Introduction to Fuzzy Set Theory, Arithmetic and Logic
Prof. Niladri Chatterjee
Department of Mathematics
Indian Institute of Technology- Delhi

Lecture 06
Fuzzy Sets Arithmetic and Logic

Welcome students to the 6th lecture on Fuzzy Sets, Arithmetic and Logic. In this lecture we shall primarily study T-conorm or fuzzy Union. We shall investigate their properties and also we will look at how to combine different operator.

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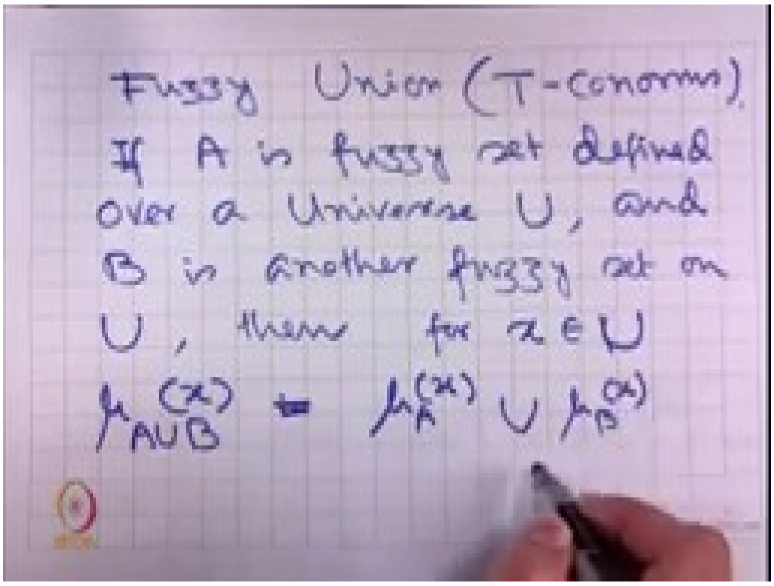

We know if A is a fuzzy set defined over a universe U and B is another fuzzy set on U then, for x belonging to U we need to define the membership of x to the fuzzy set A union B that is.

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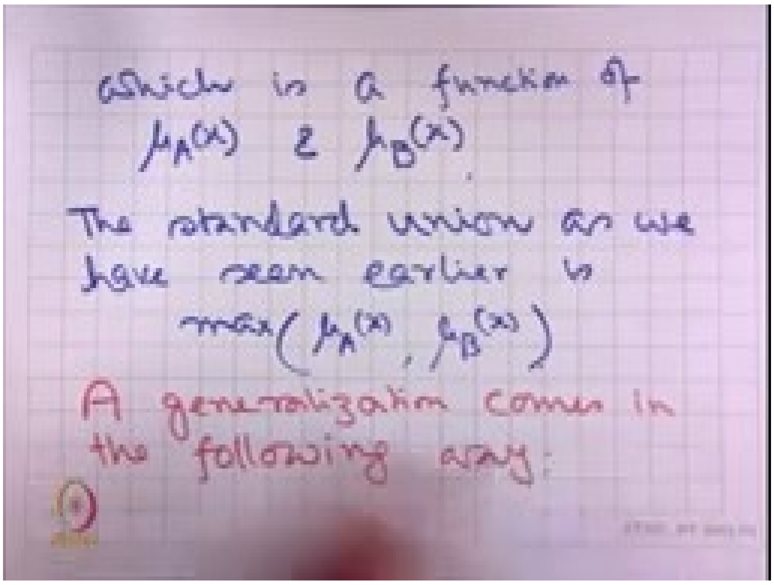

Which is a function of and . The standard union as we have seen earlier is .

But just like complement and intersection here also, we can think of generalization. So, the generalization comes in the following way.

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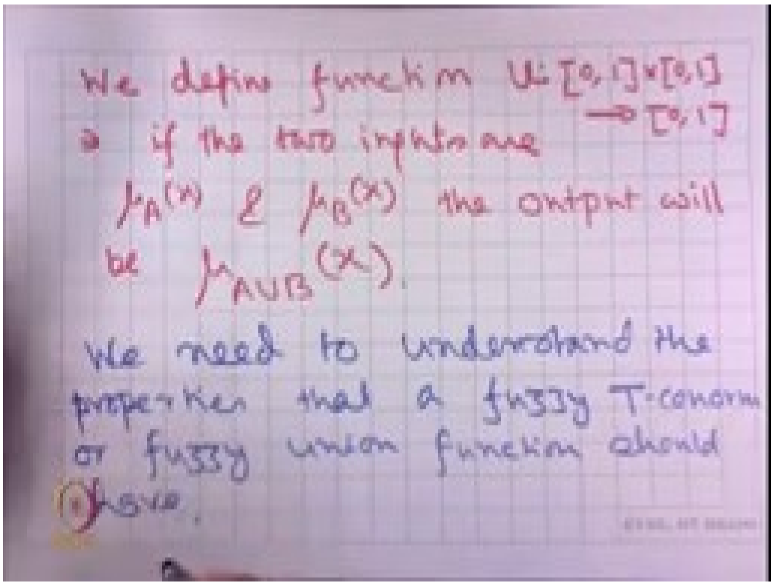

We define function such that if the inputs are and the output will be .

Obviously every arbitrary function defined from mapping into the real interval is not a union function.

We need to understand the properties that the Fuzzy T-conorm or Fuzzy Union function should have. So we enlist some of these properties which are called axioms.

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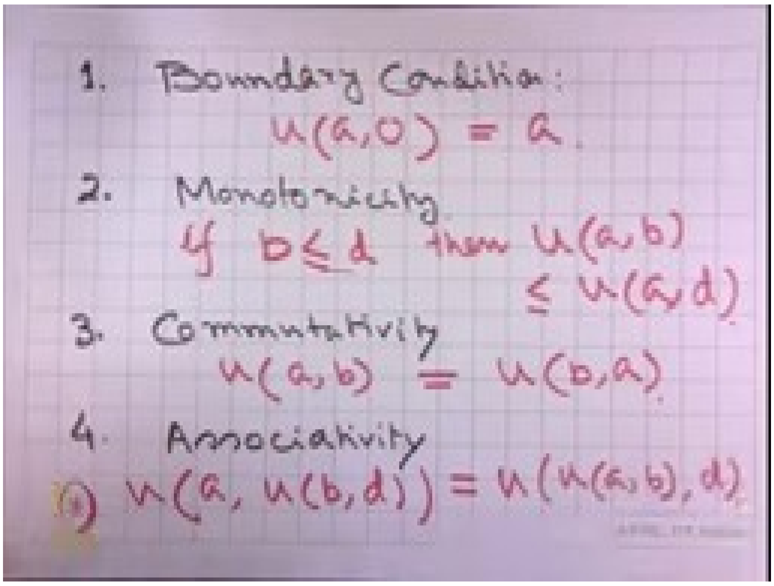

1) Boundary condition.

