Introduction to Methods of Applied Mathematics
Prof. Vivek Aggarwal & Prof. Mani Mehra
Department of Mathematics
Indian Institute of Technology-Delhi

Lecture - 01
Introduction to First Order Differential Equations

Hello viewers. So welcome to this course. This is your first lecture. So we start with this lecture with the very basic differential equation that is a first order linear differential equation.

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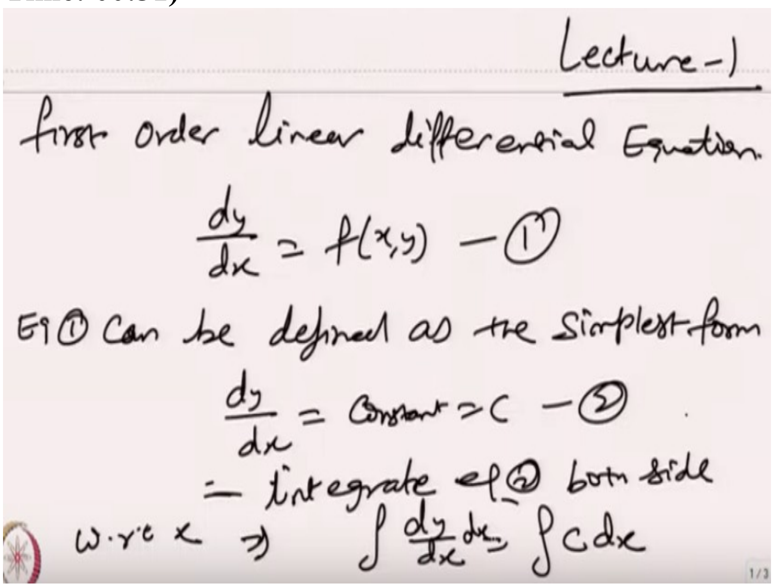

We know that a linear first order linear differential equation. So I start with the first order. So this is the simplest differential equation, first order differential equation, dy/dx = f(x, y) where y is a dependent variable and x is a independent variable and my function on the right hand side is a function of x and y. So we know that this differential equation we can solve by applying the method of integration if the right hand side function f(x, y) is integrable.

So I want to solve these differential equation. So I start with the simplest one. So I take the equation 1. Equation 1 can be, it can be defined as the simplest form. I have dy/dx is equal to some constant. That is equal to some c and I know that the constant c is a (integrable) function. So to solve this differential equation, what I do is I integrate equation (2) both side with respect to x. We get dy/dx x is equal to integration of some constant cx.

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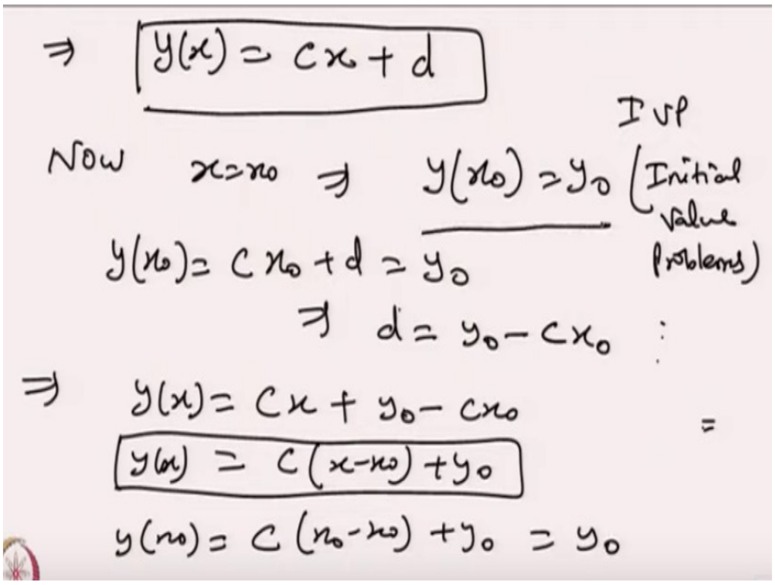

Now we know that **by** with respect to d x. Now, by the fundamental theorem of calculus, we know that on the left hand side we are taking the derivative and then integrating the function again. So from the left hand side we will get function y x is equal to c x plus constant of integration d.

So this is the solution of the differential equation 2, and I am able to solve this differential equation, because I am able to do the integration, where d is a constant of integration. Now this is the solution and for the different different values of the constant d, I can have different solutions. So in this case I can say that the solution exist and it has infinite many of solutions, because I can take the different values of d and I will get the different value of solutions.

So and if you see this is a equation of a line and if I take put equation of line. You put d = 1 there is another line, d = -1 there is another line. So these are all the isoclines we can draw and then it gives the family of the solutions. Now out of this family of the solution, suppose I want to take a solution, which starts with some points x = x naught.

So I define a condition that I want a solution which has, which start at the point x naught and it has initial value that is equal to y at x naught is a y naught. So such type of equation we know it is called initial value problems. So in the short form I will write this as IVP. Now suppose I defined this one. So the above solution I can write as is equal to c at x naught plus d and that value is equal to y naught.

So from here I can define the value of d that is equal to y naught minus cx naught. So my solution becomes yx becomes cx + y naught - cx naught, which can be further simplified and it becomes c (x - x naught) + y naught. So that is my solution. And this is a unique solution, which is starting from the point x naught. To verify this one I can put let us put in this case y at x naught.

So y at x naught become c (x naught – x naught) + y naught. So that becomes a y naught. So this is the solution corresponding to this initial condition. So after solving this equation, I will move further and I will try to make this differential equation, first order differential equation little bit harder by putting that dy/dx is equal to some function of x.

