Introduction to Methods of Applied Mathematics

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Lecture – 11
Adjoint Linear Differential Operator

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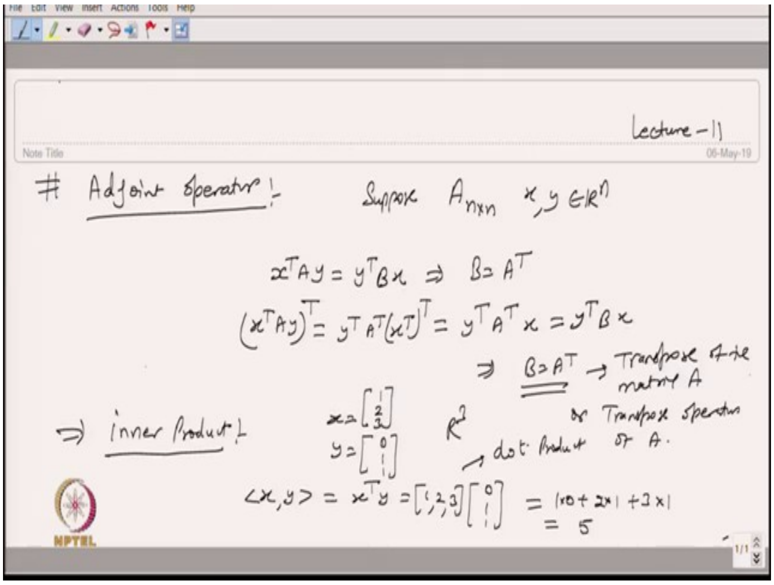

Hello viewers, welcome back to this course. So, today, we are going to start with the next lecture, that is eleventh lecture. So, in the last lecture, we have started with Green functions, how the Green functions are helpful for finding out the particular solution for a given differential equation. So, today, we will discuss another topic related to the differential equation and that is the adjoint operator.

And, then, we will discuss how the differential equation, any operator, a differential operator and its adjoint operator, and then we will find the relation between the Green function of the linear operator and the Green function of the corresponding adjoint operator. So, today, we will start with a new topic that is called adjoint operator.

So, before this one, we can start with the matrix theory, like in the matrix, so suppose I have a matrix that is of dimension n cross n and I have two vectors, that is x and y, that also belongs to R n. Then I define my x transpose A y. So, this is my matrix, I post multiply by y and the pre multiply by x transpose. And suppose, if this is equal to y transpose some matrix B x. So, this is true, if you see here, that the role of x and y has been interchanged.

So, here x transpose A y and becomes y transpose B x, then from here, if this is possible for all x and y, then we say that B is a transpose. How it is happening? Like I have an x transpose A y, I am taking the transpose of this one, so I will apply the formula of the transpose. So, it will be y transpose A transpose x transpose transpose, and this I can write as y transpose A transpose and this will be x, and now this becomes, so it is given that this is equal to y transpose B x.

So, from here, I can say that my B will be A transpose. So, in that case, this is called transpose of the matrix A or we also call it transpose operator of A because we know that the matrix can be written as linear transformations, and then it can be written as an operator. These things we have already discussed in the course of linear algebra.

Now, the same thing I want to apply for the linear operator because here we are dealing with the matrices only, then we are able to do this multiplication. But what will happen if I go by the linear differential operator and then I want to apply the same methodology to find out the adjoint operator. So, this is called the transpose of a matrix or transpose operator of the matrix.

Now, before that one, I will be introducing another concept that is called the inner product because inner productive is that, suppose, as we know that if I have two vectors like I take a vector x, so I just take vector 1, 2, 3. This is my vector and I take another vector, so, that is just take 0, 1, 1 and all belongs to R 3, and if I want to multiply these two vectors, then I have to take the dot product.

So, this dot product will be, in this case, it will be Y. So, you can say that this will be 1, 2, 3 and this vector will be. So, in this case, actually this vector, generally we take the column vector, so, I will change this one to in this form. So, suppose, this is the vector 1,2,3. So, we will take the vector as a column vector. Another vector I am just taking, 0,1,1, and then I want to define this x transpose y. So, let us see what will happen.

So, x transpose will be 1,2,3. It becomes the row vector and then it is the column vector. So, if I multiply these two vectors it will be 1 into 0 + 2 into 1 + 3 into. So, this is the multiplication. And then, from here I will get the 5. So, this is the output when I multiply or I take the dot product of the two vectors. So, this is called dot product.

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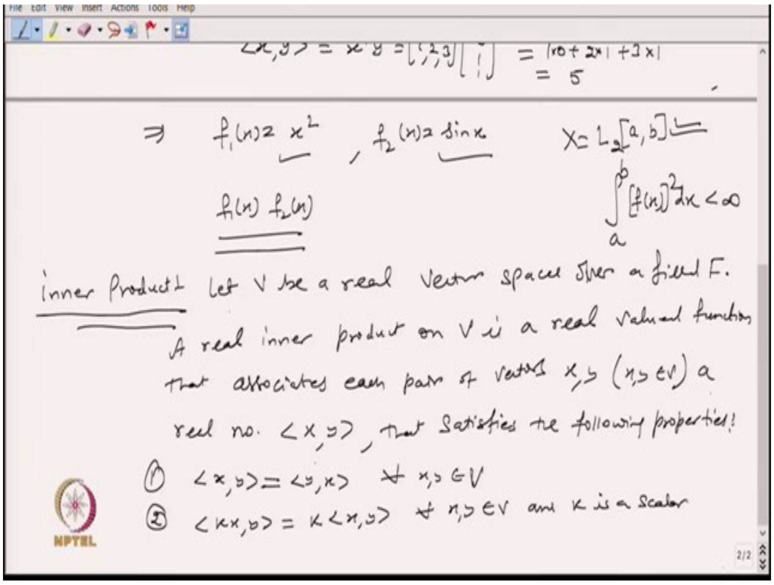

