Introduction to Methods of Applied Mathematics
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Lecture No 10
Method of Green's function for solving initial value & boundary value problems

Welcome viewers, now we are going to discuss the lecture number 10 of this course.

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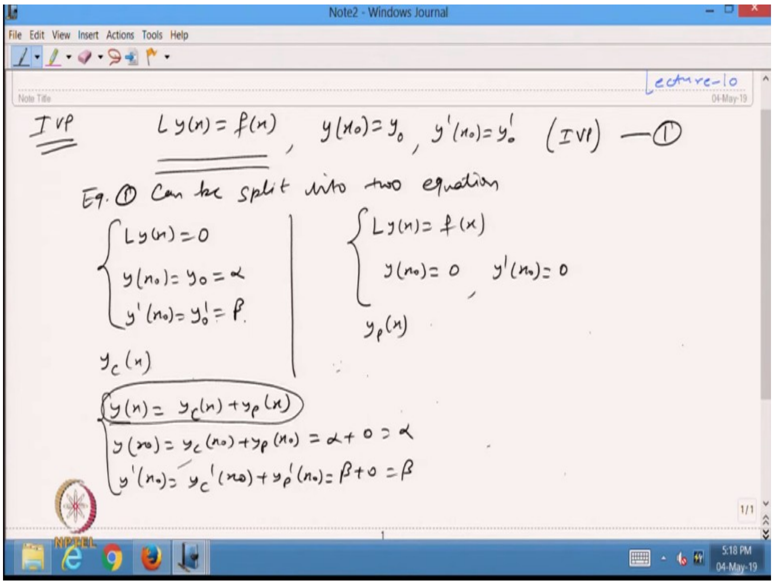
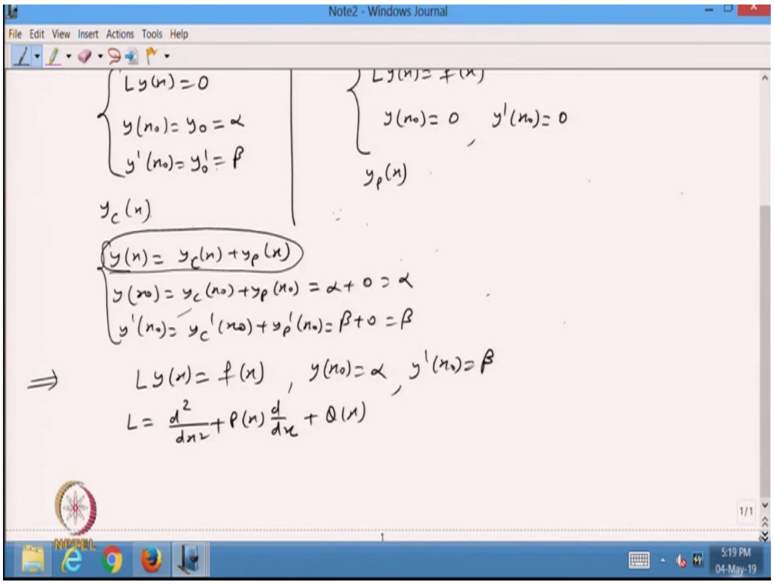




$$\Rightarrow \quad Ly(x) = 0 \qquad \text{Also} \quad LG(x,\xi) = \delta(x-\xi)$$

$$y(x_0) = \alpha \qquad \text{for } x \neq \xi$$

$$y'(x_0) = \beta \qquad \boxed{LG(x,\xi) = 0}$$

$$\Rightarrow \quad G(x_0,\xi) = \alpha, \quad G_x(x_0,\xi) = \beta$$

$$y_c(x) = C_1 y_1(x) + C_2 y_2(x)$$

$$\Rightarrow \quad C_1(x) = -\int \frac{y_2(x)}{W(x)} f(x)\, dx$$

$$y_c(x) = C_1 y_1(x) + C_2 y_2(x)$$

$$\Rightarrow \quad C_1(x) = -\int \frac{y_2(x)}{W(x)} f(x)\, dx, \qquad C_2(x) = \int \frac{y_1(x) f(x)}{W(x)} dx$$

To write the general Sol.

$$\Rightarrow \quad y(x) = y_c(x) + y_p(x) = C_1 y_1(x) + C_2 y_2(x) + \int -\frac{y_2(\xi) f(\xi)}{W(\xi)} y_1(x)\, d\xi + \int \frac{y_1(\xi) f(\xi)}{W(\xi)} y_2(x)\, d\xi$$

$$\Rightarrow C_1 y_1 + C_2 y_2 + \int_{x_0}^{x} \left[\frac{y_1(\xi) y_2(x) - y_2(\xi) y_1(x)}{W(\xi)}\right] f(\xi)\, d\xi \qquad x_0 < \xi < x$$