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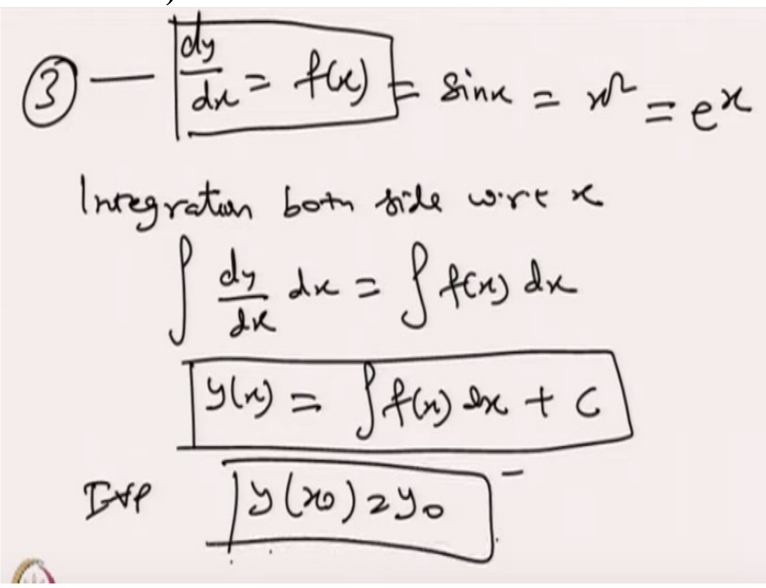

Where my fx is a function which is integrable. Integrable means which we can do the integration. It may be continuous or it will be a discontinuation then it has a finite number of discontinuities. So suppose I have fx maybe it can be sin x, it can be x square, it can be exponential function, anything it can be. So I want to solve now the differential equation of this type.

Now I in this case also my function fx is integrable, then I will do the integration both side with respect to x. So my left hand side becomes dy/dx dx = f(x) dx. Now my function is integrable. So this integration is possible, it will give you some finite solution and on the left hand side by the fundamental theorem of calculus, I will get y(x) on the left hand side becomes equal to the integration f(x) dx plus some constant of integration because on the left hand side I have taken the constant of integration.

So this is the solution of the differential equation 3. So in this case, this equation is also solvable because this is a linear equation and the right hand side function fx is integrable. So we are able to solve this differential equation in the same way and then if I apply the same initial value problem by putting some y(x naught) = y naught, then I am able to find the solution at I am able to find a unique solution by getting some value for this constant of integration c.

So this is also solvable for this initial value. I will write a general linear first order differential equation.

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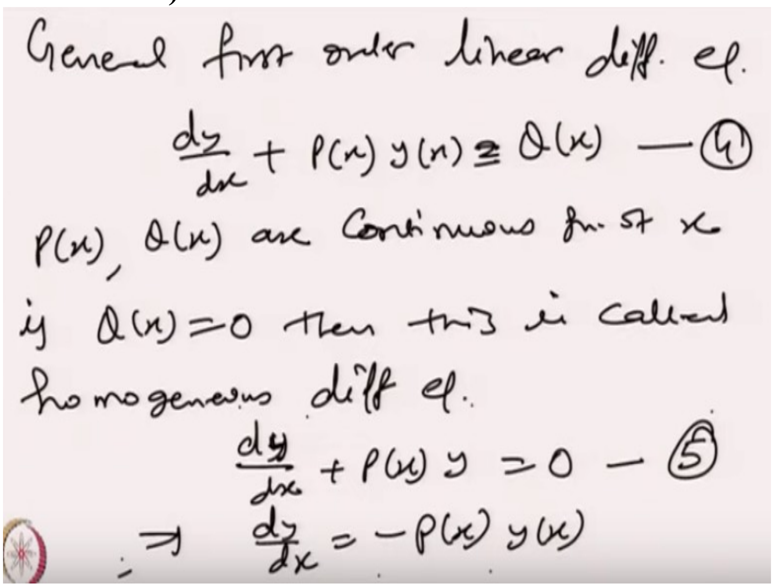

So first order, the general first order linear differential equation. So I start with the differential equation dy/dx plus some function px yx is equal to some function on the right hand side that is qx. So in this case, I consider that my function Px and Qx are continuous, or continuous function of x. And we know that if the function are continuous, then they are integrable.

So this equation, this is the fourth equation. So if Q(x) = 0 then this is called homogeneous differential equation. And then in that case, we are left with dy/dx +

P(x) y = 0. So in this case, if my Qx is 0 then I am able to solve I should be able to solve this differential equation. So this is the homogeneous first order linear differential equation. So I call it equation number 5.

Now from the from my previous knowledge, I was able to solve the differential equation, first order differential equation in which I have a function that is dy/dx equal to some function of x only. But in this case, I have dy/dx = -P(x) y(x). So in this case on the right hand side, I have some product of two functions or I can have a functions which is dependent variable also appearing on the right hand side.

So let us see how to solve this type of differential equation. So in this case, how we will proceed. So let us do this one.

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$$\frac{dy}{dx} = -P(x)\, y(x)$$

$$\frac{\frac{dy}{dx}}{y(x)} = -P(x) \qquad y(x) \neq 0$$

Integrate both side w.r.t x

$$\int \frac{\frac{dy}{dx}}{y(x)}\, dx = -\int P(x)\, dx$$

$$\Rightarrow \ln|y(x)| = -\int P(x)\, dx + C$$

$$\Rightarrow |y(x)| = e^{-\int P(x)\, dx + C}$$

So I have this differential equation dy/dx = - P(x) y(x). What I do is I take this function yx on the left hand side and on the right hand side, I have only this left with Px. So in this case, we are assuming that my yx is not equal to 0, because only then I am able to divide this one. Then I integrate both side with respect to x. So on the left hand side I have dy/dx divide by yx.

And on the right hand side I have –P(x) dx. Now on the left hand side what I have? I have the function and above that the numerator its derivative is there. So I know that the integration of this function can be written as log, log means it is I am taking the natural logarithm, yx = -P(x) dx. So that we know that how to integrate on the left hand side. And on the right hand side, I have a function Px and this is the integration we can take. So from that one now and c is the constant of integration I am taking.

So from here why I am taking the modulus value because I am taking the logarithm and we know that log is defined for the positive functions, for the positive value. So this is the positive function. Now from here I want to find my yx. So I take the logarithm, antilogarithm both side and then it becomes exponential – P(x) dx + c.