This means if has membership to the set but its membership to the set is . Then, membership of to the union of and should also be .

If we understand that then we go to the second property

2) Monotonicity

If then

3) Commutative; the commutativity property.

4) Associativity

These properties, if you notice are very similar to the properties of intersection that we have seen in the earlier lecture.

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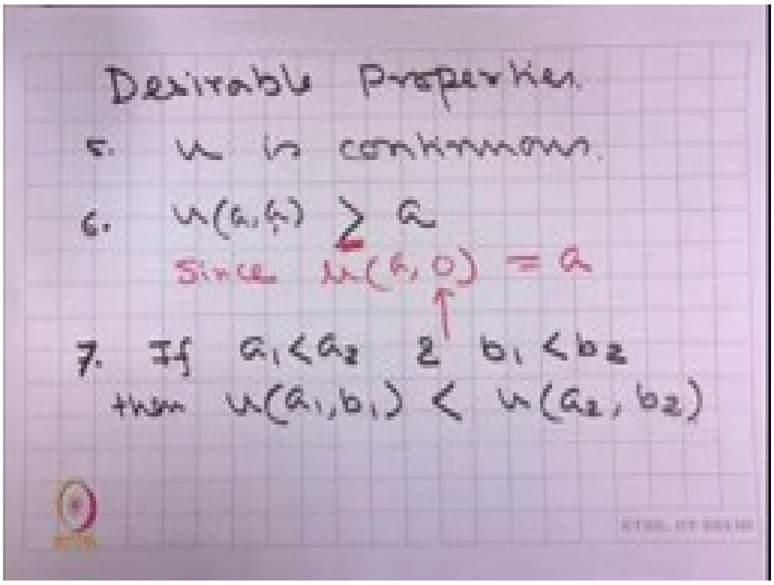

But of course there are certain desirable properties

5) is continuous

As explained with respect to complementation and intersection otherwise, small change in the membership may result in significant change in the membership of the Union.

6)

This is obvious. This is very natural since, , one would expect that if the membership to the second set is bigger than then membership value to the Union should be more. Therefore, since here we are looking at the second membership value as instead of , it is expected that, the membership to the Union will be greater than .

7) If and then

This we have seen is very similar to what we have done with respect to intersection. We have seen a theorem with respect to intersection.

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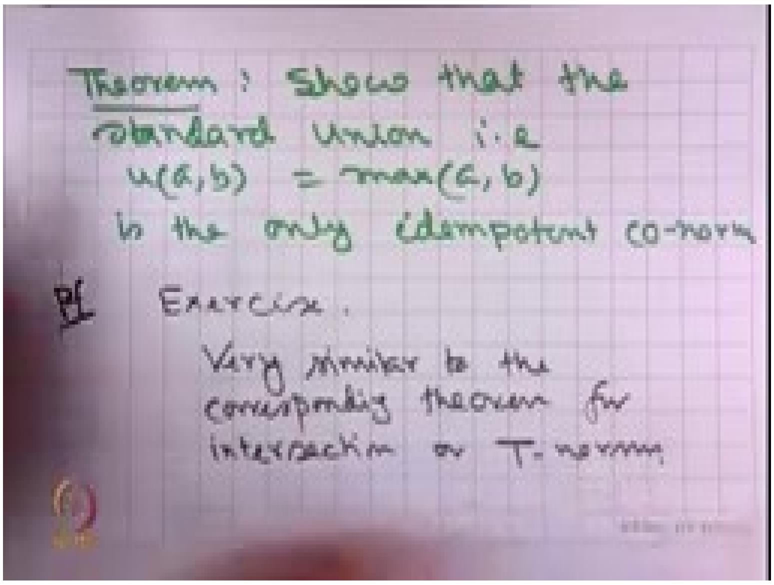

So, a similar theorem we write with respect to Union

Theorem:

Show that the standard union that is is the only idempotent co-norm.

Proof: I leave it as an exercise.

Very similar to the corresponding theorem for intersection or T-norms

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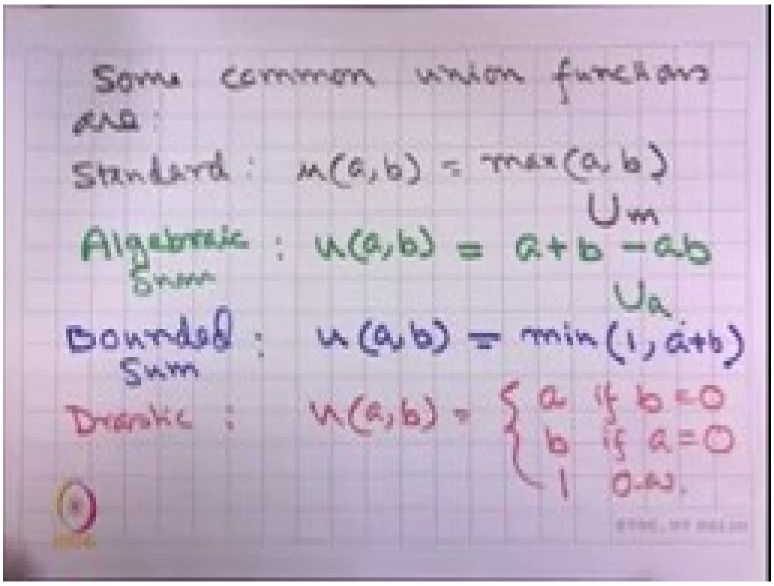

As before some common union functions are:

1) Standard:

.We will write it as .

2) Algebraic Sum:

If you remember for intersection, it was product.

Here the function is . We will write it as

3) Bounded sum:

Again very similar to the bounded product that we have seen with respect to intersection.

. We will write it as

And finally let us write

4) Drastic union:

That is, if any one of them is then the membership to the Union is the value of membership of the other one. But, if both of them are non-zero then it does not matter we are giving full membership to the Union.

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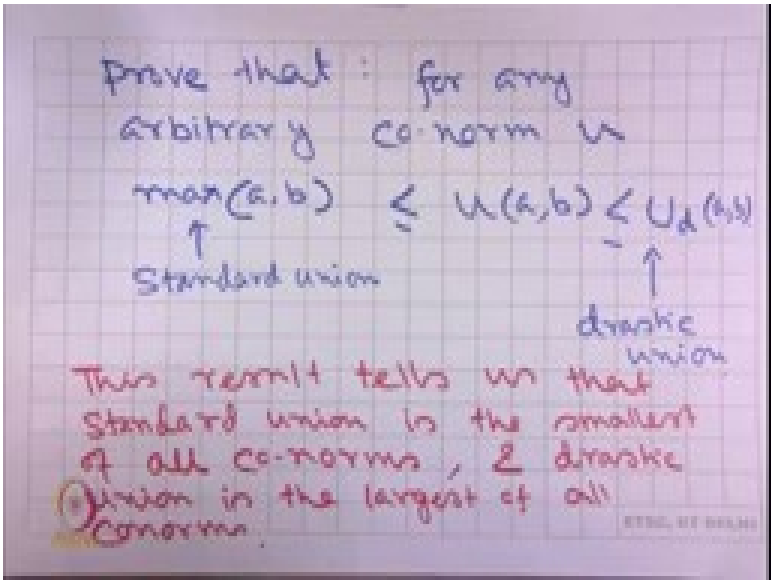

Again one result that I want you to prove is this.