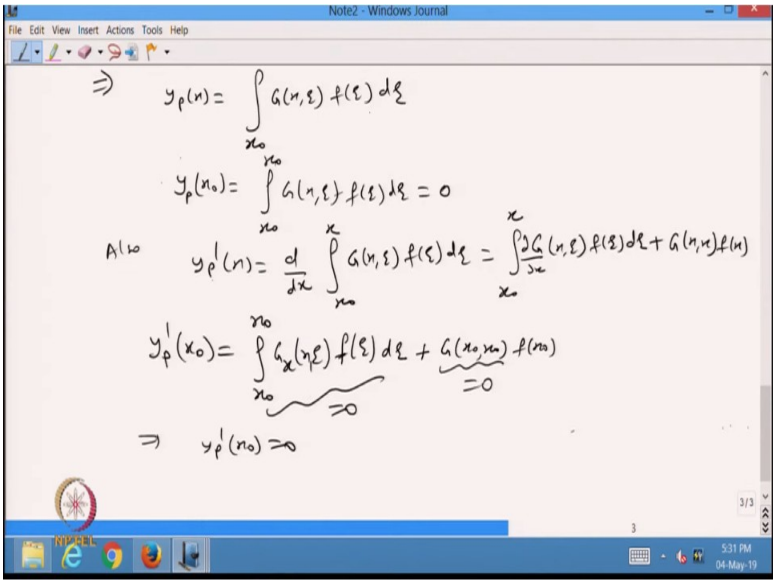
$$\Rightarrow \quad y(x) = C_1 y_1 + C_2 y_2 + \int_{x_0}^{x} G(x,\xi) f(\xi)\, d\xi$$

$$\Rightarrow \quad G(x,\xi) = \begin{cases} -\dfrac{y_2(\xi) y_1(x)}{W(\xi)} & x < \xi \\[2ex] \dfrac{y_1(\xi) y_2(x)}{W(\xi)} & x > \xi \end{cases}$$

$$\Rightarrow \quad y_p(x) = \int_{x_0}^{x} G(x,\xi) f(\xi)\, d\xi$$

$$y_p(x_0) = \int_{x_0}^{x_0} G(x,\xi) f(\xi)\, d\xi = 0$$

$$\text{Also} \quad y_p'(x) = \frac{d}{dx} \int_{x_0}^{x} G(x,\xi) f(\xi)\, d\xi =$$



In the last lecture we have started with the Green functions and now I want to develop the theory to solve initial value problem and the boundary value problem with the help of the Green function. So, I start with the first one, that is initial value problem. So, in this case, suppose I have my second order, any differential of equation, so that is Lyx = fx, this is the nonhomogeneous equation and in that case, I have the initial condition that y at x0 is some value I am taking.

So, let us take it as y0, so this is my y0 and another initial condition I have taken y dash at x0, so I call it y dash. So, this is my initial condition, and so this is my corresponding initial value problem. And I want to solve this equation with the help of the Green's function. So, this is my equation number 1. So, equation 1 can be split it into 2 equations, so we split this one with two types of equation, the first one is L of yx equal to 0.

This is the corresponding homogeneous equation satisfying the initial condition, that is y0 and y dash x0, so this is given to me. I can call it has alpha and beta also. And another equation I define as the non-homogeneous equation that is the equation we have, so this is my L(yx) = fx and which is satisfying the homogeneous initial condition with the help of, and y dash x0 = 0. Solving this equation I get my solution that will be my ycx and solving this equation, I will get the solution and I call it ypx.

So, in that case I know that my solution yx for the equation number 1 will be the linear combination of ycx+ypx. And if you see this one, then my y at x0 will be yc at x0 + yp at x0 and yc is the solution of this corresponding homogeneous, so this will be alpha and + 0, so

this will be alpha and y dash x0 will be yc dash at x0 + yp dash at x0 and this will be again, this will be beta and zero and it will be beta.
Okay, so this solution will satisfy, so my yx the solution I can write and that solution will satisfy the corresponding initial conditions.

So this is my solution of equation number 1 and this solution is satisfying the corresponding initial conditions given to me. Now, so we start with the general equation, so I have a general equation second order differential equation Lyx is equal to my fx, right. And then I have my initial condition, y at x0, I define it alpha and x dash x0, I define it beta. So, this is the, and my L in this case I am taking d square y over dx square + px d square over dx square + px dydx + qx, so these are same as we have defined the last class.

So, this is a linear second order differential operator. So, in this case, I will try to find out the Green's function. So, first I will solve the equation Lyx that is equal to 0, so this equation I solving with the given alpha and y dash x0 is equal to beta. But I know that, one thing I also know, that also I know that L of gx that is also equal to this one and I know that for x is not equal to Xi, so this equation becomes, is equal to 0.

So, it means that whatever the Green function I am going to find out, so this Green function satisfy this condition also. It means whatever the corresponding homogeneous equation I am solving, from where I will get my Green function. So, from here I can say that my Green function will also satisfy the initial condition, G dash, G dash means, because G is a function of two variable here, so from here I will say that Gx at x0 Xi, this equal to beta. So, this is I can define from my Green function.

So, now I want to solve this one, so I defined my function ycx = c1y1x + c2y2x, so this one we have defined. With the help of this y1 and y2x, we have found that my c1, that is a with the help of variation of parameter also, I found that my c1 is - y2 and this is x divided by dx and my c2x is y1x fx and, so from here, so this is the corresponding c1, c2 I found and to write the general solution, I will get my yx of the corresponding equation I call it equation number 1/

So the general solution equation number 1 I can write as ycx + ypx, okay, so ycx is c1y1x + c2y2x + ypx I can write as, so c1 is this one, so from here I can write it will be equal to - of y2, so I call it Xi, f Xi into y1x d Xi + y1 Xi f xi, w Xi, y2x dXi, so this one I can write. So, from here, we can write as, this is equal to c1y1 + c2y2 + and this integral I can write as y1Xi y2x – y2Xi y1x divided by the Wronskian, so this one f Xi d Xi.

So these are my solution, so that is my solution. So, from here I can define that my general solution is now I can say that this is c1y1+c2y2+, so this integral, actually if you see, that Xi belongs to, so in this case my Xi I can say that this Xi is, so I can take this integral between x0 to x. So, I can write this one as x0 to x my Green function as x Xi and this is f Xi d Xi, so this is my general solution for the equation number 1.