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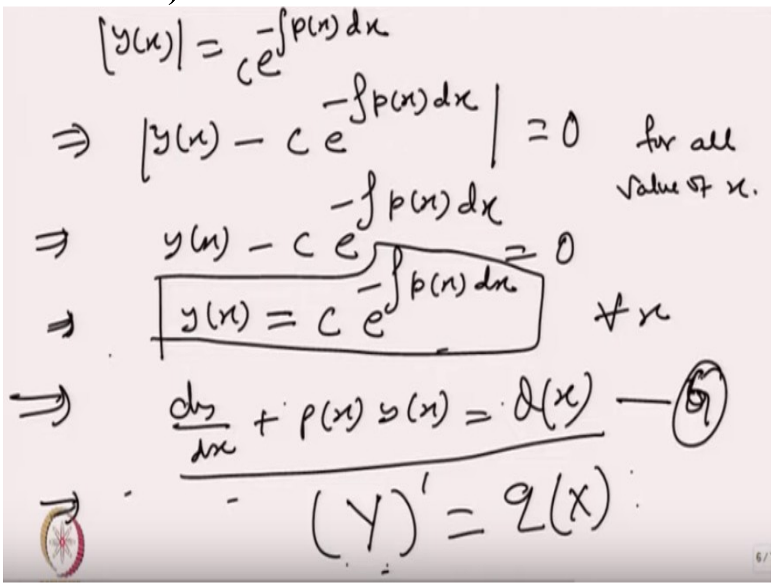

So this one I can define as yx modulus value is equal to e - Px dx. So this is a minus value, minus, and this is the constant of integration I can multiply by c, this is a new c, okay. So from here I can write from this. Now, so this exponentially is always positive value. So from here I can write yx and taking this on the left hand c e raise to power minus Px dx modulus value that is always equal to 1 in this case.

So this is true for all value of x and the function. So if the function is, the modulus of some function is equal to some constant for all value of x, then we know that then that function itself is a constant function. So from here I can define that this itself is 1 because for all value of x, the modulus value is equal to 1 and then we know that if it is happening then the function itself is equal to one.

So from here I can write that my yx is equal to 1 no sorry. What happened? So from here I have taken this one from the right side. So this one. So this is equal to zero sorry, because I have taken on the left hand side. So from here I can write, so should I erase this one. So this becomes 0 and this becomes 0. So from here, I can write this function as so yx can be written as c e px dx.

So that is my solution. So this is true for all x. So this is the general solution for the corresponding homogeneous first order differential equation. So we are going to use this one, to solve little bit harder problem. Now so after equipped with the methods to solve a simpler forms. Now I am ready to solve the general form. So that general form I have written that dy/dx + P(x) y(x) = Q(x).

So Q(x), now as in the mathematics we know that whatever the knowledge we have gained till now we want to use that knowledge to solve the tougher problem. So in this case, we know that how to solve the differential equations, which has some simpler form. So we use that knowledge to solve this equation. So what we will do that. Now we will try to convert this equation into those form which are we are able to solve now till now.

So in this case, I want to make or I want to convert this differential equation into some of the form that some of the function, some of the functions F that I am able to solve Y dash is equal to some qx. So I want to convert this equation into this form and then we know that I can integrate this one and it will be the solution of that equation. So how do we can do that?

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Multiply eq (G) with some function $\mu(x)$

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x) \quad - \otimes$$

$$\Rightarrow \quad \frac{d}{dx}\left(\mu(x)y(x)\right) = \mu(x)Q(x)$$

$$\Rightarrow \quad \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y(x) = \mu(x)Q(x) \quad - \otimes$$

$$\Rightarrow \text{ if } \quad \frac{d\mu}{dx} = \mu(x)P(x)$$

$$\Rightarrow \quad \mu(x) = C\,e^{+\int P(x)\,dx} \qquad \boxed{C=1}$$

So now what I want to do, I multiply the equation the general equation, this equation I can write as, I can write as G. So this is the general equation I want to solve. So multiply equation G with some functions. So let us call it mu x, then we have mu x dy/dx +mu x P(x) y = mu x Q(x). So that is a function I am taking.

So I have multiplied this equation by the mu x such that, on the left hand side I should be getting a function which can be put in the form of a differentiation together. Now what I want is till now I am able to solve the differential equation. So I want to put this expression in this form that d/dx = (mu x y(x)) such that it should be equal to mu x Q(x). So this one I can take the derivative by the product rule.

So I can have this mu x dy/dx + d mu/dx yx. So that should be equal to mu x Q(x). So I want to reduce my equation in this form. So this is possible if I take that d mu/dx = mu x P(x) because if you compare this equation, this equation and this equation. So in this case, if I want to convert this one into this form then my d mu by dx should be equal to mu x Px.

And this is I know that I have already solved this equation and this equation has the solution mu x is equal to some constant and then e raised to power integration. So in this case P is there, so Px dx. So we are able to solve this form. Now for the simplicity I can take my c = 1.

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$$\boxed{\mu(x) = e^{+\int P(x)\,dx}}$$

$$\Rightarrow e^{\int P(x)\,dx}\frac{dy}{dx} + e^{\int p(x)\,dx}\,p(x)\,y = e^{\int p(x)\,dx}\,Q(x)$$

$$\Rightarrow \left(e^{\int p(x)\,dx}\,y(x)\right)' = Q(x)\,e^{\int p(x)\,dx}$$

$$\Rightarrow e^{\int p(x)\,dx}\,y(x) = \int Q(x)\,e^{\int p(x)\,dx} + c$$

$$\Rightarrow \boxed{y(x) = e^{-\int p(x)\,dx}\int Q(x)\,e^{\int p(x)\,dx} + c\,e^{-\int p(x)\,dx}}.$$

So from here I can write my mu x is equal to e power plus Px dx. So I know the value of this mu now. So if I have this value then by putting this value in the equation G so we have now P(x) dx it is dy/dx + P(x) dx into P(x) y = P(x) dx into Qx right? So on the left hand side we can write this equation as P(x) dx into y(x) whole derivate and that is equal to Q(x) e raise to power P(x) dx.

So now this is the differential equation that we have a derivative on the left hand side and the integrable function on the right hand side. So we know that how to solve this one. We can integrate both side with respect to x and if I do the integration with respect to x both side then what I will get is e P(x) dx yx becomes Q(x) it is P(x) dx plus some constant of integration and which further can be written as yx is equal to, so I can divide by this factor now.

So this factor if I take on the right hand side this becomes px dx then Qx e px dx plus e minus px dx into c. So we are able to solve this differential equation the general differential equation with the help of mu function and that mu is, this is my mu function. So this mu function has some special name.