Now, what will happen if instead of this vector I have two functions, suppose, I have one function f 1 x, so, that is equal to I said just take x square, and I take another function f 2 x as sin x because I already know that these functions, these are the continuous function, and they also make the vector space, generally we call that vector space x, and we call it the L 2 space and these functions are defined in the interval.

So, that interval I just take a, b. So, this is the space of functions such that, the L2 means, such that if I take the integration of square dx, then it should be finite value. So, all the continuous functions we were discussing for solving the differential equation that also belongs to this space.

So, in this space, this is a vector space L 2 space. Now what will happen if I want to multiply the two factors. Suppose I want to multiply, what will the answer of this f 1 x into f 2 x. So, in that case, we just extend the idea of the dot product and then we define the inner product. So, let us define inner product.

So, this is the definition. So, let v be a real vector space over a field F. Then, a real inner product on v is a real valued function that associates each pair of vectors. So, I will call it x and y. So, this x and y belongs to this vector space v, a real number, that we represent by this sign, that satisfies the following properties. So, what is the first property?

The first property is that if I take the inner product of x and y then it is same as taking the inner product of y x. So, this is true for all x and y belongs to v. It means I can just interchange the x and y. So, it should be true for this case. And the second one is that for K x,y I can take the K common, then it should be equal to x, y. So, it should be also true for x and y belongs to v and K is a scalar. So, in this case we are taking real numbers.

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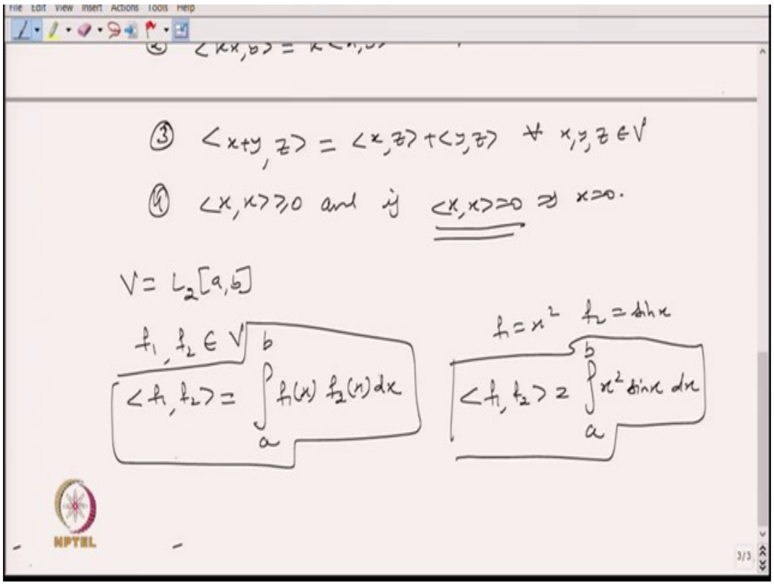

So, this is a scalar. Third property is that if I have summation of two vectors and I am taking the inner product with the Z, then I should be able to get x Z y Z. So, this should be true for all x, y Z belongs to v. And the fourth one is that if I take the inner product of a vector with itself and this is always greater than or equal to 0, and if it is equal to 0, that will imply that x = 0, because I know that if I take the dot product of a vector with itself, then it gives the length of vector.

So, in this case, the length of the vector is equal to 0, that means the vector itself is 0. Whenever we are dealing with the functions, in that case, so the inner product in this case becomes, so now let us suppose that I am taking v as L 2 space, then in this case if I want to take the inner product of two functions, so suppose f 1 and f 2 belongs to v, then I want to define the inner product of f 1 and f 2.

Then this will become the integration from a to b F 1 x F 2 x dx. So, that will be the inner product in that case whenever we are dealing with the functions. So, like, suppose, I have two function as we have taken that f 1 is x square and after f 2 is sin x and this two belongs to my L 2 a b, so in this case, and if I want to define what will be the inner product of f 1 f 2, then it will be from a to b x square into sin x dx. So, this will be the inner product of the two function in L 2 space.