Prove that:

For any arbitrary co-norm ,

So, very similar result we have seen with respect to fuzzy T-norms.

I want you to emulate that proof to prove this statement.

This result tells us that Standard Union is the smallest of all unions and Drastic union is the largest of all conorms.

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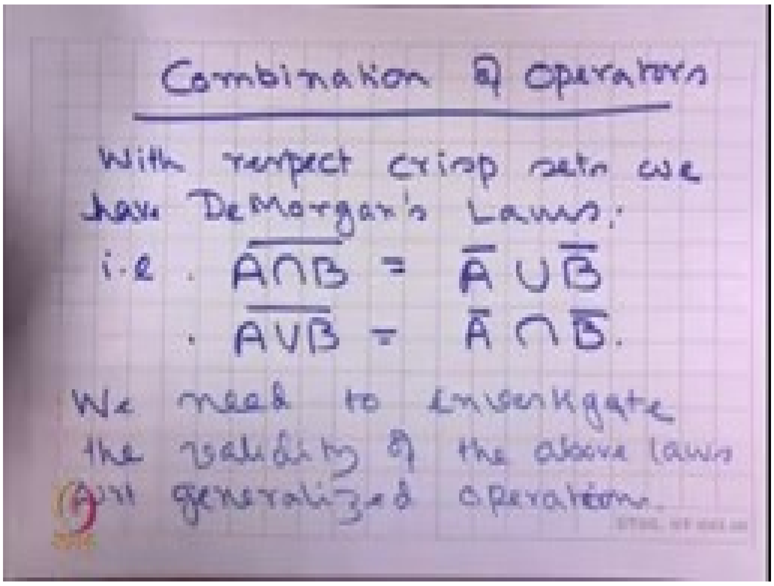

Combination of operators:

With respect to crisp sets, we have De Morgan's law, that is,

and

We need to investigate the validity of the above laws with respect to generalized operators or operations.

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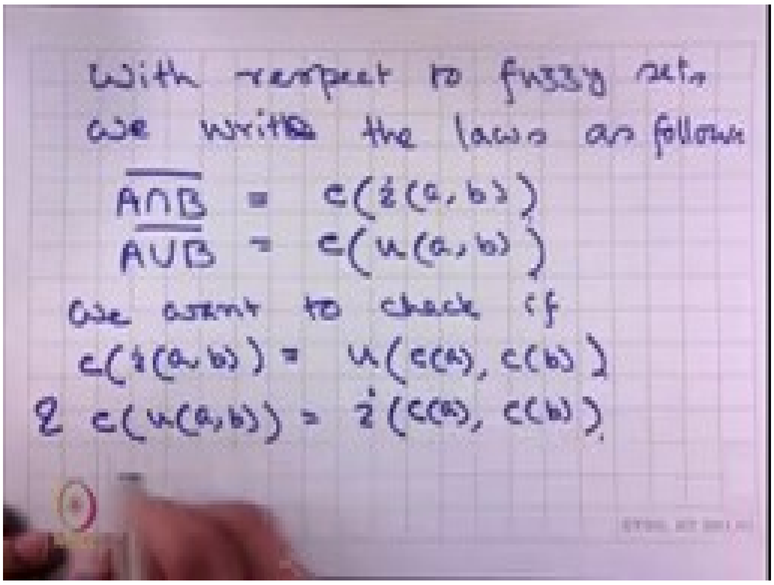

So, when we look at the laws with respect to fuzzy sets, we write them as follows:

We want to check if:

and

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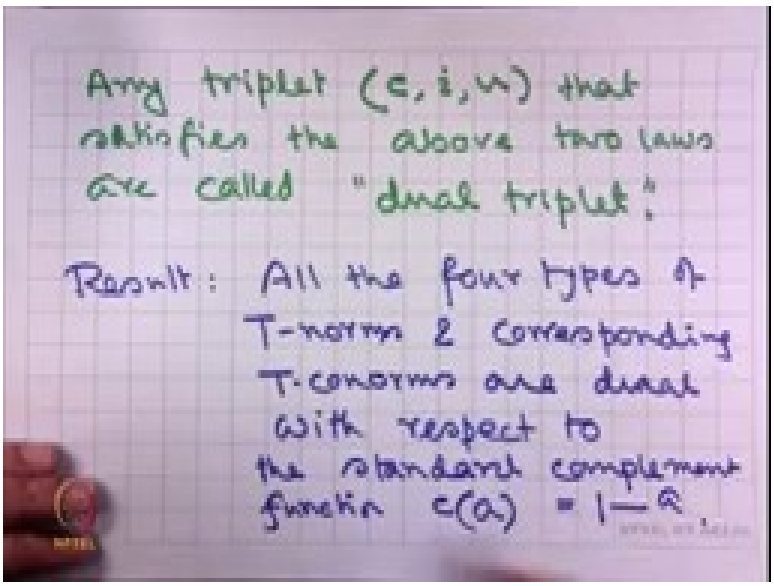

Any triplet , where is a complementation function,is an intersection function and is a union function, that satisfies the above two laws are called dual triplet.

Result:

All the four types of T-norms and corresponding T-conorms are dual triplets with respect to the standard complement function

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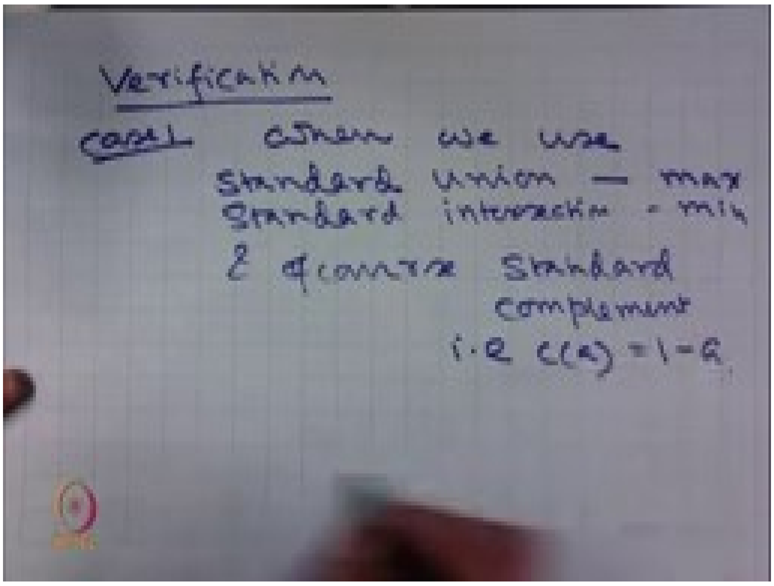

Verification:

Case 1: Let us first verify it when we use standard Union, that is , standard intersection, that is and of course standard complement that is .

