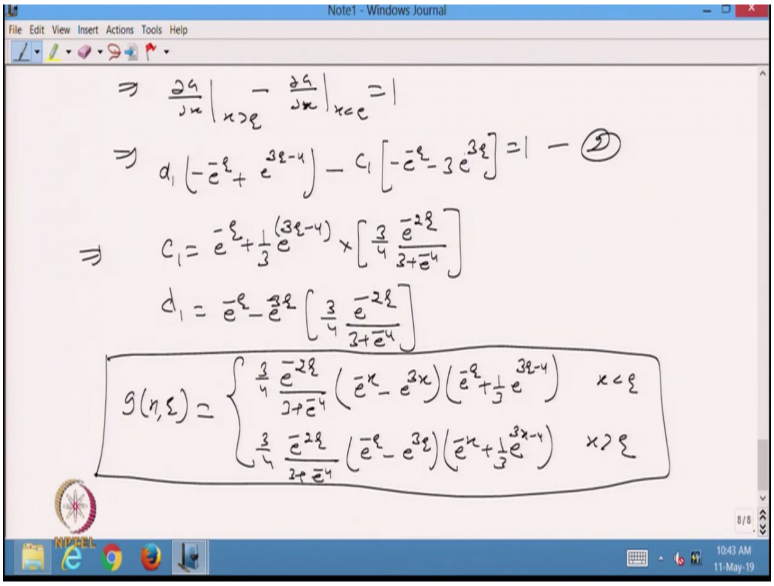

So, from here I can write down. Now, because I have two constants c 1 and d 1, so I need two equations to find out. Then we apply another property of the Green function, that, del g by del x at x greater than xi jump discontinuity at x equal xi. So, del g by del x at x less than xi and it has a jump at of width that is 1 at x = xi. So, from here I can find out, so x is greater xi, so this one I can find out, d 1 – e – xi + 3 into 3 9.

So, this one I can find out. So, my d 2 will be 1 by 3. So, this should be 3 instead of this one, because this will go down and then it will be 1 by 3, okay. From here if I change this one, it becomes 1 by 3 and then I can change this also. So, this will be three times into 1 by 3, right. So, this is a continuity condition, this one. This is putting the continuity condition and then we are putting this one.

So, this will become d d 1 – of this and e 3 xi – 4, because I am taking the derivative of this one. So, 3 will be coming here and it will cancel out this one, this value, – c 1, and then taking the derivative of this one, so it will be – e raised to power – xi – three times e raised to 3 xi. This value should be equal to 1.

So, if I solve these two equations and then if we do some calculation, so from here I can write my values of c 1 and d 1, so if we solve these equations together, then after solving this one, I will get the value of c 1, so this one I can write directly. So, c 1 becomes e – xi + 1 by 3 e raised to power 3 xi – 4 into 3 by 4 e raised to power – 2 xi divided by 3 + e – 4.

So, this is the value of c 1 we can find out, and then my d 1 will be e raised to power – xi – e raised to power – 3 xi into 3 by 4 e raised to power – 2 xi, and then 3 + e – 4. So, this is the value of c 1 and c 2. So, from here I can write down my Green function. So, the Green function g x xi in this case will be, so I can write this one as the final Green function.

So, this is 3 by 4 e raised to the power – 2 xi divided by 3 + e raised to the power – 4, and then e raised to the power – x – e raised to the power 3 x into e raised to the power – xi + 1 by 3 e raised to the power 3 xi – 4, so this is when x is less than xi, and 3 by 4 e raised to the power – 2 xi 3 + e raised to the power – 4, this one.

And then it will be e raised to the power – xi – e raised to the power 3 xi, and then it will be e raised to the power – x + 1 by 3 e raised to the power 3 x – 4 when x is greater than xi. So, this is the Green functions we are getting for the given differential operator. Now we want to find out the Green functions for the corresponding adjoint operator.