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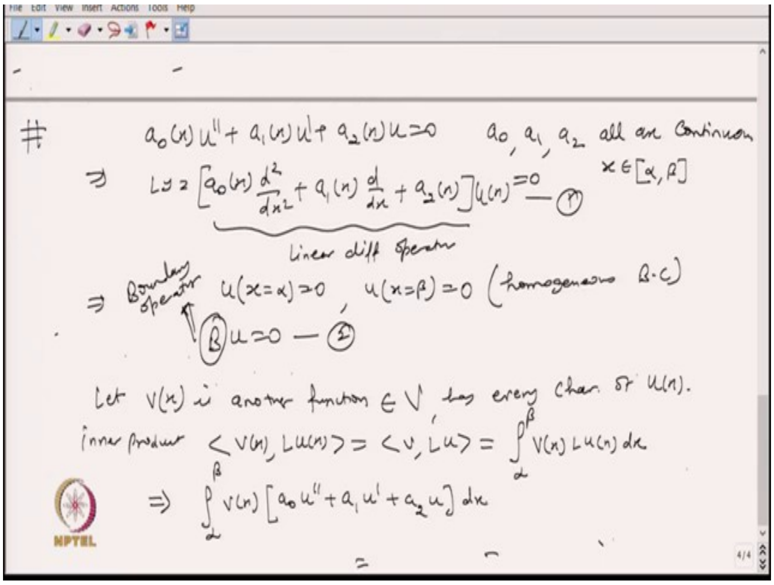

Now, with the help of this one, now I will define what do you mean by the adjoint operator. So, we already know that in the previous classes we were dealing with the second order differential equation of this type. So, this is a 1 x y dash a 2 x y = 0. And in this case we know that this a 0, a 1, a 2, all are continuous in the given interval x belongs to a b. So, these are they are continuous in the interval, it means they are belonging to the L 2 space and y x is the solution and the second derivative is also there.

So, it means that y x itself is also a continuous function so that y x, the solution, also belongs to the same space that is L 2 space. So, from here I can define the operator L y as a 0 x d square by dx square + a 1 x d by dx + a 2 x y x. So, this is my operator I am taking. So, this is my differential operator, I call it linear differential operator, and this is my differential equation and this is my differential operator.

Now, what I want to do, I want to find out what will be the adjoint operator for this linear differential operator. So, that one I want to find out. As we have found that whenever I was taking the matrix we can find out the transpose of the matrix and we have seen that the transpose of the matrix in that case is called a transpose operator or we can also call that as adjoint operator.

In the matrix theory we know that the adjoint of the matrix is a different concept. So, that is why in the form of the matrices we always say the transpose operator. So, let us find out how I can find the adjoint operator for the given differential operator. So, for this one, this I take as 1. So, this is my differential equation.

So, this is my operator. And let this differential equation have boundary or initial condition. So, I just take the boundary condition as u x = 0, and x at, suppose I take x at alpha = 0, and u at x = beta = 0. But in this case we are taking this, so I jus, here I should, because I am taking a here, so I should define here alpha beta.

So, I define the boundary condition u at x = alpha is 0 and u at x = beta is 0. So, this I am taking as a homogeneous boundary condition. So, in that case, if I take B as the boundary operator, so I can say that B of u is 0. So, then this B called boundary operator. So, this is my boundary operator. I call it 2.

Now, based on this one, I want to define, so I just choose, let I take a function v x is another function which belongs to that same vector space v from where that y is there, a 0 is there, and has every characteristics of y n. So, in this case, actually this y is a function of x and v is another function I am choosing.

So, instead of y I just take, just because here we are dealing with two types of function, so instead of y I just define my function as u because in that case it will be easy to define. So, this is also u. Then, my u is this one. So, I take that v x is another function defined and it has every characteristic of u x. It means that v x is another function and it has the same characteristics as of the u x.

Then I define the inner product. So, let us define the inner product. So, inner product I define as v that v x square taking know product I define as v, that v x we are taking with L u x. So, just short, I will call it, because these are the functions, so instead of this one I just write this is v L u. And I just told you that in this case the inner product will be from alpha to beta.

So, v is a function I am taking, so I am defining that v x and then I am defining L u x dx. So, this is my inner product of the two function. So, from here I can find that this will be alpha beta v x and then L, this one, so this will be a 0. So, from here I can write a 0, it is a function of x, and then I can say u double dash + a 1 u dash + a 2 u and then taking the integration with this back to x.

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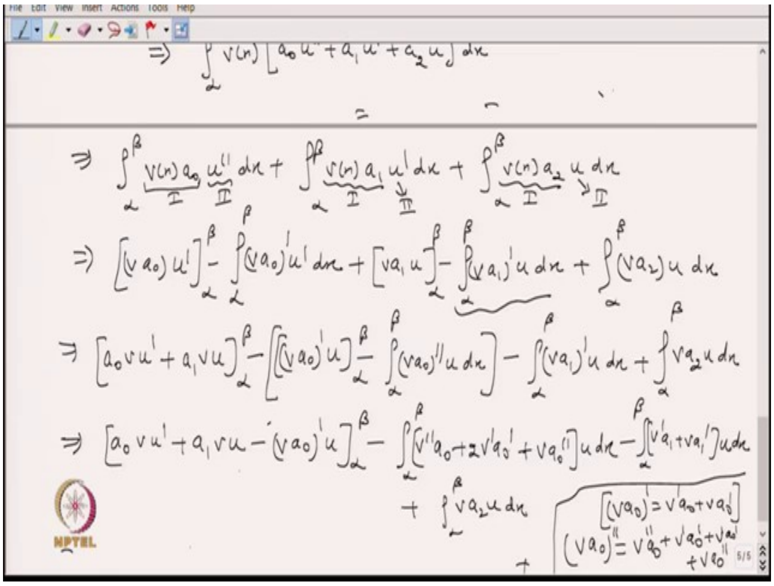

Then I can solve this one and this becomes alpha beta, then we multiply this v x taking inside. So, I will call it v x a 0 u double dash dx + alpha to beta I am taking the integration. So, it will be v x a 1 u dash dx + alpha beta v x a 2 u dx. So, this one we are defining. So, these are all function of x. So, from here then what I do is that I want to solve the first integral. So, in this case, if you see I will apply the product rule.