So, from here, I can define that my Green function will be - y2 Xi when x is less than Xi and y1 Xi over Wronskian when x is greater than Xi. So, this is my Green function in that case. To y1x and this will be y2x, so this one my. So from here, so my particular solution I can say that my ypx will be x0 to x, f Xi d Xi. So, this is the particular solution I found. Now from here, what about if I apply my yp at x0. So, if I apply this one, this integral from x0 to x0, d Xi and this value will be 0.

Also, yp dash, so I am taking the derivative with respect to x. So, with the help of the Leibnitz theorem, this can be written as from x0 to x, so this will become del G by del x f Xi d Xi + putting the value of x here and this 2 to Xi will be the value of this, so that will be G xx and it will be equal to fx and this x naught is a constant so we will get this. So from here I will get this one, so this can be written as, so from here if I want to apply what will be value of yp dash at x0, so this will be again x0 to x0, that is my Gx, this one + G of x0 and my f of x0.

So this one I will continue, so this value is equal to 0, and what about this one, this is also 0, because I know that my x0 is the initial condition and Gx is the solution of the corresponding homogeneous equation, so this is equal to 0 and so from here I can say that yp, whatever the particular solution I am having that solution corresponding homogeneous initial condition that is, this is equal to 0 and this is equal to 0.

So we start with the, if you see this one, we have started with a problem, this one and then we split this one into two parts so this one we have used to find out the Green functions and the corresponding values of c1 and c2, and then with the help of the particular solution we found we have showed that they satisfy this homogeneous initial condition, and then that is ypx is our particular solution.

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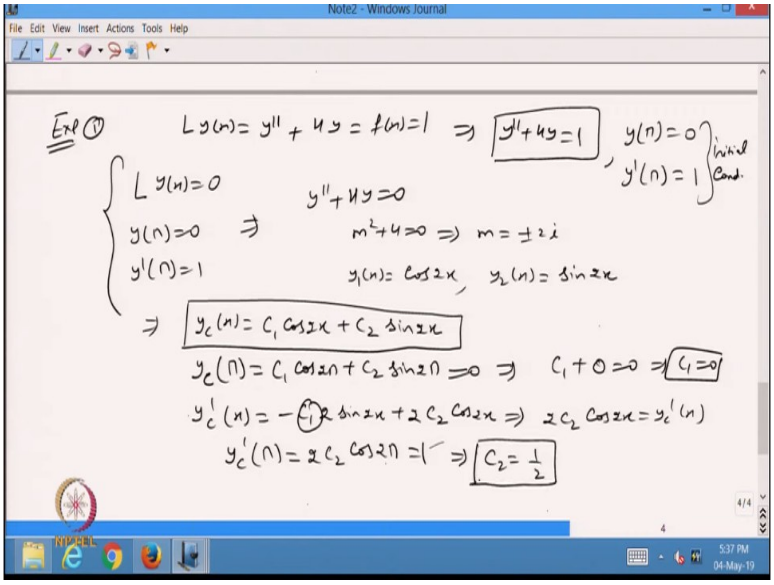

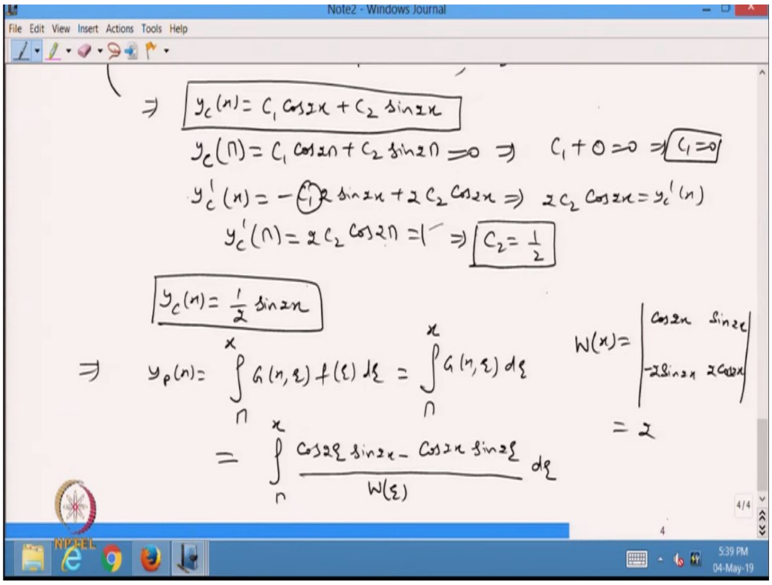

So, now, let us take one example, then the things will be more clear, so let us take one example, example number 1, I want solve my equation, so L yx I am taking y double dash + 4y = fx and that I am choosing just 1. So, basically I am solving, if you see, y double dash + 4y = 1, so this one we are solving. And I already know that how to solve this one because this is a linear second order equation with constant coefficients and the right hand side function is just 1.

But I want to solve this one with the help of Green function. So, first I will solve the equation, L(yx) = 0, now I putting the initial condition also, so I will define at y Pi = 0 and y dash Pi = 1, so this is my initial condition. So I want to solve this one, so this is initial condition, y dash Pi = 1. So I want to solve this one. So if I solve this one, then from my previous knowledge, I know that 4y = 0 and then I can find out my corresponding characteristic equation that will be m square + 4 = 0.