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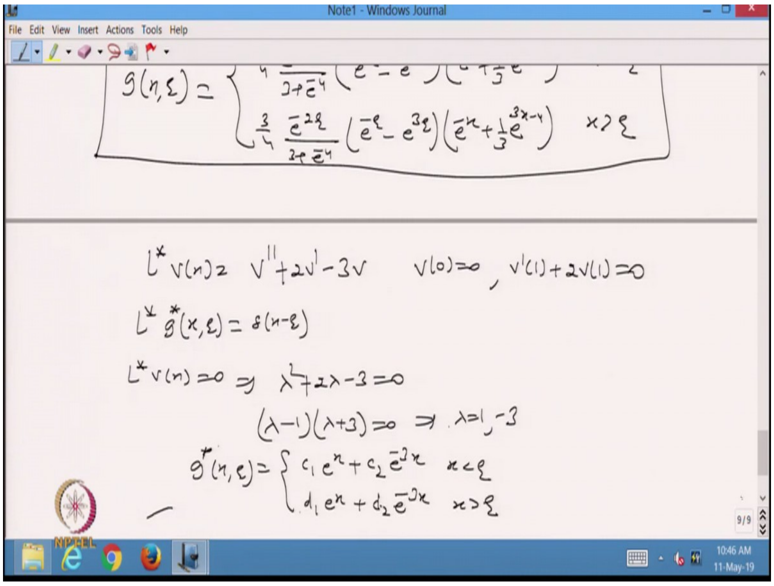

So, if I do the calculation my L star v x in this case is v double dash – 2 v dash – 3. So, this is the differential equation we got. So, after solving this one, this is my 3 v with the boundary condition v 0 = 0, and v dash 1 + two times v 1 = 0. So, this is the adjoint operator we found out, and from here I want to find what is the Green function.

I know that the L star and the g, so I represent by star here, so this is equal to x – xi. So, from here, just to find out G xi, let us solve this one, L star v x = 0. So, from here I will get my characteristic equation that is lambda square – 2 lambda – 3 = 0, and from here I can have the relation + 3 = 0, because lambda square 3 lambda –, so it should be + here, so this is + sign.

So, if I solve this one, then it is a + 2 lambda and – 3. So, from here, I will get my lambda = 1 and – 3. So, my Green function in this case will be c 1 e raised to power x + c 2 e raised to the power – 3 x when x is less than xi and this is d 1 e x + d 2 e – 2 x when x is greater than xi.

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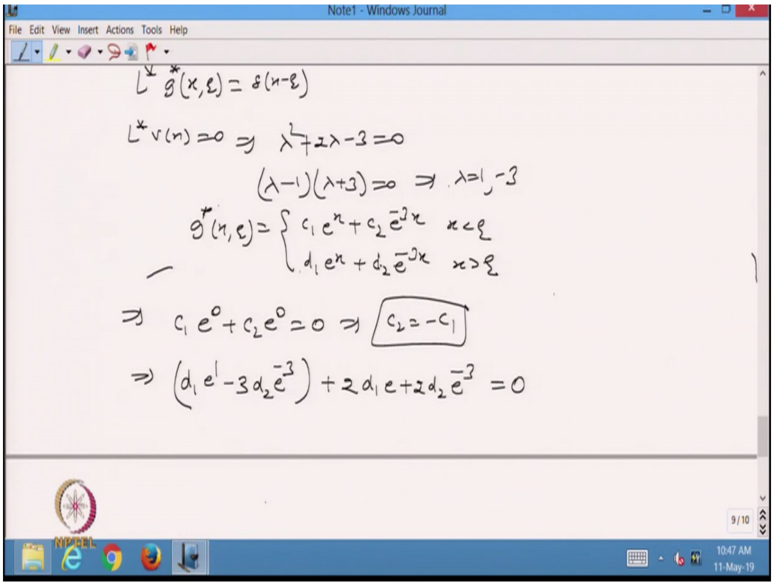

Then, I can find out the constants here. So, if I further solve this one, then I know that c, because the initial condition the value at v 0 is 0 and this is the boundary condition at v 1, so v 0 will give me c e 0 + c 2 e 0 that is equal to 0. So, again this is giving me c 2 = – c 1. So, this is the value at 0.

And then we also found that the v dash 1 + 2 v 1. So that gives me, so I take the derivative of this one. So, this should be d 1 e 1 – 3 d 2 e – 3. So, this is v dash at 1 + two times v at 1. So, it is d 1 e + two times d 2 e raised to the power – 3 = 0. So, putting this boundary condition at 1 gives me this value.