So, now, what I do is that I will choose this one as the first function and this one has the second function. Similarly, this one as the first function and this as the second function, this is first and this is second. So, let us do this one. Now, from here, if I take the integration, so this I will take as a v, just I am putting v a 0 the integration of u double dash, so it will be u dash, putting the limit alpha beta – alpha beta and taking the derivative of v a 0, and then u dash dx.

So, this will be the result for this integral, + the same thing I will do here. So, I will take it v a 1 and then u. So, I am taking from alpha to beta – alpha beta. Now I am applying the same integral of this one. So, it will be v a 1 u alpha taking the limit alpha beta – the derivative of v a 1 dash u dx and keeping this integral alpha to beta the same. So this is v a to u dx.

Now, from here, I will keep this quantity here. So, it will be a 0 v u dash + a 1 v u. So, this quantity I will separate this one, then this is the integral and I again want to solve this integral. So, this integral if you want to solve, then again I will apply the product rule. So, in that case, it will be v a 0 dash u. So, this will be from alpha to beta – alpha beta and taking the double derivative of this one.

So, it will be the double derivative of this one and then u dx. So, this will be there, then – and then applying the same thing here, so it will be again v a 1 dash. So, this will be remaining the same way. So, here there is no to, just keeping the same thing here. Then it will be – v a 1 dash u dx + alpha beta v a to u dx.

Now, if I want to solve this one further, then what I will do is that, so I want to solve this one, so this factor is again I can separate. So, it will be a 0 v u dash + a 1 v u – v a 0 dash u. So, this one I just take alpha beta –, so now I want to keep this one alpha beta. So, what I need to do now is that I have to take the second derivative, two derivative of this one. So, if you see from here I just write down that the v a 0 dash.

So, that will be v dash a 0 + v a 0 dash. And if I just take the two derivatives of this, so v a 0 double dash, so this will be v dash double dash a 0 + v dash a 0 dash + v dash a 0 dash + v a 0 double dash. So, from here, I can write that this will be = v double dash a 0 + two times v dash a dash + v a 0 double dash u dx – alpha beta, so it will be v dash a 1 + v a 1 dash u dx and then + alpha beta v a 2 u dx. So, this is the quantity I will get. Now if I further solve this one I will get the quantity.

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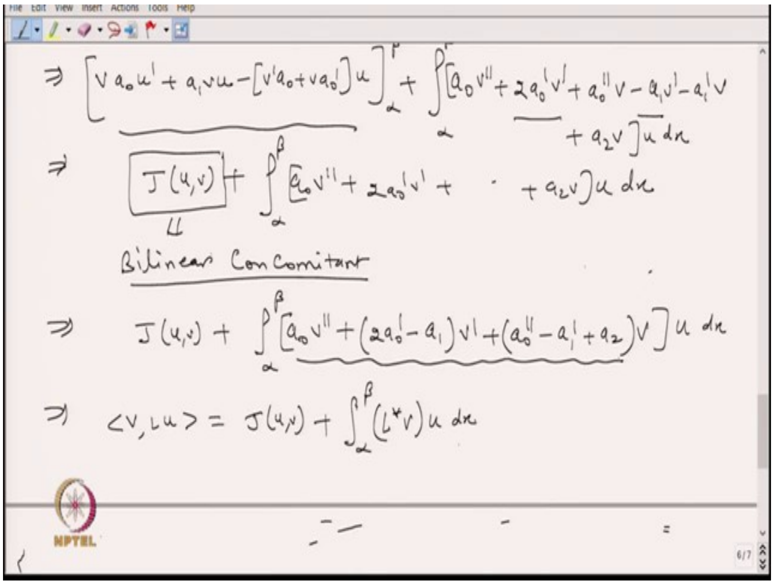

So, this quantity will be now v a 0 u dash + a 1 v u –. So, this quantity I will find out. So, this is v dash a 0 + v a 0 dash u alpha beta +, and this quantity will become alpha beta. So, I am taking a 0 v double dash + two times a 0 dash v dash, so this is that quantity, + a 0 double dash v – a 1 v dash – a 1 dash v + a 2 v. So, this quantity can be written as this one. So, I am just taking the terms together.

So, this one I have taken, two times of this one, this is this quantity – – and the last is this one. So, I got this quantity. So, from here I will just write that this quantity whatever the quantity I am writing, so that I will write J that is a function of u and v, if you see, it is a function of u and v, + this integral, whatever the integral we just got it to a 0 v dash this, this this, a to v u dx.

So, this factor that is written here, so this factor is called, it has a special name and that is called bilinear concomitant. Why it is called bilinear, because it is a linear in the form of u and v. So, this is called the bilinear concomitant. So, from here, if you see, then this quantity if I just write, so I can write this as J of u v + alpha beta, then I am taking the quantities together, so this will be a 0 v double dash + 2 a 0 dash – a 1 v dash.