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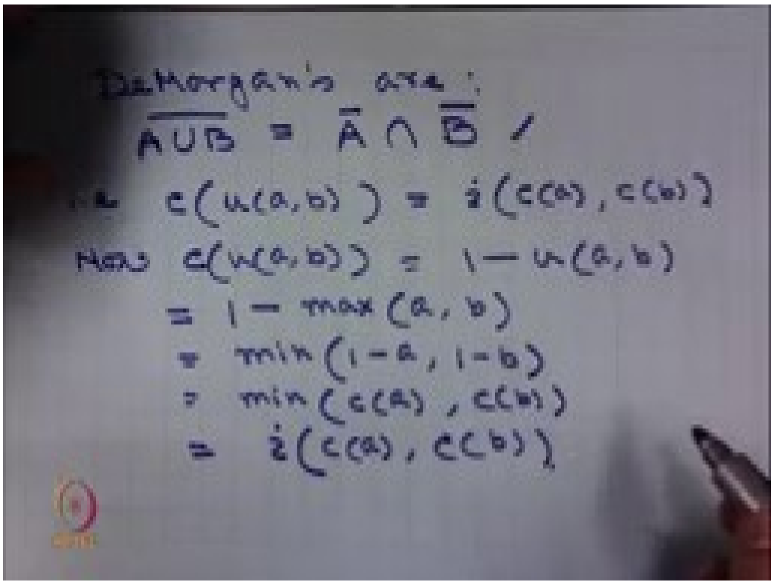

De Morgan's laws are:

, that is with our notation,

Now,

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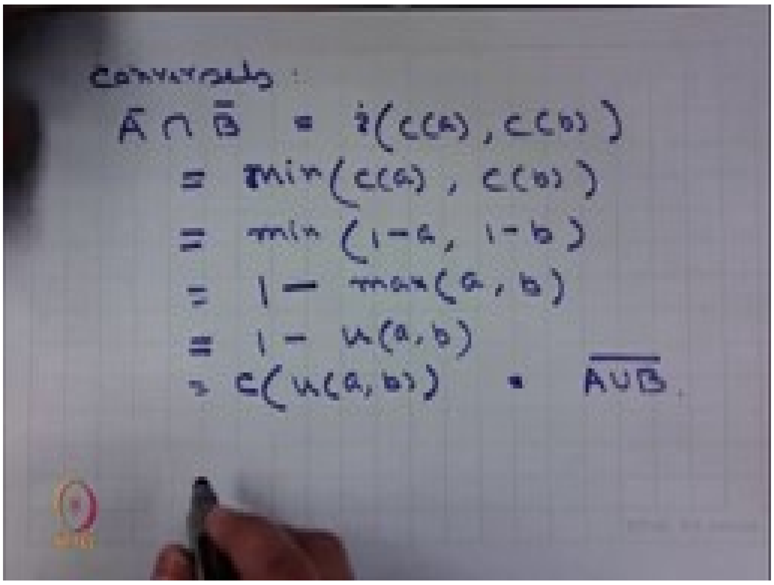

Conversely,

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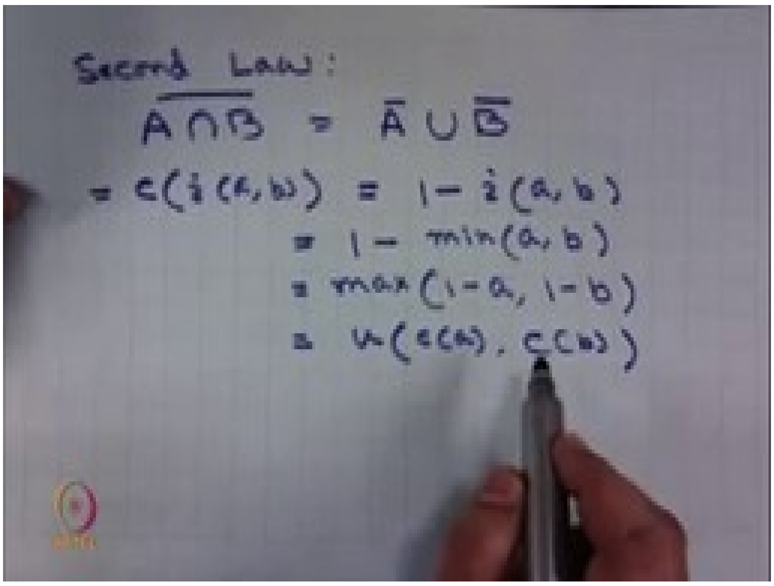

So this establishes the first law.

Second law:

Therefore, we establish that is actually .

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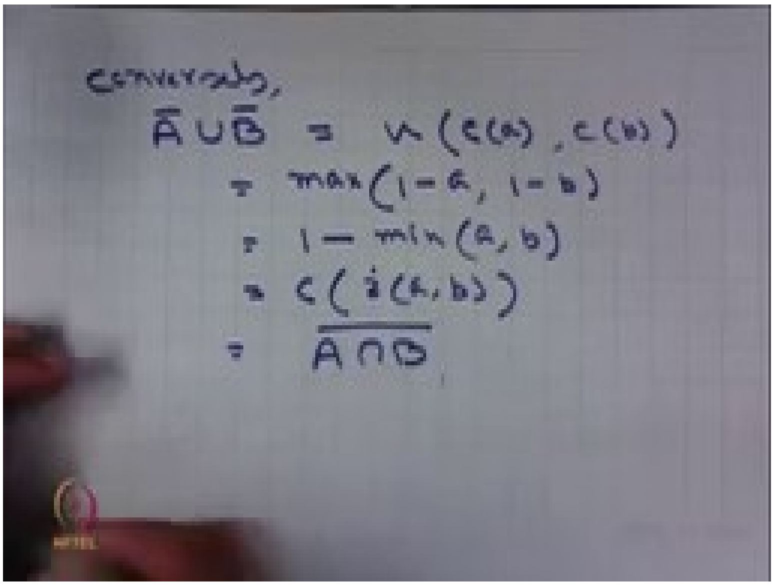

Conversely,

So, this verifies our statement with respect to Standard Union and Standard Intersection. We shall now verify with respect to some other Union and Intersection functions.