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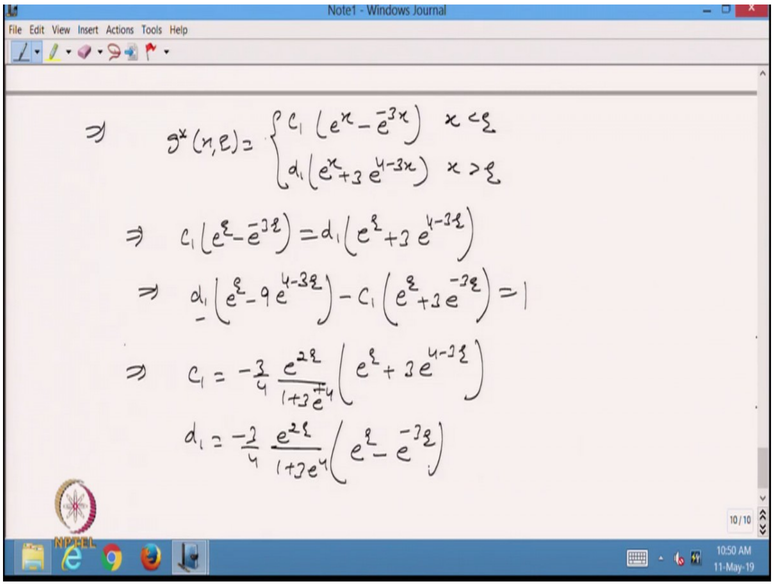

So, if I solve it further, then from here we can find the Green function. So, from here, I can write my Green function g star x xi is equal to, so this will give me c 1 e raised to power x – e raised to the power – 3 x when x is less than xi, and this would be d 1 e raised to power x + 3 times e raised to power 4 – 3 x when x is greater than xi. So, this is the Green function. And then, c 1 and c 2 again we can find out with the help of the properties of the Green functions.

So, from here, if I apply the properties of Green function, then it will be c 1 at e xi – e – 3 xi = d 1 e xi + three times e 4 – 3 xi. So, this is the first condition, the continuity condition. And the second one is the derivative. So, from here if I write down, so it will be d 1 e xi +, so this is – 3, so it will be – 9 e raised to the power 4 – 3 xi – c 1 e xi + 3 e – 3 xi, that value is equal to 1.

So, from here, if I solve it further and try to find the value of c 1 and c 2, so from here I will get the value of c1, it is – 3 by 4 e raised to power 2 xi over 1 + 3 e raised to power + 4 into e xi + three times e raised to power 4 – 3 xi. So, this is the value of c 1 we got. And the d 1 we can find out, so the value is coming 3 by 4 e raised to the power 2 xi 1 + 3 e 4 and this is equal to e xi – e raised to the power – 3 xi. So, from here, I can find the value of g star.

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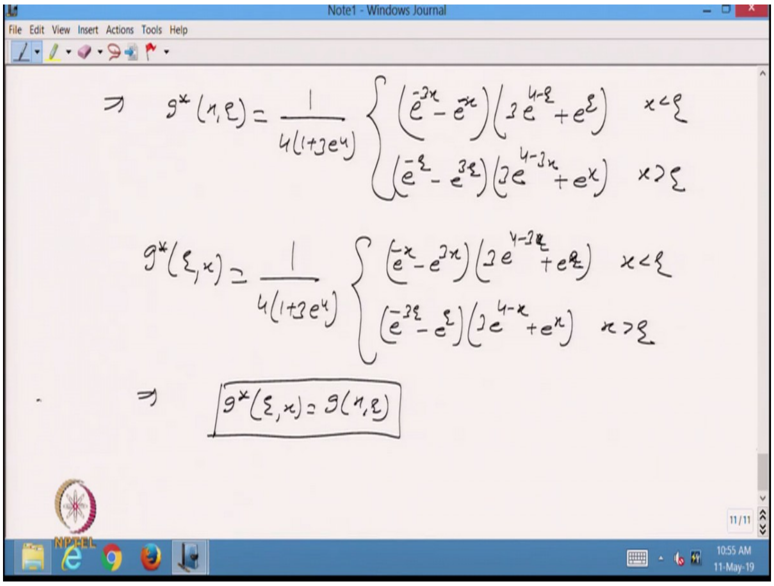

So, g star in this case will be x xi. So, if you write the g star, so g star will be 1 over 4 1 + 3 e raised to the power 4, this one. So, c 1 is this one. So, it will be e – 3 x e x into three times e 4 – xi + e xi when x is less than xi. And this is equal to e raised to power – xi – e raised to the power 3 xi 3 e 4 – 3 x + e x when x is greater than xi. And from here, if you see, now in this case what I do is I will try to find out what was the value of g star xi x, just changing the variables x and xi.