So, I am taking this quantity and this and this + a 0 double dash – a 1 dash + a 2 v, so this quantity I am taking now and then u dx. So, from here what I have done, I have started with v and then L u. So, from here I will get J of u and v +, and this quantity if you see, this is another differential operator. So, from here I can write this as alpha beta and I call it, some new name is there, so from here I will just write this name, so I will call it suppose L star of v and then u dx.

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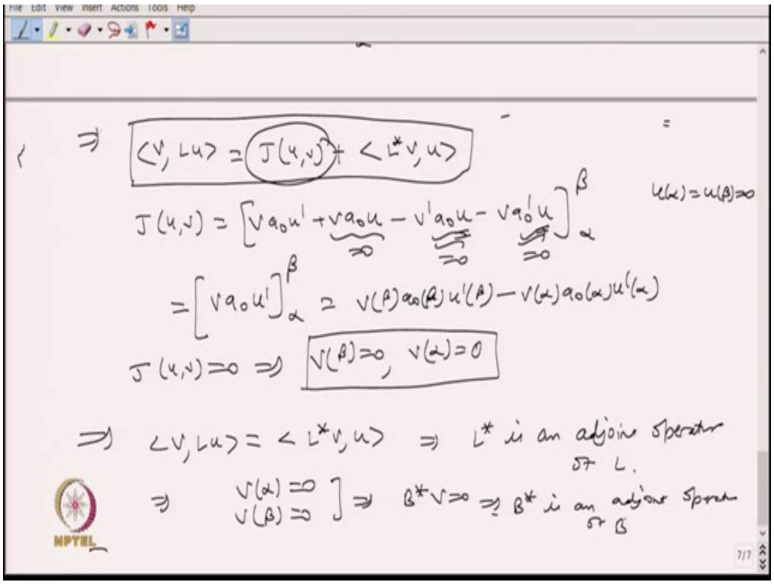

So, from here I can write v of L u = J of u v +, and this quantity I can write as L star v u. This one I can write. So, now this is the quantity we got. Now, the thing is that, if you see clearly, so from here now we have come with a new differential operator and that is represented by the L star. Now, what about this quantity? So, let us check this one. So, our J of u v is basically, so this = v a 0 u dash + v a 0 u – v dash a 0 u – v a 0 dash u.

So, this quantity I am taking. Now, if I substitute the value here, then if you see, from here, this u and this u, and I know that the u at alpha and u at beta both are 0. So, from here if I put the limit here for beta and alpha, then you will see that, and this quantity also has a u, so all this quantity becomes 0, this is also become 0, and this also become 0. So, we will come up with only v a 0 u dash alpha beta.

So, this will be v at beta a 0 u dash at beta, so, this is beta, sorry, – v of alpha a 0 alpha u dash at alpha. Now, my concomitant u and v is 0, if I take that this is equal to 0, then if I choose my v beta = 0 and v alpha = 0, from there I will get a new type of condition on the function v that v alpha is 0 and v beta is also 0.

So, if I have this one, now I can have my v L of u is same as L star v u. So, by the method of inner product from here I can say that the L star is an adjoint operator of L, so in this case I can say that. And then I also have v alpha = 0 and v beta = 0. So, from here I can say that my B star v = 0. So, from here I can say that the B star is an adjoint operator of B. So, this is the boundary the adjoint operator B.

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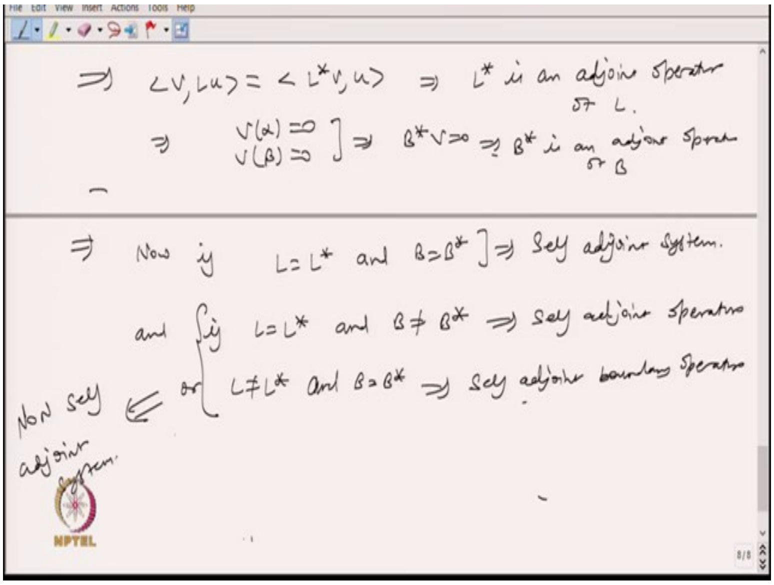

So, from here, now we have come up with the condition that, now what will happen if I will find out that my L = L star and B = B star, because if you see here the L star in this case is this one, and L was our differential operator and B was the 0 boundary condition and if you see B star, the B star is also the corresponding adjoint operator and they are the same type. So, in that case, now if I take if L = L star and B = B star, then in that case we say it is self-adjoint system.

Because in the system we have the operator and the boundary conditions. So, in that case, we will say that this system is a self-adjoint operator system. And if L = L star and B is not equal to B star, so then we call it as self-adjoint operator or I have L = not L star and B = B star, so in this case we call it self-adjoint boundary operator. But in both the cases if you see, then this is non-self-adjoint system. So, it is not an adjoint system.