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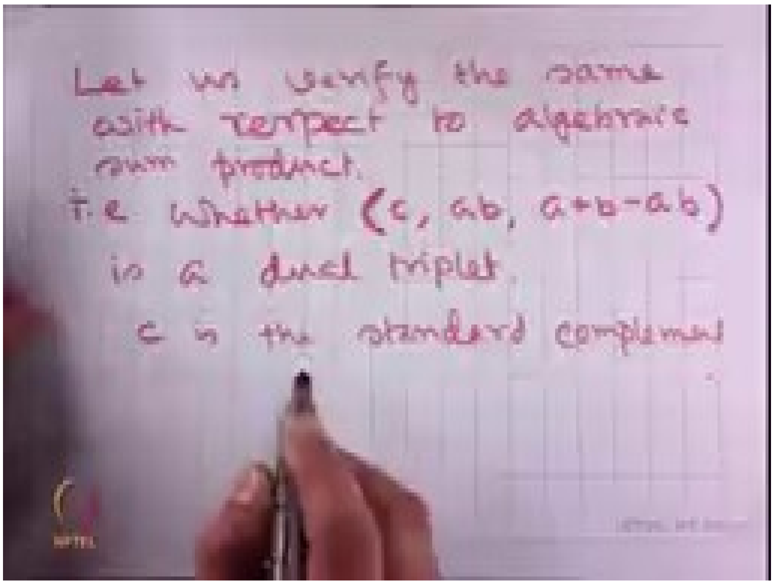

Let us verify the same with respect to Algebraic Sum and Algebraic Product. Therefore, we want to check whether the triplet   is dual triplet. Mind you, is the standard complement.

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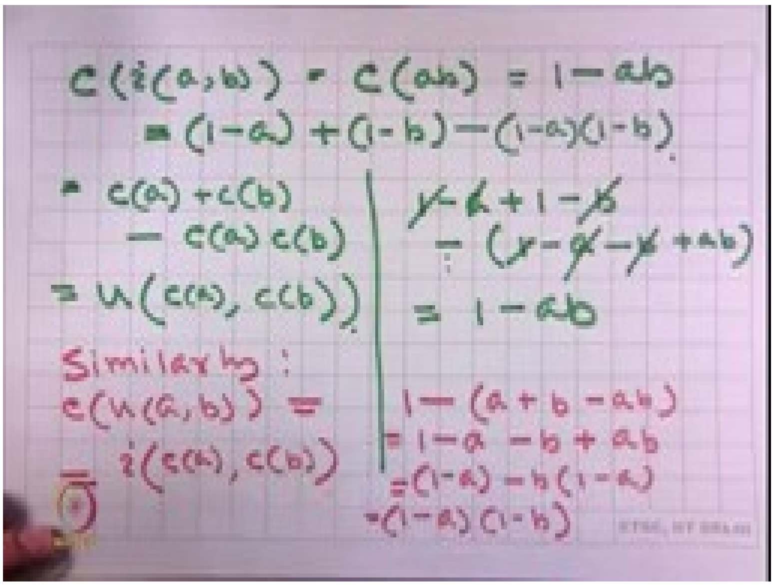

Therefore,

And,

Therefore,

Similarly,

I like you to verify the same results with respect to the other two types of union and intersections that we have mentioned namely bounded sum and product, and drastic.

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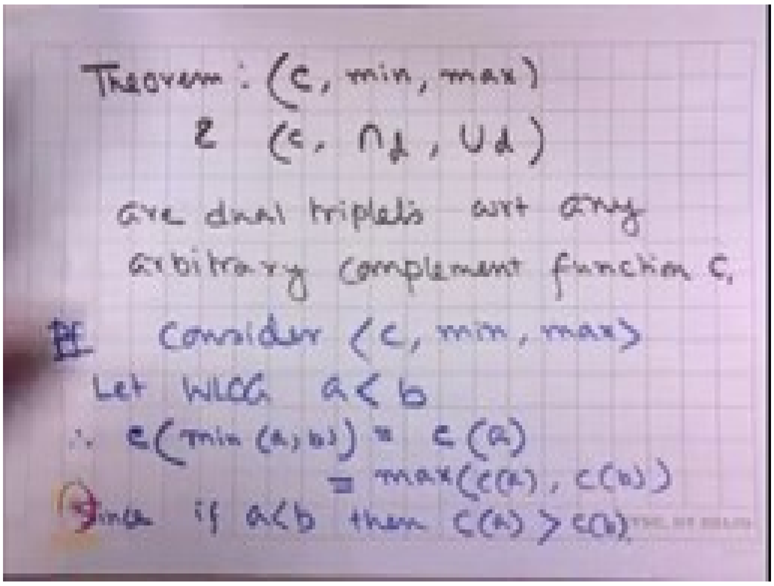

Theorem:

and are dual triplets with respect to any arbitrary complement function .

Proof:

Consider .

Let without loss of generality

Therefore, .

Since if then .

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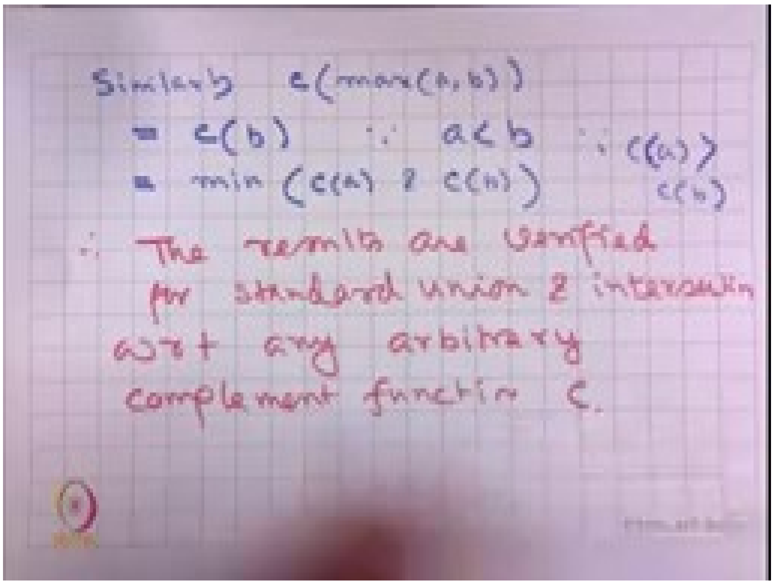

Similarly,

Therefore, the results are verified for Standard Union and Intersection with respect to any arbitrary complement function c,

I suggest that you verify the same results with respect to drastic union and intersection.

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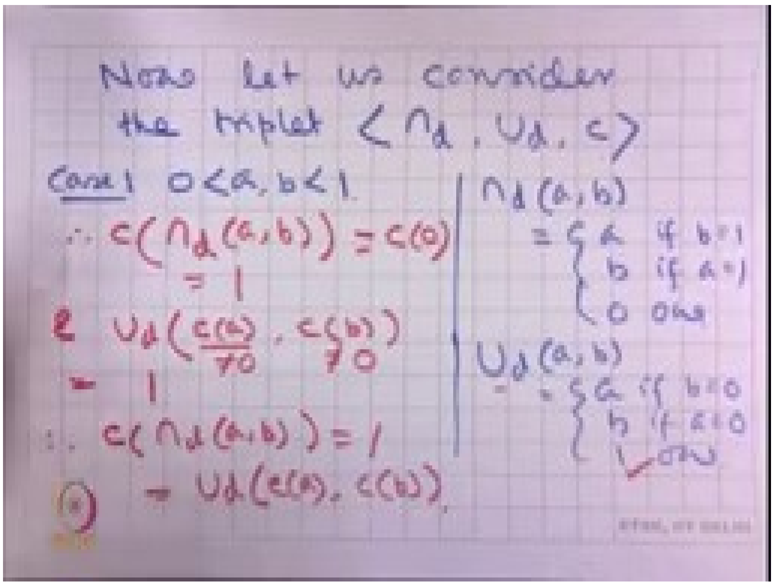

Now let us consider the triplet , where is drastic intersection, is drastic union and is any arbitrary complement.