And from here, I will get my m is + - 2 iota and from there I can define my solution y1x as cos 2x and y2x as sin 2x. So from here, I can define my linearly independent solution that is cos 2x and sin 2x. So from here, I will get my solution ycx will be cos, so I want to define this one, so this one will be c1 cos 2x + c2 sin 2x. Now, I will try to find out the value of c1 and c2 from here. So, this is my corresponding complimentary solution.

So complimentary solution I can find out, and then from here I can find my, it is given that yc at Pi that is giving to me, so this will be c1 cos 2Pi + c2 sin 2Pi and that value is given equal to 0. So, from here I know that, cos 2 Pi will be 1, so it will be c1 + and then it will be 0 = 0, from here I calculate that c1 = 0. And then, I will also define yc dash at x. So yc dash = x, I will define, so this will be - c1 two times sin 2x + 2c2 cos 2x, but this value is already, because c1 is already 0.

So from here I will get, 2 times c2 cos 2x = 1. So this the derivative I have **is** that is equal to my ycx because c1 is already 0 from this initial condition, first initial condition. So, from here I can say that my yc at Pi will be 2c2 cos 2Pi and that is equal to 1. So, form here, I can say, and the cos 2Pi is 1. So from here I can say, that my c2 will be 1 by 2. So from here, I am able to find my ycx, will be c1 is 0, so it will be, of sin 2x.

So this is I am able to find. Now, I want to find the corresponding Green functions with the help of this one, so from here I can say that my corresponding Green functions, so now I want to find my ypx, so ypx will be from Pi to x, d Xi. So f Xi is 1 here in this case, so it will be Pi x d Xi, where I know that my Green function will be in this case, it will be cos 2 Xi into sin 2x – - cos 2x sin 2 xi divided by the Wronskian, so the Wronskian into d Xi.

So, from here, the Wronskian I know, that the Wronskian in this case will be, so this is my cos 2x and this is sin 2x, so this will be - 2 sin 2x, and this will be 2 cos 2x and that value will be equal to 2, so in this case I will get the Wronskian that is equal to 2. So generally we take the positive value if it is coming negative.

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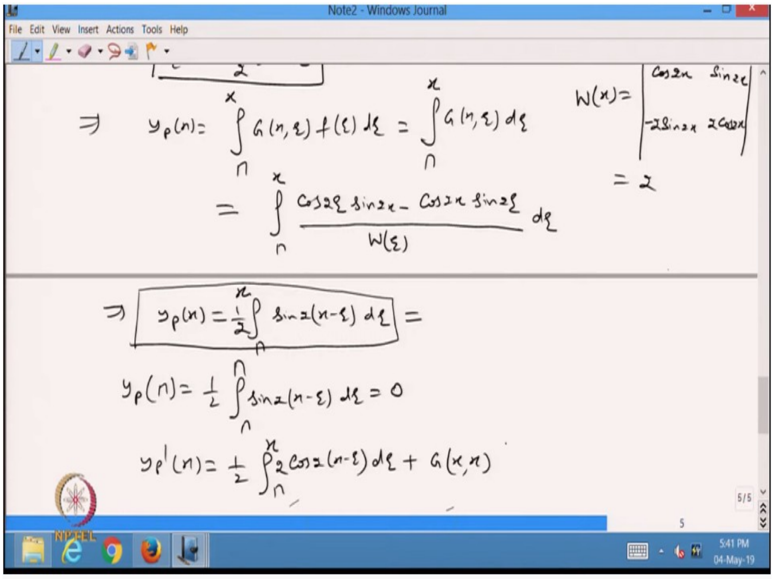