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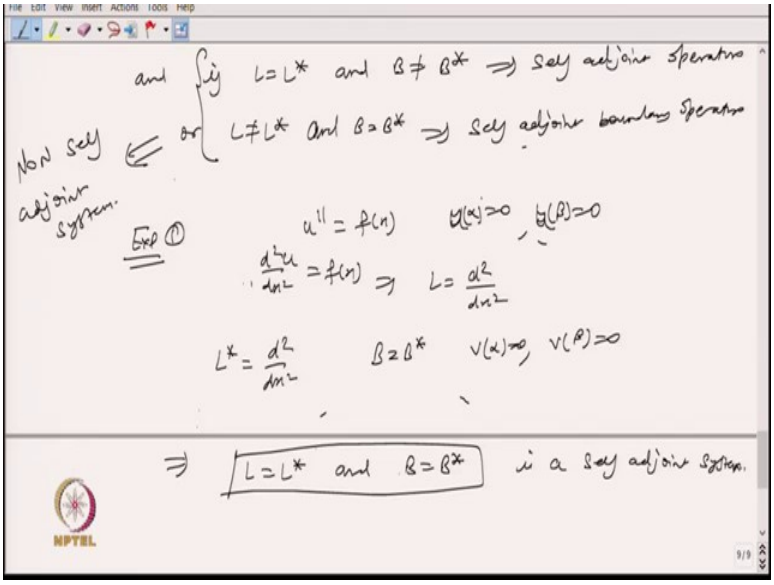

So, by using this one we are able to define the adjoint operator for any linear differential operator. So, let us take one example. So, I will take the example 1. I will start with the very simple equation. So, let us take y double dash = some function f x with some initial condition that y at alpha is 0 and y at beta is 0.

So, in this case, I can say that d square over dx square, so let us take it u not y, so this is also u and that is also u. So, d square u by dx square = f x. Now, in this case. I want to take, so my L is this one, now from here I want to define what will be my L star. So, if you see the L star from here, so L star is this, it is depending upon the coefficients of the L.

So, from here if you see our equation was this one a 0 u double dash + a 1 u dash + a 2 u, so this was my differential equation. Now, my a 0 and a 0 is same in this case, but here the coefficient of v dash become two times a 0 the derivative of this one – a 1 + two time derivative of a 0 – one time derivative of a 1 + a 2. It means the coefficient of these are the same.

So, from here, my L star can be again the same, d by dx, and in that case also my B and B star both are same, because just now we have seen that v of alpha is also 0 and v of beta is also 0. So, from here, if I get these values, then from here I can say that my L = L star and my B is B star. So, these things I can write here. So, I can say from here that this is self-adjoint system. So, this is the self-adjoint system.

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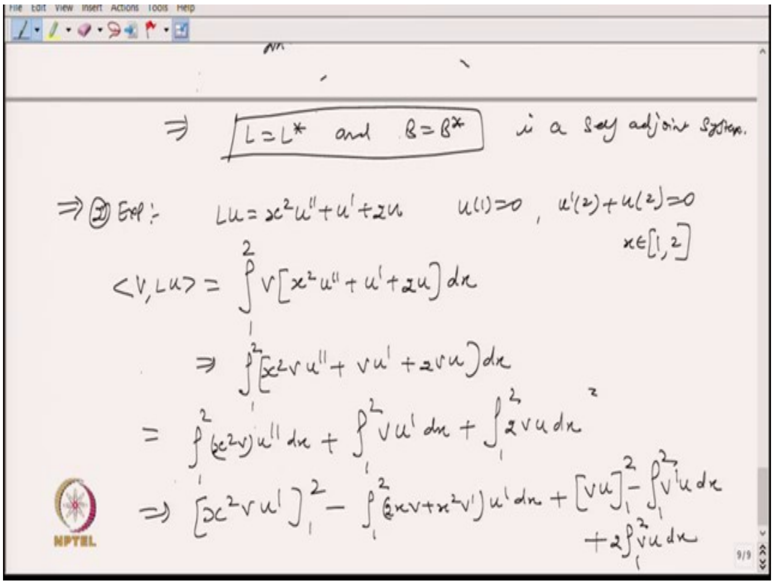

Now, I go with another example. So, this is example 2. So, let us take this equation. My L u I am just taking, x square u double dash + u dash + 2 u. So, this is my operator and then I am applying, this is my boundary condition, u dash 2 + u 2 = 0. So, this is the boundary condition I have taken. So, in this case I can say that my x belongs to 1 and 2. So, this is my corresponding linear operator.

Now, from here, I want to find out what will be the L star. So, now I will define this one. So, let us see this one. Now what I want, I want that let there is a function, I choose v and then I take the inner product of this one. So, this one will be from 1 to 2 v and then I will define this one x square u double dash + u dash + 2 u dx.

So, from here I just take 1 to 2, so I can just take x square v u double dash + v u dash + 2 v u and taking the integration. So, again from here I can write down from 1 to 2 x square v u double dash dx + integration from 1 to 2 v u dash dx + integration from 1 to 2 two times v u dx. Now, the main thing is that I want to solve this one.