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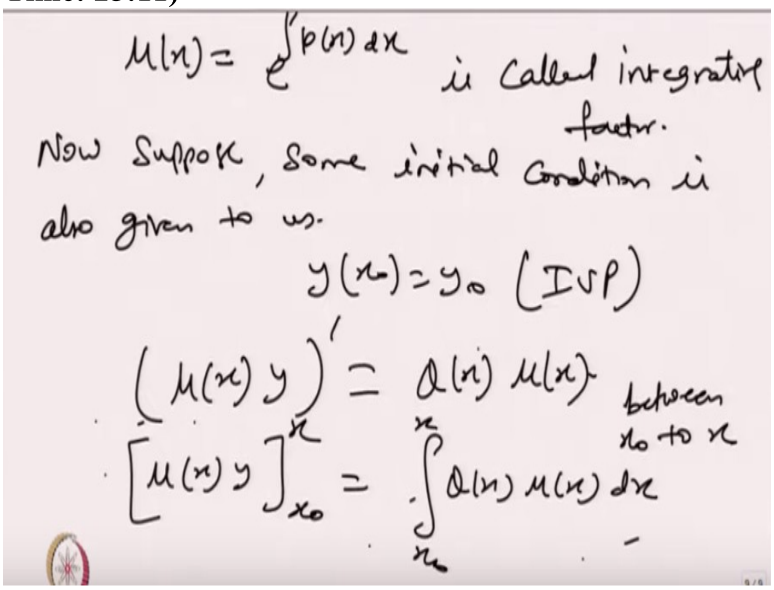

So this mu whatever we have written here, in this case Px dx is called integrating factor**.** So because it is helping us to make the differential equation the left hand side the differentiable. So this is the general solution for solving the general linear first order differential equation. Now, suppose some initial condition is also given.

So let us suppose I have a function yx, let us suppose I have a function y at x naught is equal to y naught. So this is my initial condition given to me. So I have the same general first order equation with this initial condition. So in this case my solution becomes now I, so I want just to let you know that how we can solve when the initial value is given to me.

So in this case I know that I already had the factor that is mu x so mu x is given to me. S I have mu x y dash is equal to Qx mu x. So this is I have. Now I integrate both side with respect to x from between because I now starting with the point x naught. So between x naught to x. So in this case I have to integrate starting from the x naught. So integrate this one.

Then I will get mu xy from x naught to X is becoming on the right hand side I will get x naught to x Qx mu x dx. So this is the definite integral we know.

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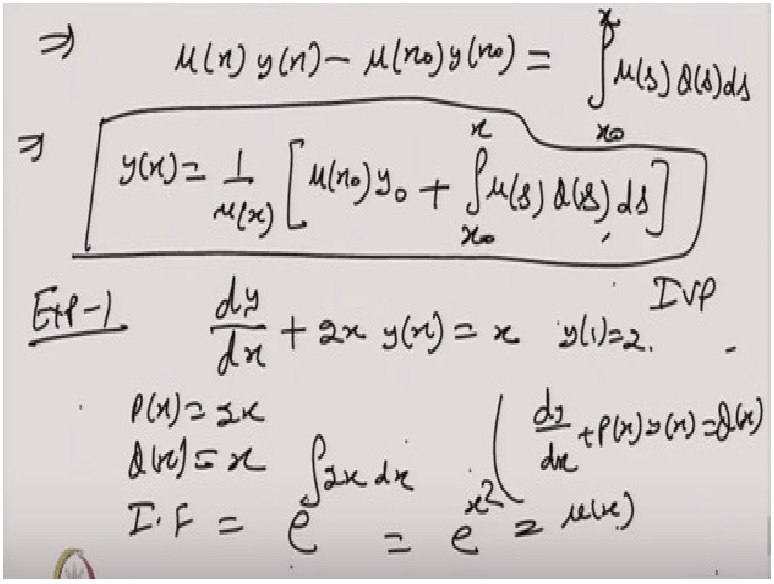

So the left hand side now if we put the limit in that case, so it becomes mu x yx - mu x naught, y yx naught and on the right hand side, I have factor x naught to x. So now I can change my internal variable, so I can write as mu s Qs ds because now in this case, after putting the limit, I will get the function in x. So from here my solution becomes yx is = 1/mu (mu x).

Then mu x naught and why naught is given to me so why at x naught is given to me plus x naught to x as Qs ds. So this is my solution for the initial value problem. So now we are able to solve any initial (value) problem, which has a general form of linear first order equation. So we are after developing this theory, just we want to solve some examples and we want to apply this theory that we are able to solve the differential equation or not. So now start with the example 1.

So let us I want to solve dy/dx + 2x yx = x with the initial condition that y1 = 2. So this is my differential equation initial (value) problem, and it is starting with the point 1 and passing that y at x equal to 1 is equal to two. So now this is this differential equation is of the form dy/dx + Px yx = Qx. So in this case my Px is 2x and my Qx is x.

And from here we can see that Px is 2x and Qx is x and they are the continuous functions. So it is satisfying the condition based on that one. So from using this one I can define my integrating factor. So integrating factor is e into Px, Px is 2x dx. So it becomes e x square. So this is my integrating factor that is the mu x we have defined in the last class, in the last slide.

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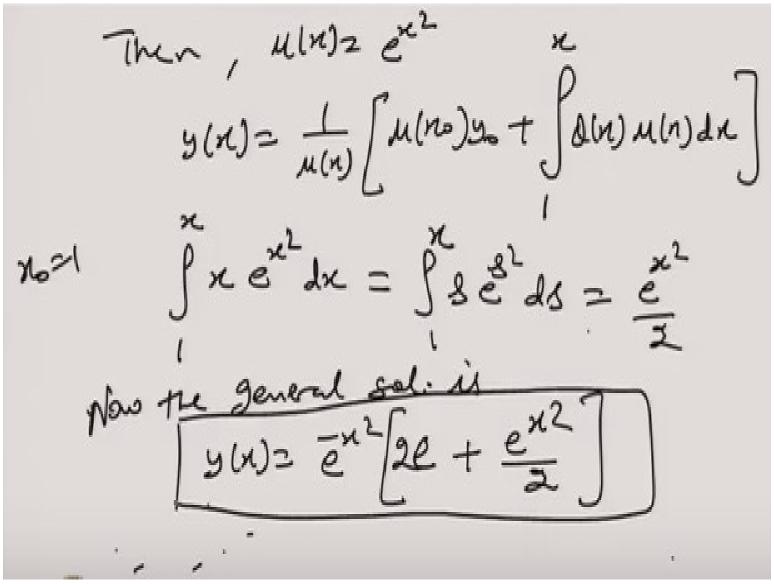

So once I have the integrating factor, then we know that I have the solution yx is = 1/ mu x and then mu x naught y naught plus. So in this case, my x naught is 1. So from 1 to x and then we have Qx and the integrating factor mu x dx. So this is my solution. So mu x is I already know, that mu x is e raised to power x square. So now this is already known, so I want to find out the right hand inside.