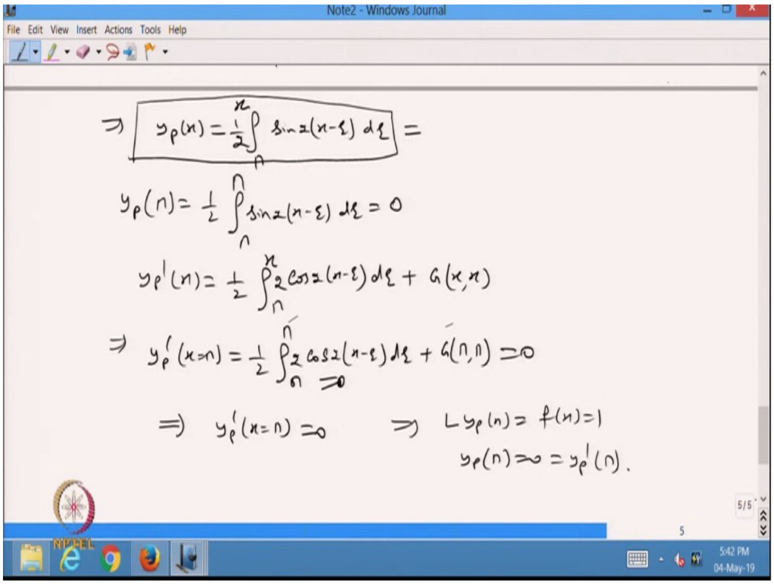

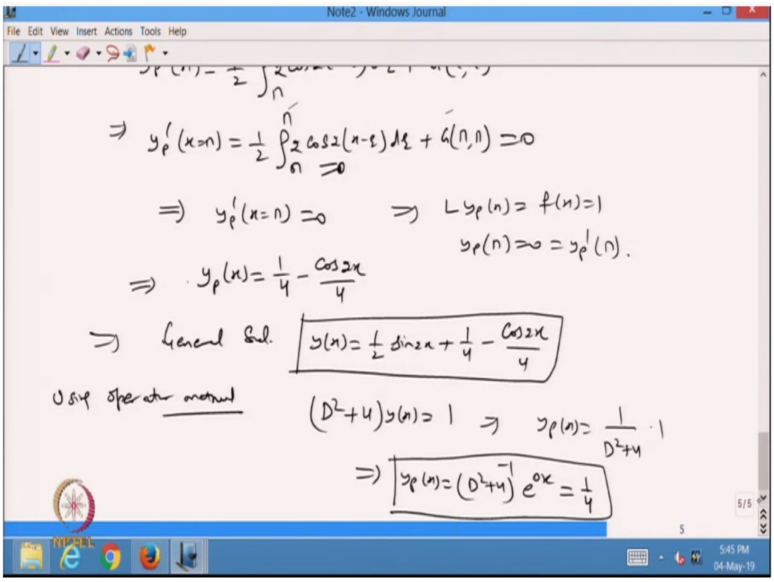

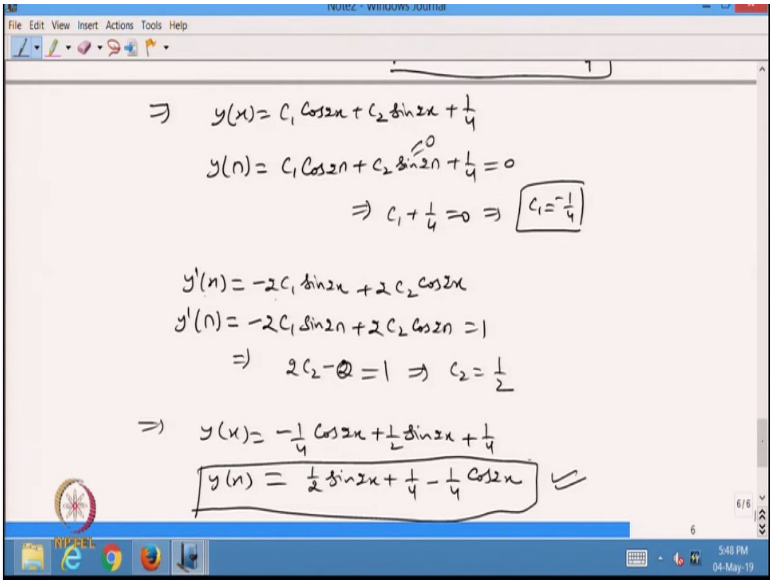

So, from here, I can define my ypx will be from Pi to x and this will be half, so 1 by 2 and this one I can write as sin 2x – Xi d Xi. Okay, so this one I can define from here and if I solve this one, so this is my ypx. So, ypx I am able to found, so this ypx now I take, what will be the value of yp at Pi? So yp at Pi this will be the integral. From Pi to Pi, sin to x - Xi d Xi and this will be value equal to 0, then yp dash x will be again.

So, in taking the derivative of, so it would be cos 2x – Xi into 2 d Xi + then we found the Green function, so this will be xx that will be the value. Because I am putting this upper limit in the function and then the function fx in this case it will be 1. So, from here, I just define that what will be the value of yp dash at x = Pi. So, in this case, this integral become 0, 2 cos + G Pi Pi and this is equal to 0 and this will be equal to 0 because G Pi Pi is the solutions of the corresponding homogeneous equations.

From here, I can say that my yp at x equal to Pi will be 0. So, from here I can say that my ypx Lyp that my L of yp = fx that is equal to 1 and it is satisfying the homogeneous initial condition and that is this equal to 0 and this is again yp dash = 0. So from here, I am able to solve this one and now, if I solve the integral value, then my yp will be, if I just want to take the integration of this one, so from here, my yp x will be in this case, it will be 1 by 4.

So, if I just, ycx is half so it will be 1 by 4 – cos 2x by 4, so then, my general solution yx will be 1 by 2 sin 2x + 1 by 4 – cos 2x by 4, so that will be my general solution for the given equation. If I solve the same equation with the way we have solved in the previous lecture and you can verify also that, if I use the operator method, then I can find this equation as d square + 4 yx = 1 and from here, my ypx I can define as 1 over d square + 4 into 1 and this one I know that we can write as d square + 4 inverse and 1, this 1 I can write as 0x.

And from here, I find that putting the value of 0 instead of d, so it will get 1 over 4. So my ypx in this case it will be 1 by 4.

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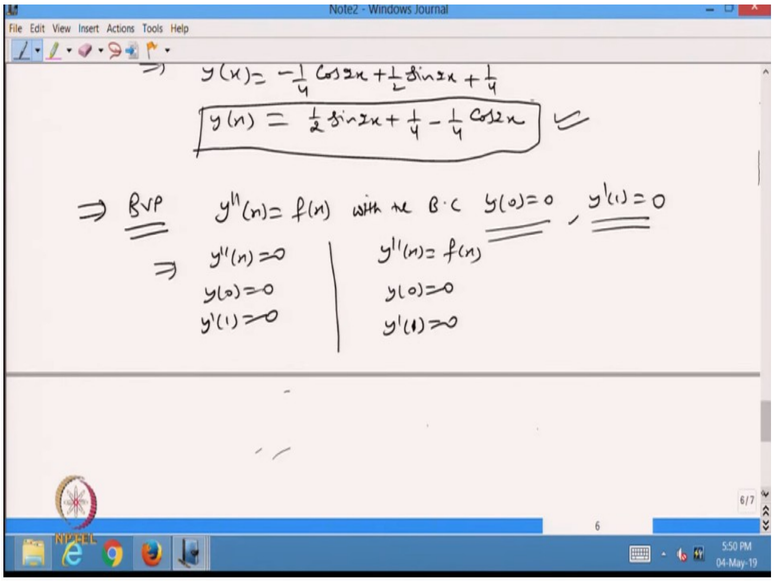