So, it will be again x square v u dash taking integration from 1 to 2 –, so taking the derivative of this one, so the derivative will be 2 x v + x square v dash. So, this is a derivative I have taken, u dash dx, +, now from here I am taking again the product rule. So, I will call it v u from 1 to 2 – the derivative of v dash u dx from 1 to 2 and the last quantity is as such I am taking, so 1 to 2 two times v u dx.

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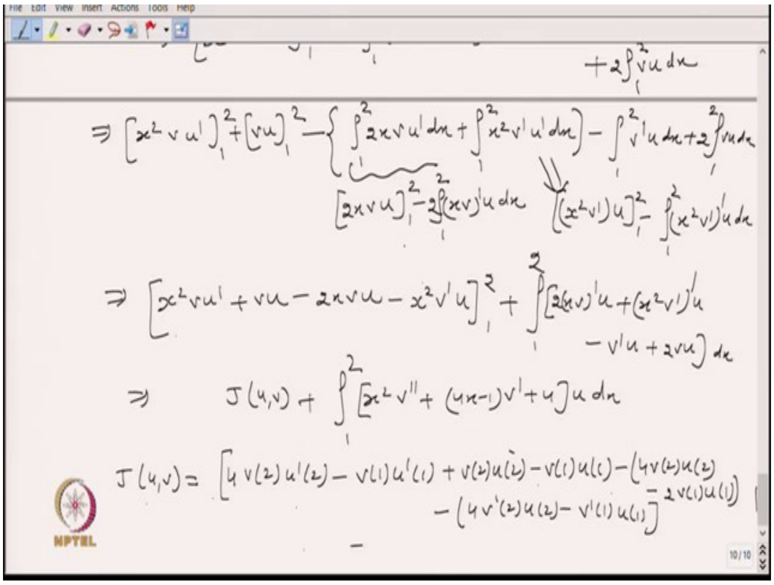

Now, again, I am solving this one. So, this quantity will be same. So, it will be x square v u dash 1 to 2 and this quantity + v u –, now this quantity I want to solve, so if I want to solve this quantity, so this is 2 x v u dash and x square v dash u dash. So, from here I can separately write this one. So, this will be 2 x v u dash dx + 1 to 2 x square v dash u dash dx. So, this is separating, I am doing this one, – 1 to 2 v dash u dx + dx.

So, I am solving this one now. So, if you solve this one, it will be 2 x v u 1 to 2 –, so this is just 2 I am taking outside. So, it will be x v dash u dx. So, this is the correspondent integral and now if I solve this one, this will be again x square v dash u 1 to 2 – 1 to 2 x square v dash u dx, and the other quantity will be same.

So, if we go further, so from here I can say that if I solve it further, than this quantity x square v u dash + v u from here – 2 x v u from here it is coming, and from here it is coming – x square v dash u. So, this will be my bilinear concomitant, +, so this this – + will be there. So, then, I will get the integration from 1 to 2 and this will be +. So, it will 2 x v dash u from here and – sign, – –, + will be here. So, it will be x square v dash u – v dash u + 2 v u dx.

So, I hope I have made all the calculations right. So, this is my J of u v + this quantity. So, I can just expand this one and then solving, so from here if I just want to write it directly, then it should be of the form x square v double dash + 4 x – 1 v dash + 4, this is 4 u dx, because this quantity will come from here, the x square taking the derivative v double dash + v dash 2x.

So, this will cancel out and it will get this value. Now, the thing is that what is the value of J u v. Now, if you see it here, I have the boundary condition, my u 1 is 0 and u 2 is this one. From here I can write that u dash 2 will be – of u 2. So, if you see that here my u 1 is 0, u 1 is 0, right? So, from here, I can define my J u v. I am just putting the limit.

So, if you are putting the limit here, it will be 4 v 2 u dash 2 – 1 v 1 u dash 1. So, this will be the first quantity + v 2 u 2 – v 1 u 1, okay, –, so this quantity will be there, so 2 to 4 v 2 u 2 – 2 v 1 u 1, so this quantity is there, – 4 v dash 2 u 2 – v dash 1 u 1. So, that is my given quantity.

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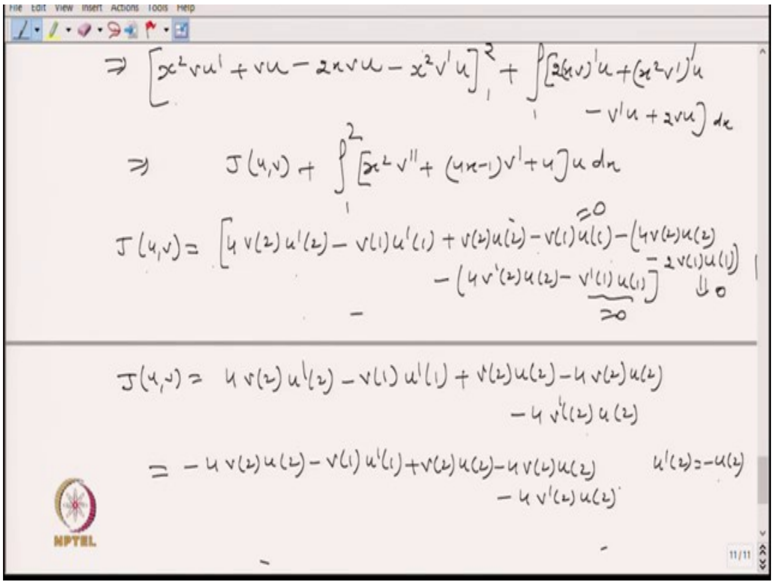