So this factor I want to find 1 to x Qx is x. So e raised to power x square dx. So this one I want to find. So this can be also written as 1 to x s e raised to power s square ds. Now I know that this is a function and this is the derivative. So it can be written as e raised to power s square by 2. So this will come e x square, so this is written as e x square by 2. So this is my factor I have solved.

Now the general solution is yx is equal to, so mu x is already known. So this becomes e raised to power minus x square and mu at x naught, what is the mu at x naught? What is x naught? So x naught is 1. So mu at x naught will be e only. So this will be e because putting just instead of x, I am putting x naught equal to 1. So once I put the x equal to 1, it becomes e and y naught is 2.

So it becomes 2, because in the previous you can see that this is y1 = 2. So once I have y1 = 2 then this value becomes 2e + e x square by 2. So after solving this equation, so this is my general solution for this initial value problem. So that is my solution. Now, so this is the example we have solved with the help of the our knowledge of solving the first order differential equation. So let us take some another example.

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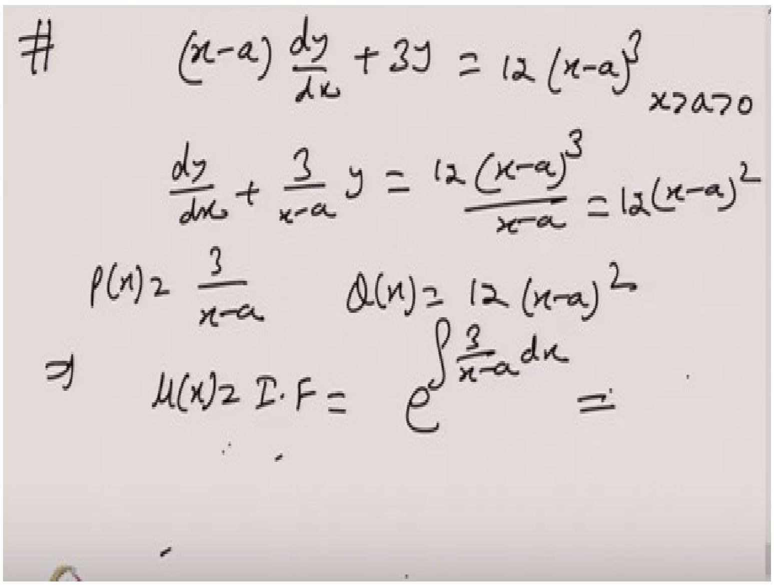

(x –a) dy/dx + 3y = 12 (x-a) power cube such that I have x is greater than a and greater than 0. So this is given to me that x is greater than a and both are greater than 0. So this differential equation is the first order linear differential equation and I want to solve this one. So in this case, first I want to put this one in the standard form. What are the standard form we have? So in this case, I know that x is not equal to a, so I can divide by this factor.

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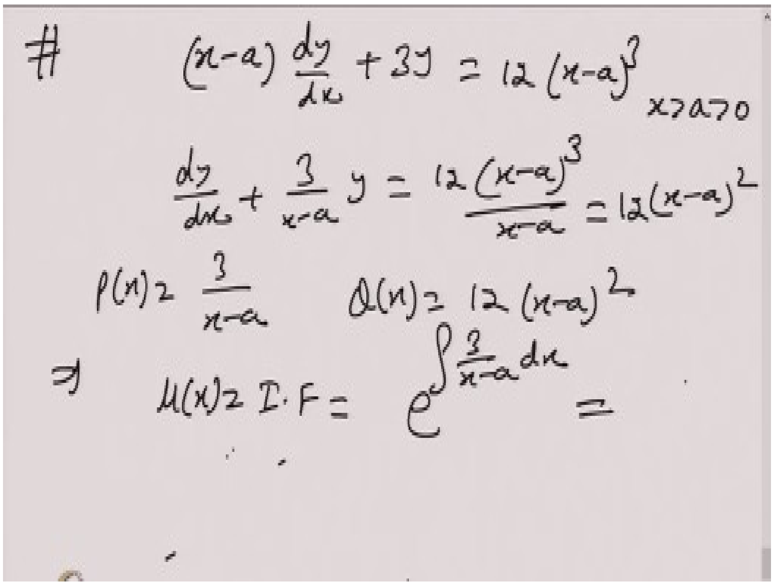

So I can divide the whole equation with x - a and I will get 3 x – a y = 12 x – a cube by x – a. So that will becomes 12 x - a power 2. So now in this equation we are able to convert into the standard form. So from here I know that my Px is 3/x – a and my Qx is 12 x – a whole square. So this Px and Qx also continuous function because in this case x is never equal to a.