Let me recollect,

and

Case 1:

Therefore,

And because and as .

Therefore, we see that

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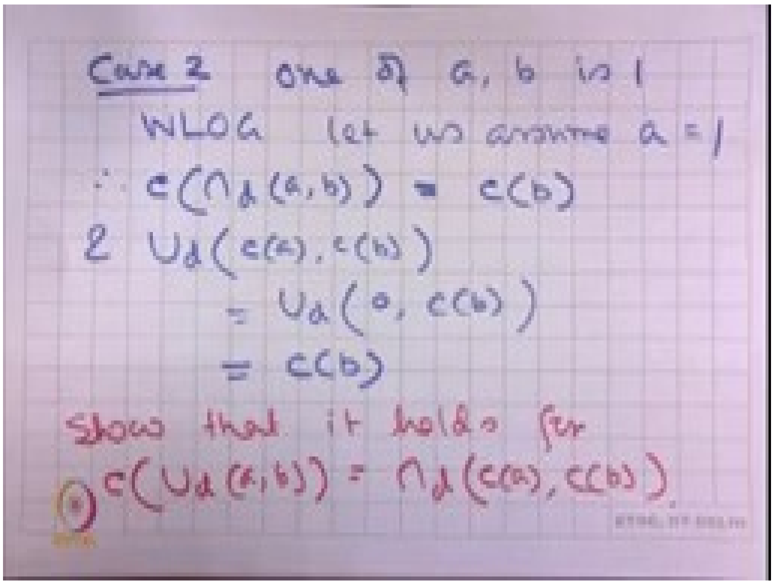

Case 2: At least one of is .

Without loss of generality let us assume

Therefore, , since

And

Show that it holds for:

I leave that as an exercise.

The question comes therefore, how do we know or how do we construct a dual triplet?

The following theorem helps us in designing that.

When we have a T -norm and we have an involutive complement function then we can construct a T-conorm with the help of them in the following way and not only that the newly constructed T-conorm will be forming a dual triplet with respect to the T-norm and the complement function.

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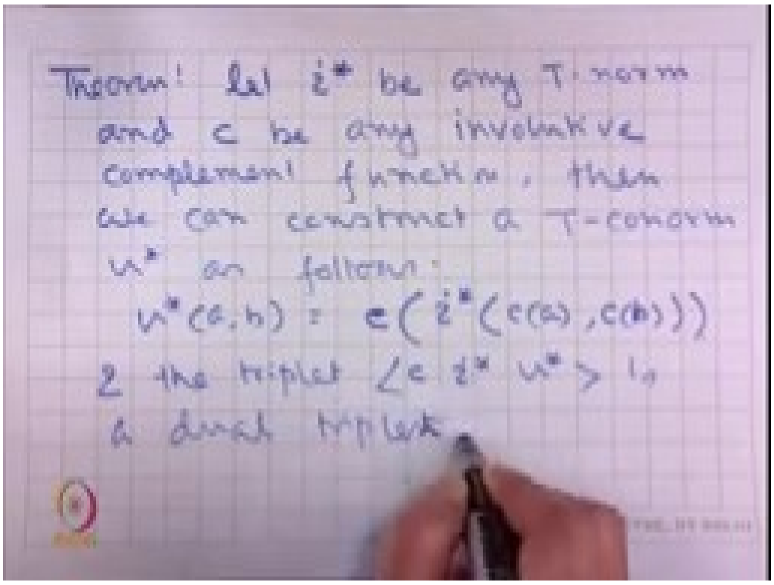

Theorem:

Let be any T-norm and be any involutive complement function, then we can construct a T-conorm as follows,

and the triplet is a dual triplet

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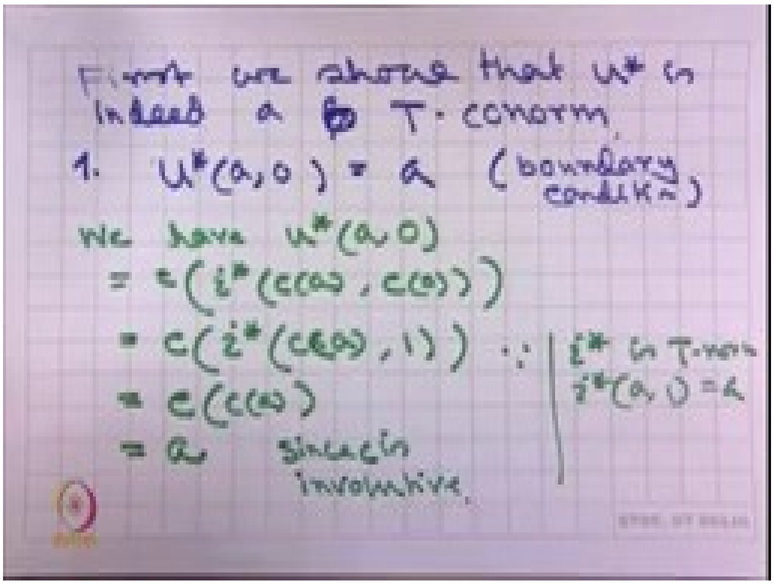

First we show that is indeed a T-conorm.

For that we need to verify the four necessary properties:

1) . We know that it is the boundary condition.

We have

is a T-norm,

is involutive

Therefore, property 1 is verified.

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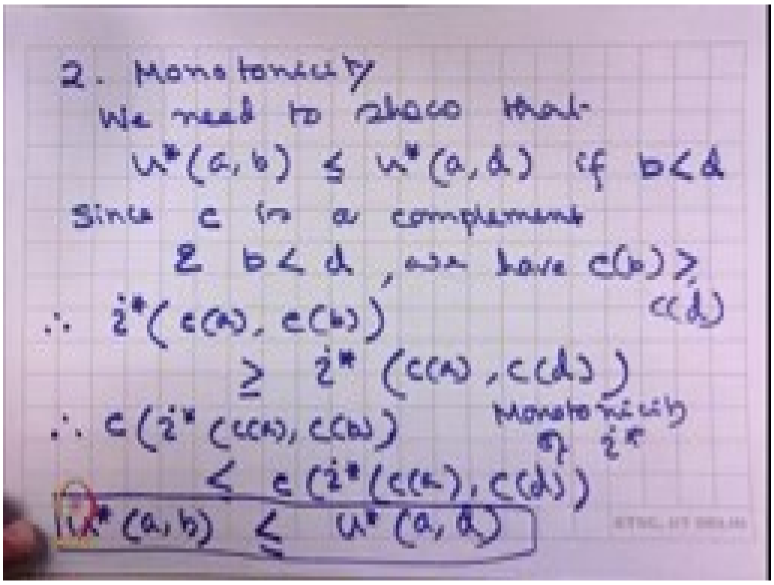

2) Monotonicity

We need to show that if

Since is a complement and , we have .