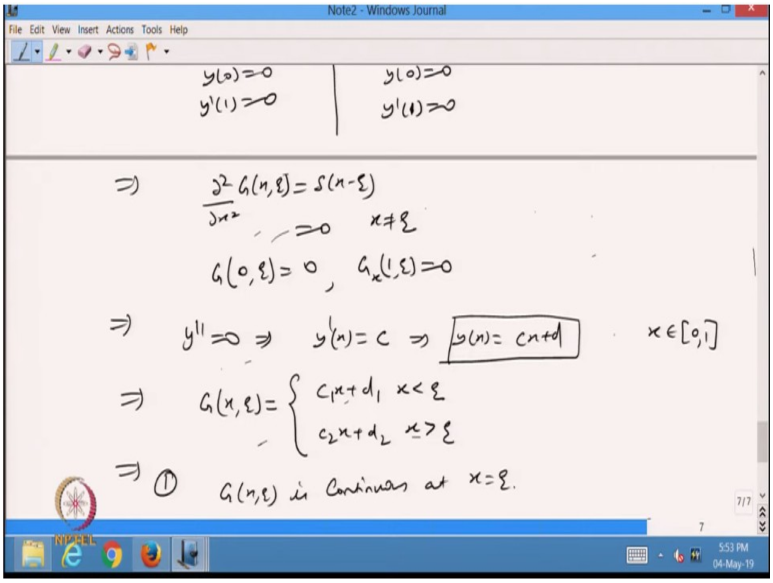

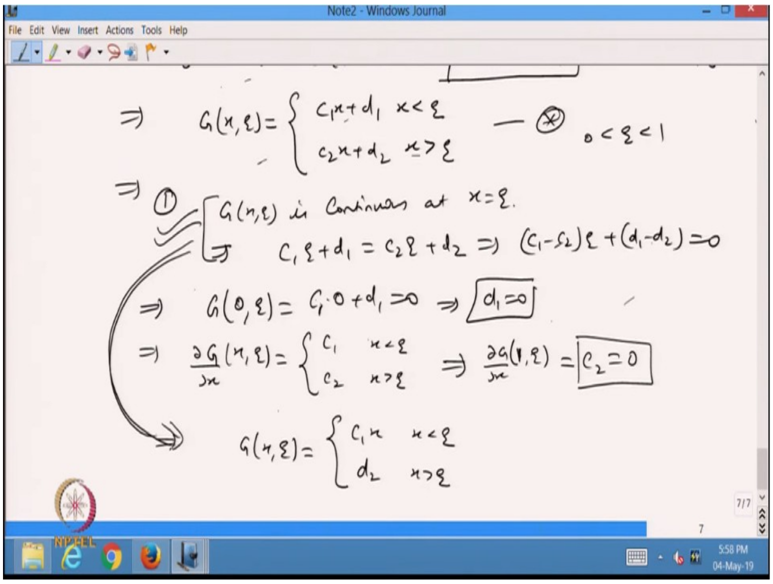

So from here I can find my solution and then, so if I solve this one, my general solution will be, so this general solution in that case will be c1 cos 2x + c2 sin 2x + 1 by 4 and then apply my initial condition, so my y of Pi will be c1 cos 2 Pi + c2 sin 2 Pi + 1 by 4 and this value is given to me is equal to 0. So, from here, so this is 0, this is 1. So from here, I can get my c1 + 1 by 4 = 0. So my c1 will be -1 by 4.

And then, I will take the derivative of the solution, y dash at x, so this one will get -2 c1 sin 2x + 2 times c2 cos 2x, so this is my derivative and then I put value equal to Pi. So, in that case, it will be -2 c1 sin 2 Pi + 2c2 cos 2 Pi, and that value is given to me, that is equal to 1. So, from here, I can say, so my c1 is already -1 by 4. So, this one I can write as 2 times c2, so cos 2 Pi is 1, so I can write this as -2. So, c1 is but this value is always 0, so this is equal to 0 and from here, I will get 2c2 and that is equal to 1.

So, from here, my c2 is 1 by 2. So my c1 is -1 by 4 and c2 is 1 by 2. So, from here, I can write my general solution that is yx is equal to from here, so c1 is -1 by 4, so it is -1 by 4 cos 2x, c2 is half so half sin 2x + 1 by 4 and from here I can write that it is equal half sin 2x + 1 by 4 -1 by 4 cos 2x, so this is the same solution we got with the help of Green function. So, this solution and the solution we have solved using the Green function, so this and this, both are same.

So from here we verify that, whether we are solving this equation with the help of the operator method or we are solving the same equation with the Green functions, we are getting the same solution. Now, I take the, so this is the initial value problem that I have taken, now let us start with the boundary value problem. So I take the boundary value problem. So, in the boundary value problem, so let us see that how we are able to find the Green functions.

So let us take one example of that boundary value problem, so let us solve this equation. I am going to solve the equation y double dash x is equal to some function fx with the boundary condition, so boundary condition I am taking y0 = 0 and y dash 1 = 0, so this is the boundary condition I am taking. So this is the mixed boundary condition. It is, if we see, this is a Dirichlet type boundary condition and this is a Neumann boundary condition. So it is a mixed type of, we have a function and its derivative.