So this is also continuous function well defined function. Now in this case, to solve this one I apply the again the integrating factor. So I find out my mu x that is the integrating factor. So this will be exponential P dx. So this is the integrating factor I want to solve. So if you solve this integrative factor, so it becomes

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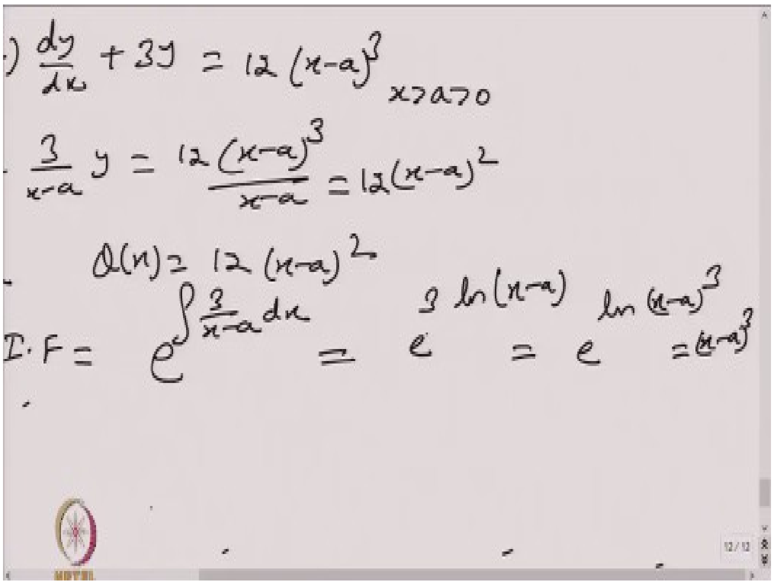

So e 3 and x - a I can take, so this becomes ln x - a. So that will become, because in this case my x – a is always positive because x is greater than a. So I have this value. This can be written as e ln x - a power cube and this is further can be written as x - a power cube. So now I have my integrating factor.

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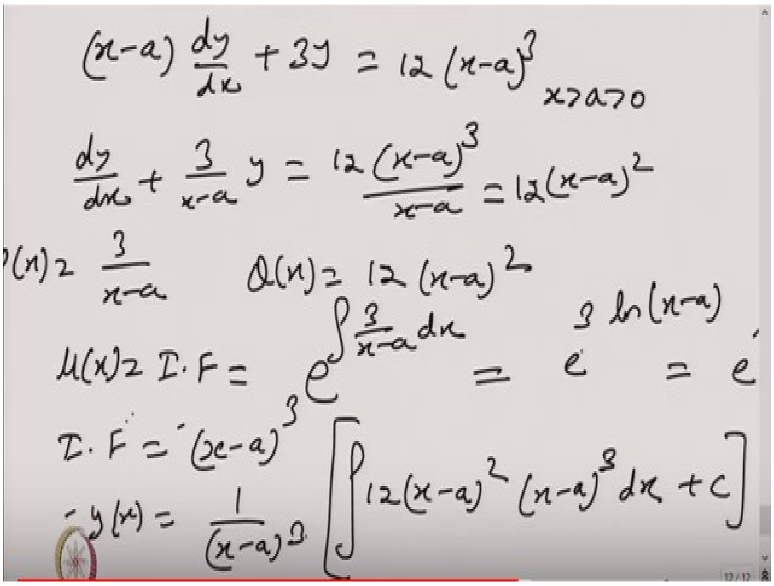

So integrating factor in this case is x - a power cube. So that is my integrating factor. So the general solution yx can be written as 1/x – a power q and then integrating with my Qx is 12. So it will be 12 x - a square with the integrating factor x – a Q dx + c. So that is the constant of integration. Because in this case, we do not have any initial condition. So this is a indefinite integrals.

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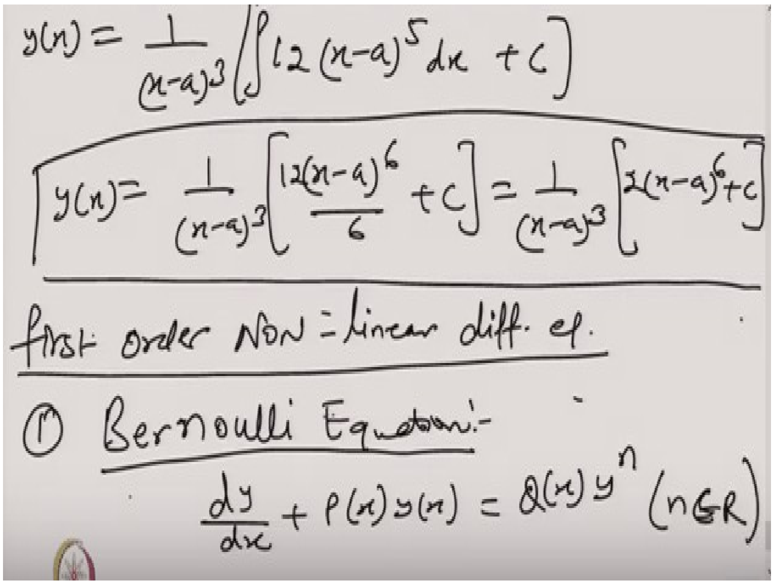

Now if I solve this equation further, it can be written as my Yx becomes 1/x – a power cube and this becomes 12 x – a power 5 dx + c. So this is the solution and which can be further solved. So this is the power 6, So it becomes x - a power 6 by 6 into 12 + c. So which can be further simplified and this equation becomes 2 x - a power 6 + c where c is the constant of integration.

So this is my general solution for this equation and c is the integration, the constant of integration. So using this one we are able to solve the first order, linear first order equation which have the standard form. Now I will also try to solve few nonlinear equations, which are very important and they are the first order equation. So I will just give you the example, because till now we have done only the linear equation which have the standard form.

Now I will give you some example, which has a nonlinear form. So first order nonlinear differential equation. So the most important is we start with the very important is first one is the Bernoulli equation. So Bernoulli equation we know. So Bernoulli equation is can be written as dy/dx + P(x) y(x) = Q(x) y raised to power n where the n belongs to real line. So this equation is looks similar to the standard form of the linear equation.

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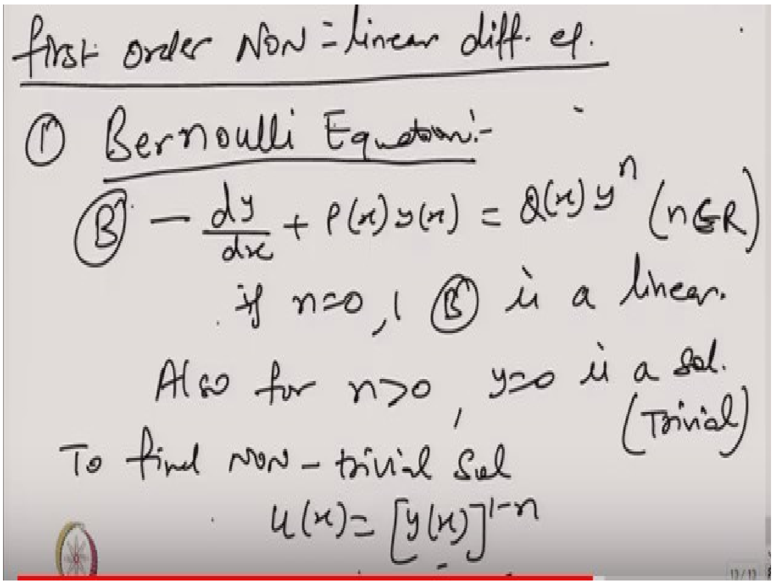

And now in this case if n is = zero, then it becomes just the simpler form of for n = 1, for n = 0 and 1 the equation. So this is write as B. B means Bernoulli equation. This B is a linear. So linear form and that we can solve very easily. Now for also for (n) greater than zero, y = 0 is a solution. So that is a trivial solution because if you put n greater than zero y = 0 is also one of the solution.