Therefore, . This is because of monotonicity of . Therefore, because we are now taking the complements therefore inequalities will be reversed.

Therefore, we get that

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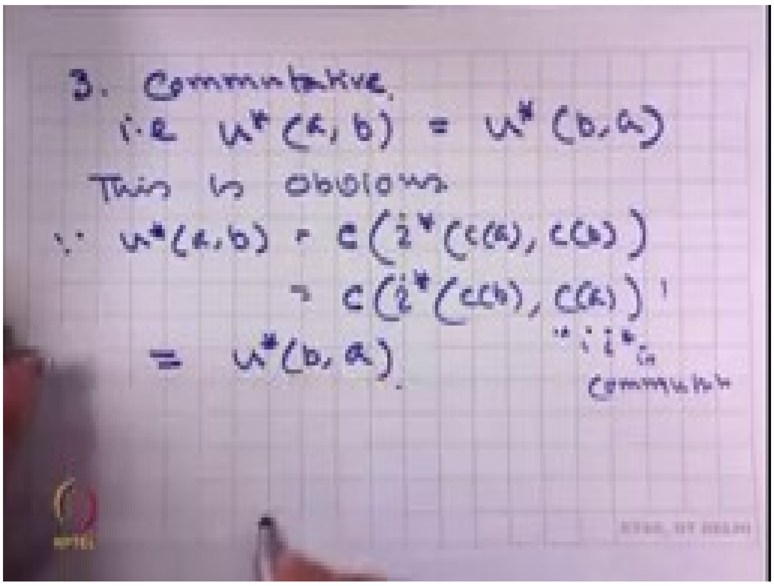

3) Commutative.

That is, This is obvious.

Since

because is commutative.

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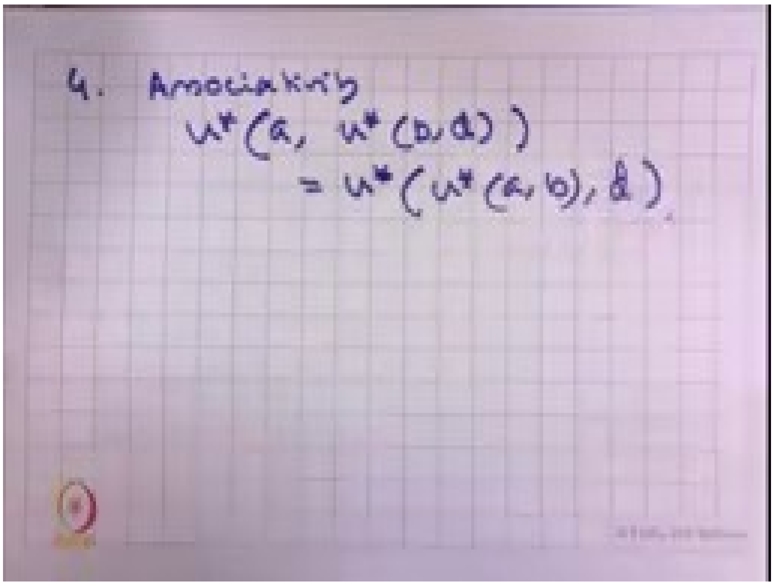

4) Associativity.

That is

I leave this as an exercise and I want you to apply the definition of and different properties of and to verify this result.

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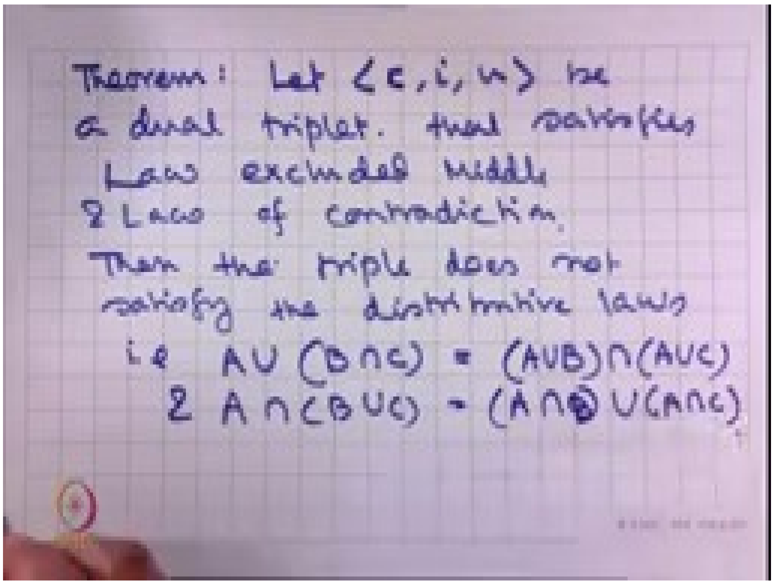

Before I stop today I state and prove another theorem:

Let be a dual triplet that satisfies Law of excluded middle and Law of contradiction. Then the triple does not satisfy the Distributive Laws, that is

So, the theorem suggests that if and are (norms) T-norms and T-conorms such that along with , which is a complement function it satisfies the law of excluded middle and law of contradiction, then the distributive property does not hold.

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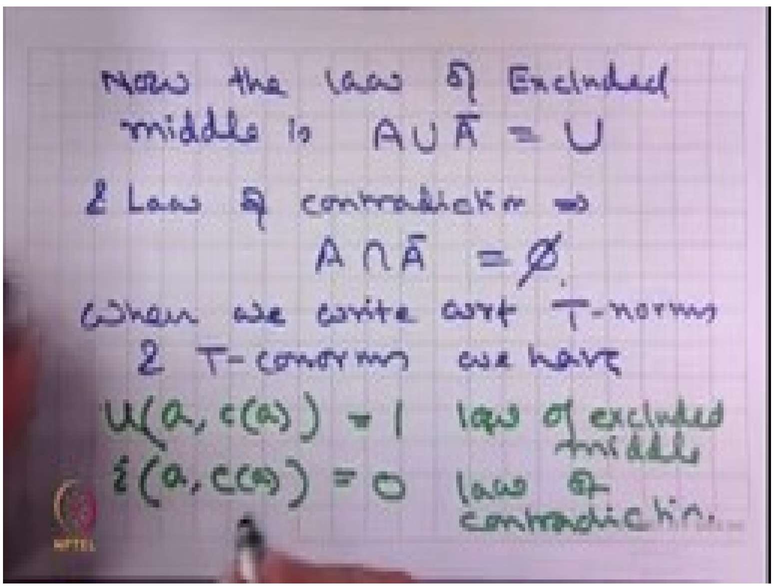

Now the law of excluded middle is:

And Law of contradiction implies

When we write with respect to T-norms and T-conorms, we have

which is the law of excluded middle and ; law of contradiction.