So, in this case, I will solve the Green function. So, the same way we will split this one into two parts; first one I will solve like this one y double dash x = 0 with our boundary condition, whatever the boundary condition we have and then I will solve the whole equation, yx = fx with the homogeneous boundary condition. But in this case, the boundary conditions are already homogeneous, so the same boundary condition I am putting for the both the equation.

Now from here, and I also know, that the second derivative remains del square over del x square, this will be equal to the Dirac Delta function, okay. And this will be equal to 0 when x is not equal to Xi. It means that in that case also, I can say that 0 will be 0 and Gx at 1 Xi, this will be 0 in this case. Now, if I want to solve this one, so solving my y dash = 0, taking the integration one more time, so I am taking my y dash is equal to some constant c and from here I will get my yx I will write as a cx + d. So, this is my equation of the line, okay.

So, this my yx = cx + d. So from here, and now in this case my x, so if you see here, my x is lying between 0 and 1. Because its value is given at this one. From here, I can say that this is my solution of the equation and x belongs to, I can say, it is belonging from 0 to 1. Now, from here, I can define my Green's function. So, this Green function will be cx + d when x is less than Xi and so I can just define as c1 d1 and then c2x + d2x is greater than Xi.

So, this one is a Green function we know, I can define from here and then I can find the value of c1, d1, c2, d2. So, this is I am doing with the help of the properties of the Green functions, and from here, I will apply the properties. The first property is I am applying that my function G is continuous at x = Xi.

Because you can find out the Green function by solving this homogeneous equation also and then we put the boundary condition and then we can find the corresponding coefficients. But here I am applying the another method that is with the help of the properties of the Green function. From here, I can write that c1 Xi + d1 = c2 Xi + d2. So from here, I can write my c1 – c2 Xi + d1 – d2 that is equal to 0, okay.

Now, I also know that this conditions, the boundary conditions are satisfied for the given function, so because in this case, I already know that I can, from the another equation, the jump discontinuity, I can find the another equation and using those equations, we can have **a** two equation and the 4 variables, c1, c2, d1, d2 to find.

So, what I can do is that, I can apply this boundary condition for the given function also. So this the corresponding Green function, so from here I can say that G of 0 Xi, so G of 0 Xi because x is less than Xi I am taking and Xi is in this case, will be lying between 0 and 1. So I can say that this will be c1, so it will be c1 into 0 + d1 and that value is given to me. So, from here, I can say that d1 is equal 0. So, from here I can able to eliminate the d1.

Now, I also know that G, the del g by del x, so let us find out this one, that what will be the value of this one, so I just want to find out this one. So this value equal to, so it will be c1, I am taking the derivative with respect to x, when x is less than Xi and it will be c2 when x is greater than Xi. And from here, I am able to find this value, so from here, I can put my del G by del x at 1 Xi. So it means the, so I am putting x = 1 here.

So I am taking this condition that x is greater than Xi. So, in this case, I can say that my c2, this is given to me, and this value is equal to 0. So from here, I can also able to eliminate c2 = 0. So from here, I can now write my Green function. So this Green function is c1x when x is less than Xi and c20, so it is d2 when x is greater than Xi.

So, now we have only two c1 and c2. Now, what I will do, I will take the help of this properties, because just I wanted to use this one so with the help of this continuity equation and jump equation, I can find the value of c1 and d1.

So this part I will do now for this Green function because this is the Green function after putting the boundary condition.

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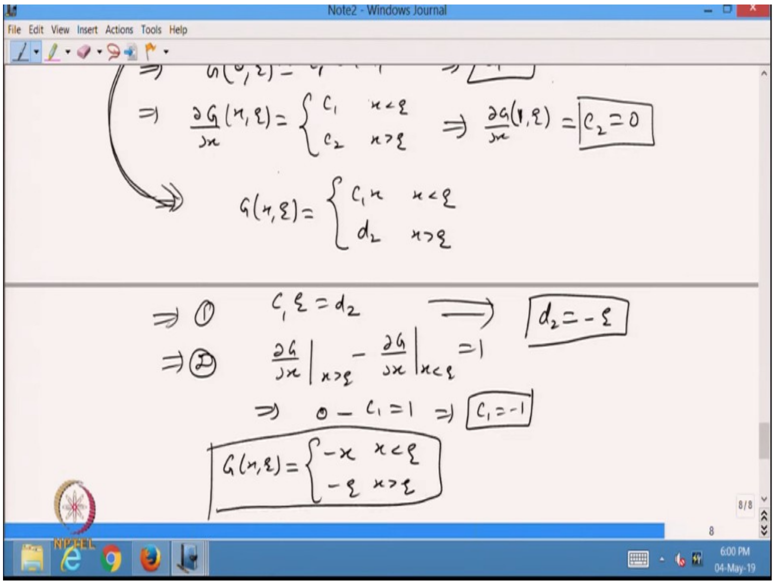