So for the non-trivial solutions now to find non-trivial solution, so what do we want? We want to convert this one into the form we are able to solve from the previous methods. So because this is the things we are doing then now this is a nonlinear equation. So by some transformation I want to transfer this equation into the known form which I have just solved. So further to find non-trivial solution.

So let us consider one transformation. So let I take a function ux is equal to some yx. So yx is the solution of this power 1 - n. So this is a transformation I am taking.

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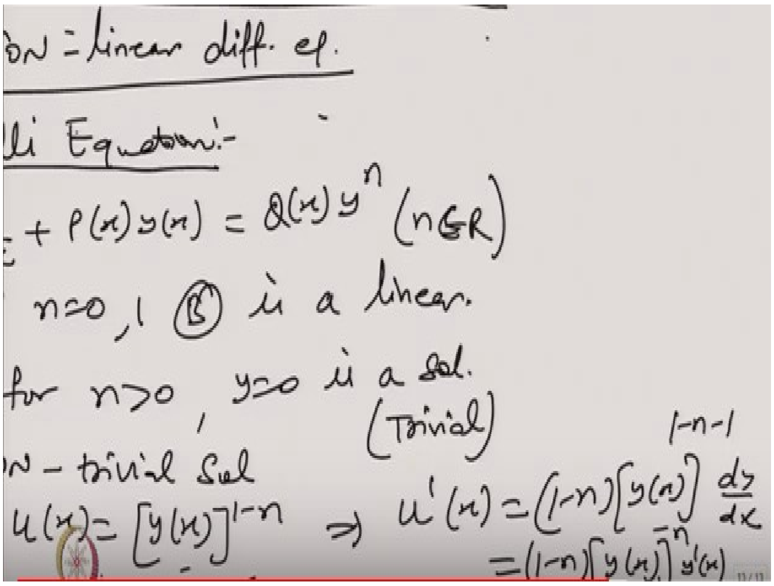

And using this transformation so from here, I can define that my u dash x it will be 1 – n yx power, so 1 – n – 1 and that is dy/dx. So this is my derivative. So which can be further simplified and it becomes 1 – n yx power –n and this is y dash. This is y dash x. Now which can be further solved, so this can be further solved.

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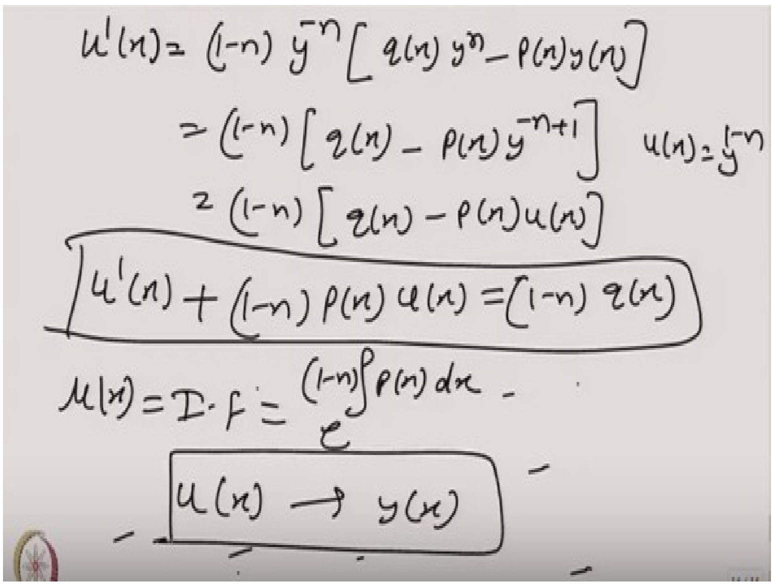

And we have you dash x becomes 1 – n y raised to power - n and the y dash. So y dash I already know that my differential equation. So y dash can be written as qx y to power n – px yx. This is because I have just in the y dash I have put the form right hand inside. So which can be further simplified and if I take the Y minus raised to power minus (n) inside, so I will get qx – px y – n + 1. So this is I am getting.

So which can be further written as 1 – n qx – px and what is (this) y? So I know that ux is y 1 – n. So this is 1 – n, so it becomes ux. So from here, I have my differential equation u dash x is equal to, so this form I will take on the left hand side. So this one becomes plus 1 – n px ux = 1 – n qx. So this is again the same form. So now this is my standard form, whatever the Px we have taken the previous and this is the capital Qx on the right hand side.

So this is the linear equation and I can solve this linear equation very easily and then by applying this one, so in this case my integrating factor because whenever I want to solve this type of linear equation, I need an integrating factor. So integrating factor in this case, that is my mu x will be e raised to power. So integrating factor in this case will be e raised to power 1 – n px dx.

So once I know the integrating factor, I can solve this differential equation. And then once I know the value of the solution ux, so because this is a linear equation ux, so from here I will get the value of ux. So once I know the value of ux from here I can find the value of yx by just putting back into the transformation. So let us do one example that how to solve a Bernoulli equation. So let us take one example. So let us take.