So, from here if you see then this quantity will be 0, this quantity will be 0, this quantity will be 0. So, we are left with only 4 v 2 u dash 2 – v 1 u dash 1 + v 2 u 2. So, this is 0 – 4 v 2 u 2, this is 0, – 4 v dash 2 and u 2. So, this is the quantity we are getting. And now I also have that u dash 2 = – u 2. So, if I put this value here, so from here I will get, so u dash 2 is here only, this one, right? So, I will get from here, so it will be – 4 v 2 u 2 – v 1 u dash 1, then + v 2 u 2 – 4 v 2 u 2 – 4, so this – v dash 2 u 2. So, this quantity will be remaining.

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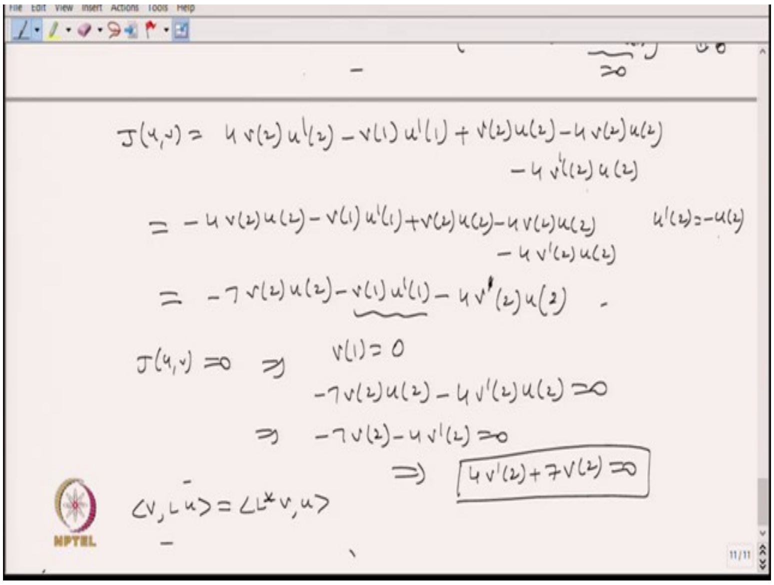

And from here, so v 2 u 2, v 2 u2, v 2 u 2, so it will be – –, so – 7 v 2 u 2. So, this quantity will be there, – v 1 u dash 1 – 4 v dash 2 u 2. So, from here, if I want my J u v = 0, so from here what I choose is that I choose v 1 = 0, so this quantity will be 0, and then from here I want my – 7 v 2 u 2 – 4 v dash 2 u 2 = 0.

So, from here I will get – 7 v 2 – 4 v dash 2 = 0 because u 2 I do not want to be 0, I do not know about the u 2. So, from here I will get this value and from here I will get the condition that 4 v dash 2 + 7 v 2 = 0. So, this is another boundary condition I got. So, from here I will get my v L u = L star v u, right, and that L star.

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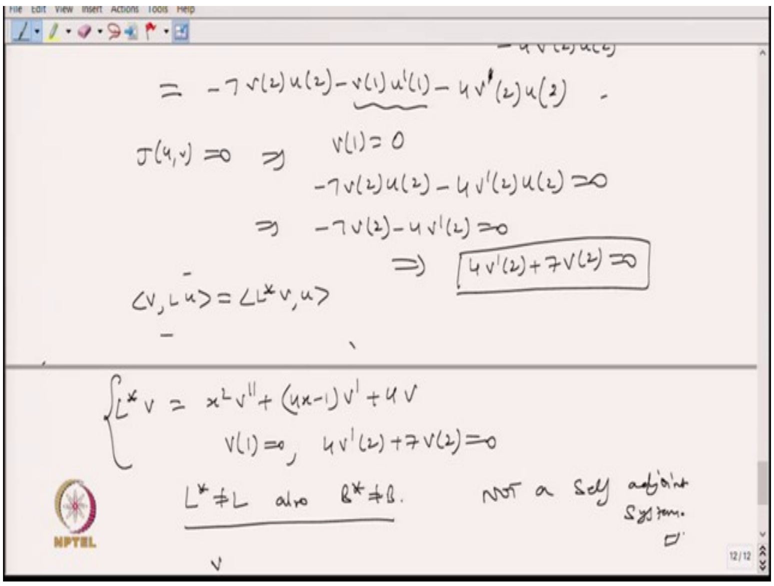

So, now my L star v in this case is x square v double dash + 4 x – 1 v dash + 4 v, so that is this one, and I get the bounding condition that my v 1 is 0 and 4 v dash 2 + 7 v 2 = 0. So, this is my corresponding adjoint system and in that case you can see from here that my L star is not equal to L, also my B star is not equal to B. So, in this case I can say that this is not a self-adjoint system.

So, in this lecture, we have discussed the concept of the adjoint, that how we can define the transpose of the matrix, and then extended that concept to the functional space and then we defined the adjoint operator for the linear differential operator. So, in the next class will go further and we will try to find out the relation between the Green's functions of the linear operator and its adjoint. Thank you very much for watching.