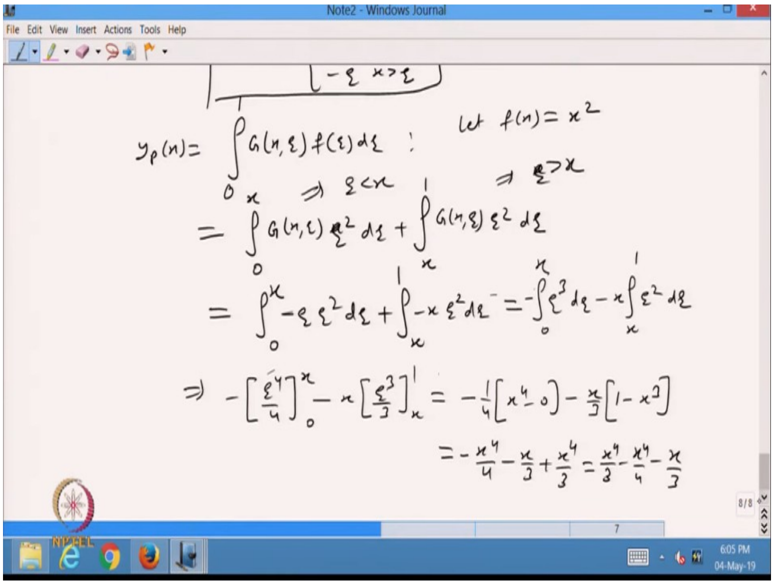

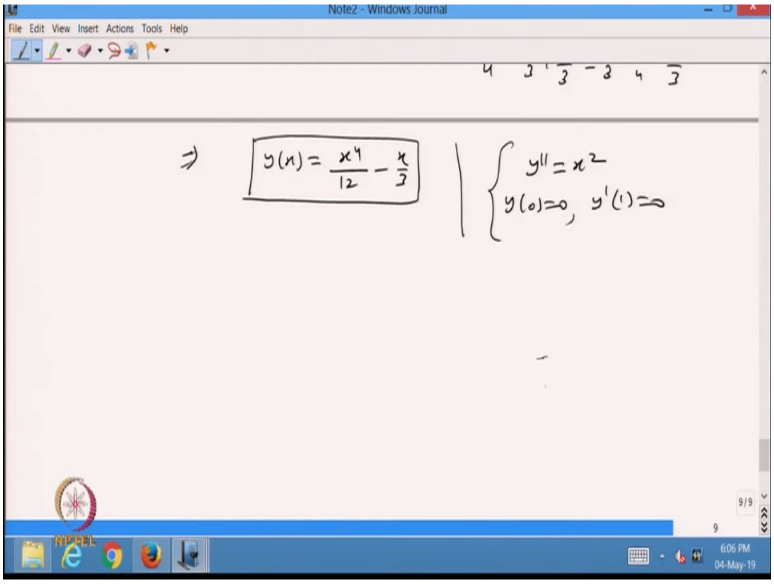

So from here, first property it is continuous at Xi so from here I can say that c1 Xi into Xi = d2. So this is one of the equation and from the second property, that second property is that del G by del x has the jump discontinuity of magnitude 1 about x = Xi. So, from here, I can say that del G by del x – del G by del x = 1. From here, I can say that, so this is my d2, so it will be 0 – c1 and that is equal to 1.

So, from here, I can that my c1 is -1. So once I have c1 = -1, from here with the help of this one, so c1 = -1, d2 in this case will be - of Xi. So with the help of this one, I am able to find my Green function now, so c1, this is the Green function. So c1, so I can define this function as –x when x is less than Xi and d2 is –Xi so this is –Xi when x = Xi, so this is my Green function.

So, once I am able to define this Green function, so now, I can find my particular solution ypx, so ypx will be from 0 to 1 G, d Xi. So this one I want to find. Now, so, with the help of this one, ypx is equal to from 0 to 1, Gx this one, I am able to find my ypx. So, in this case, because if you see we have started with the equation with fx is unknown, so now I define the fx, let us take a simple one. So let us take fx equal to x square.

So if I take x square, then my ypx I can define as from 0 to x, my Gx into x square into Xi square d Xi + from x to 1, d Xi. Why I have done that, because the Green function is defined for in the p's y's form when x is less than Xi, this value and this value so from here, I can define my, so this will be, so when my Xi is lying between 0 to x, it means when my Xi is, so from here I can say, in this condition, my Xi is less than x and in this one, my Xi is greater than x, because Xi is lying between x and 1.

So when x is greater than Xi my this is value Xi, so I can put from here 0 to x and then this will be –Xi into Xi square and d Xi + x to 1 and this value is –x, so it will be –x Xi square d Xi. This one can be written as Xi square d Xi. So, I will just solve this one, 0 to X - of Xi cube d Xi -x to 1, so this x I can take outside, this will be Xi square d Xi, okay.

Now, I will solve this further, so from here, I will get, so it will be Xi 4 by 4 applying the conditions here, -x and this will be Xi cube by x to 1. So, from here, I will get 1 by 4, so it will be x power 4 – 0 – x by 3 and this will be 1 – x cube, so from here this one, I am putting this limit. So once I get this value, I will get – x power 4 by 4 – x by 3 + x4 by 4. So, from here, if you see, I can write this value, x4 by 3 – x4 by 4 – x by 3 and this one can be written as that my yx will be x power 4 12 – x by 3.

So, this is the solution for the corresponding equation with the help of Green functions. You can solve this equation, because if you see, that I have solved this equation, y double dash is equal to x square, with initial condition that y at 0 is 0 and y dash at 1 is 0. So, you can also solve this equation with the previous method where we were solving the differential equation with constant coefficients and the right hand side function is a polynomial, so that is x square.

So, if you solve this one with the previous methods, then we should get the same solution as this. So, this is you can take as homework to verify whether you are getting the same solution for this one with the operator method or not. So thanks very much for this lecture, so in this lecture we have discussed that how we can develop the Green function for initial problem and the boundary value problem.

And then we also solved few examples and verified that, that if we are applying the Green function or the methods we have solved with the help of operators, then we found that the both results are coming same. So in the next lecture, we will go further and we will try to find and apply the Green function for some other problems. Thank you very much.