Then, from here, I can write, this is 1 over 4 1 + 3 e raised to the power 4. So, in this case x is greater than xi and if I put xi here and x here, so this will be this part. So, in this case, I can write down e – x – e 3 x three times e raised to the power 4 – 3 x + e x when x is less than xi and this will become e – 3. So, this will be x and this will be xi, and it should be e raised to the power – 3 xi – e 3 xi 3 e 4 – x + e x when x is greater than xi.

And then it should be the same as the g value. So, e – x – 3 x, so if you see the g we found out, so this is the g we go, e – x and this is e – xi 1 by 3 e raised to power 3 – xi. So, it should be same as that one. So, if we just do some calculation, we will get the value from here. And from here we can say that my g star x xi should be same as x xi. So, this is we are able to find the Green functions for the linear operator and the adjoint operator.

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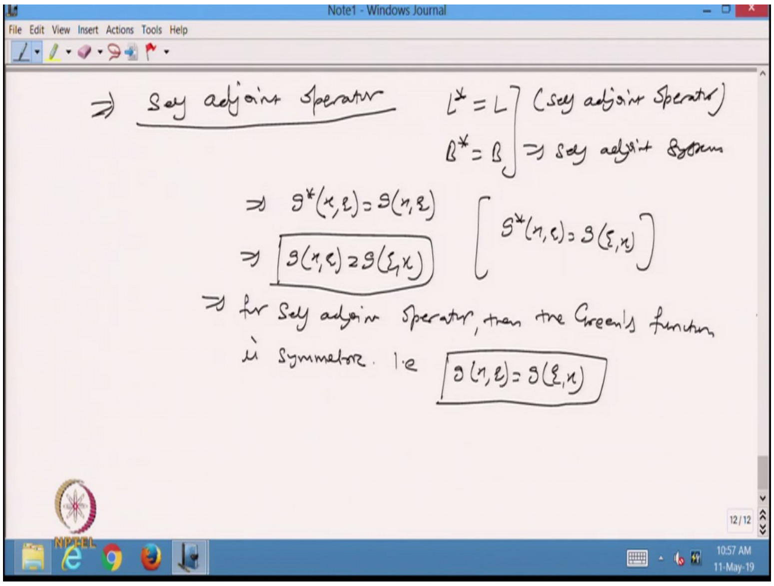

Now, what will happen if I take the self-adjoint operator. So, let us start with another form of the operator and that is called the self-adjoint operator. So, self-adjoint operator we know that when my L star, that is the adjoint operator, is equal to the L itself, then we call it the self-adjoint operator. Now, suppose, we have the boundary condition also B star = B and then we call it self-adjoint system.

So, in this case what will happen? If I have a self-adjoint operator, then from here we can say that my g star, the corresponding Green function of the L star, this will be g of x xi and this by the relations from here we know that, from here I can write that the g x xi = g xi x, because I know that for the Green function of the adjoint operator and the operator my g star x xi = g of xi x. This already we know from here.

But, when the operator is self-adjoint, then my g star x xi will be g x xi. In that case, from here I can write that the g x xi = g xi x. From this one we can say that for self-adjoint operator the Green function is symmetric, that is, my g of x xi = g of xi x. So, this is true for the self adjoint operator.

So, in this lecture, we have developed how the adjoint operator we can define for the linear equation and linear differential operator, and then we found the Green's functions corresponding to the linear differential operator and its adjoint, and then we found the relation between the Green function that is g star xi = g xi x, and for the self adjoint operator we are able to show that the Green's functions are symmetric in nature. So, that is all in this lecture. Thanks for watching.