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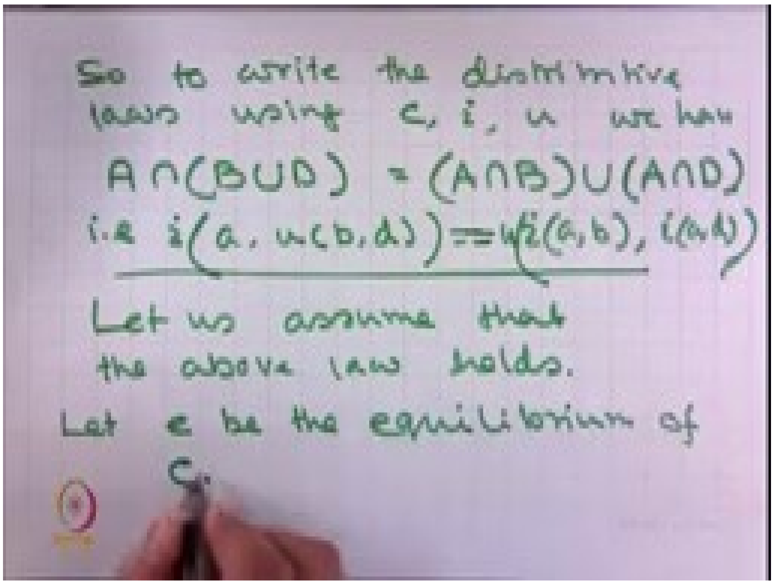

So, to write the distributive laws using and we have

So, we need to show that this does not hold.

Let us assume that the above law holds, let be the equilibrium of .

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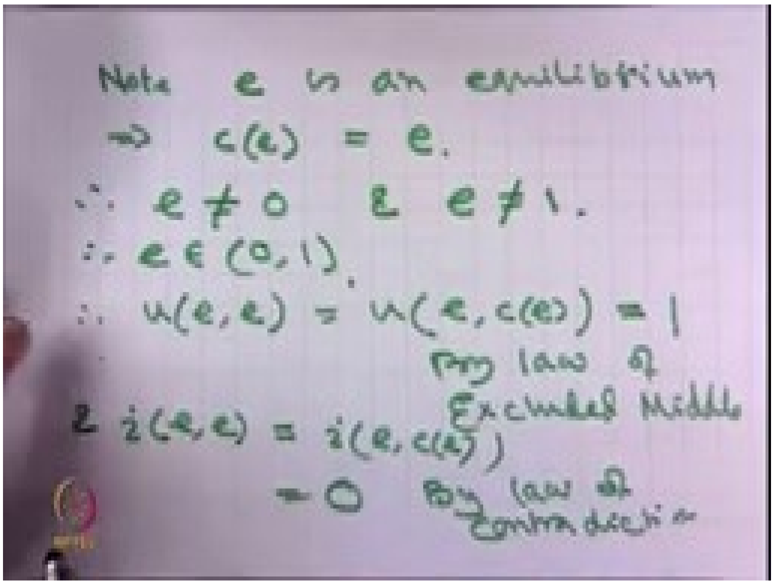

Note that is an equilibrium.

Therefore, . Therefore,

Therefore, by law of excluded middle and by law of contradiction.

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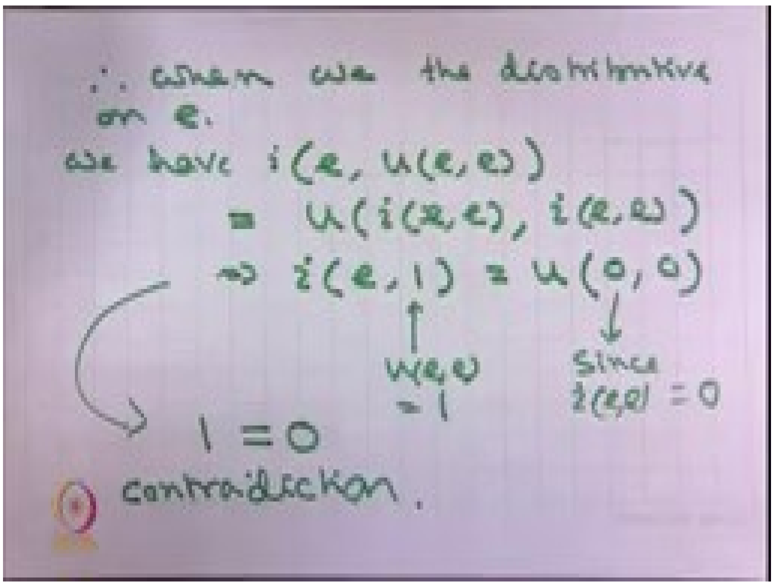

Therefore, when we use the distributive law on .

We have

Therefore, this implies that which is a contradiction.

Therefore, if we assume that the distributive law holds we arrive at a contradiction.

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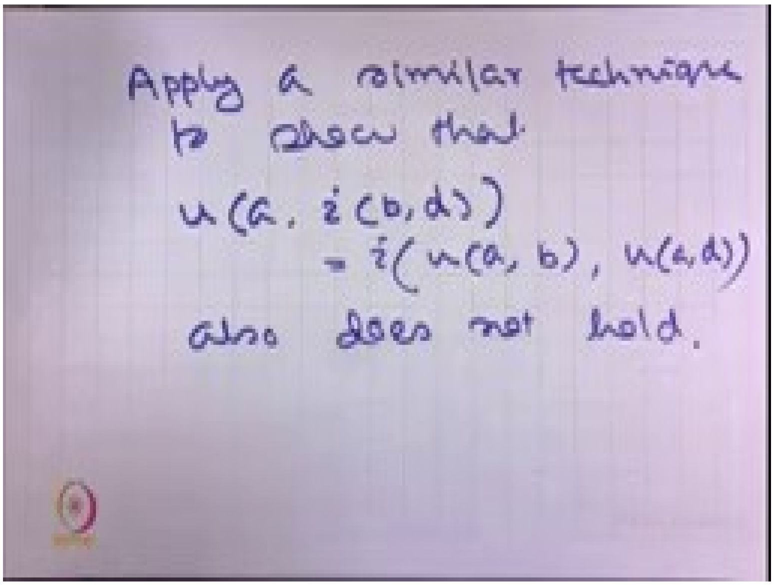

Very similar contradiction we will get if we consider the other distributive law, that is

Apply a similar technique to show that also does not hold. I leave that as an exercise.

Okay students. I stop here today so in this set of 3 lectures we had looked at different set operations, their properties and certain properties related to combining them.

Now I will change the topic and from the next class I will start fuzzy arithmetic, that is arithmetic using fuzzy numbers. Thank you so much.