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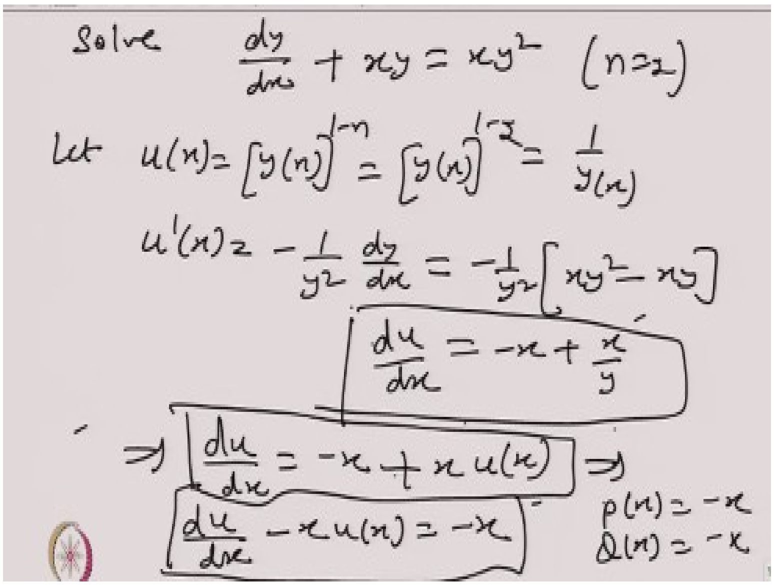

So let us solve the differential equation dy/dx + xy = xy square. So this is a nonlinear equation because nonlinear because we have y square on the right hand side and this is a Bernoulli equation for n = 2. So now I want to solve this differential equation. So

I will take the same transformation. So let my ux so I define ux, ux is = yx power 1 – n. So in this case my n is equal to 2. So that becomes yx 1 – 2. So it is 1/yx.

So this is my transformation ux. And in this case my u dash x will be -1/y square dy/dx. And dy/dx is already given to me. So from here I will get, so dy (by) dx can be written as xy square - x y. Now I take the factor inside, so I will get - x and + x/y. So I have a differential equation of this type now. So this differential equation can be written as again I will, so minus further I have du/dx = - x.

Now what I have taken, ux = 1/y. So from here I can write, so this can be written as x ux. So this is the equation I want to solve. So from here which can be solved, so this equation can be further written as dy (by) dx – x ux = - x. So this is again in the form of p qx. So in this case my Px is – x and my Qx is also - x. So if I solve this differential equation further then

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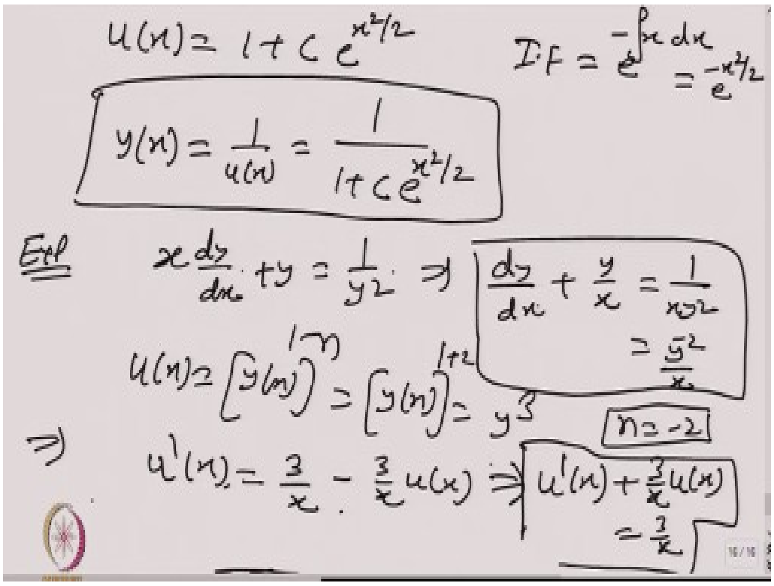

So if I solve this one, I will get the solution. So this you can solve yourself, then I will get the solution ux = 1 + c e raised to power x square by two, because integrating factor in this case is e – x dx. So that becomes e raised to power –x square by 2. So this is my integrating factor. So this is a solution in the terms of u and if I take the solution in the yx, so yx can be written as yx is 1/ux.

So this will be 1 + c e raised to power x square by 2. So that is the solution of this Bernoulli equation. So by the transformations, we are able to convert or transform in

(a) nonlinear differential equation into the linear differential equation and the linear differential equation we know how to solve that differential equation. So we can take another example of the Bernoulli form and that we take x dy/dx + y = 1/y square.

So in this case this is a Bernoulli form for n = - 2. But first I will transform this one into the standard form. So here we take that x is never equal to 0. I convert this one into divide by x. So I can write this as y/x + 1/xy square and which can be written as y – 2/x. So this is the again the binomial Bernoulli form with my n = - 2. Now this is also a nonlinear equation and this is in the Bernoulli form. So I want to solve this equation.

Now the criteria is same. I will take the transformation, transform this nonlinear equation to the first total linear equation and then I will solve with the help of integrating factor. So in this case the transformation let I have my ux and that is the yx 1 – n. So in this case my yx becomes 1 – 2. So it will becomes yq. So that is my transformation. So you square is equal to y cube.

So if I apply the same transformation, I will get my differential equation. So differential equation this case becomes u dash x = 3/x – 3/x ux. So this is a differential equation we are getting which can be further solved and which can be written as u dash x taking this one on the left hand side plus 3/x ux = 3/x. So this is my linear first order differential equation of the form of the standard form, which can be further solved and if I solve further, you can solve this equation by yourself.

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$$\boxed{u(x) = 1 + \frac{c}{x^3}}$$

$$\boxed{y(x) = [u(x)]^{1/3} = \left(1 + \frac{c}{x^3}\right)^{1/3}}$$

And in this case, I will get the solution ux = 1 + c x cube. So this is the solution I got. So by the transformation I know that my yx = ux power 1/3. So my yx solution becomes 1 + c over x cube power 1/3. So that is the general solution for the given Bernoulli form. So we are able to solve the very famous Bernoulli equation, which is highly nonlinear by taking the transformation and converting that one into the linear form.

So in the today lecture, we have discussed that how to solve the linear differential equation and then we have extended to the Bernoulli equation which is highly nonlinear by taking the transformation we are able to convert the Bernoulli equation into the linear equation and which can be solved with the help of integrating factor.

So thanks very much for listening me and from the next class we will go further and solve the Riccati equation and some important classification like separable equations and exact differential equation. Thank you very much.